ON TWO PRESENTATIONS OF THE AFFINE WEYL GROUPS OF THE CLASSICAL TYPES

JIAN-YI SHI

Department of Mathematics, East China
Normal University, Shanghai, 200062, P.R.C.

and

School of Mathematics and Statistics, University
of Sydney, Sydney, NSW 2006, Australia

ABSTRACT. The main result of the paper is to get the transition formulae between the alcove form and the permutation form of $w \in W_a$, where W_a is an affine Weyl group of type $\widetilde{B}_m$ $(m \geqslant 3)$, $\widetilde{C}_n$ $(n \geqslant 2)$ or $\widetilde{D}_k$ $(k \geqslant 4)$. On the other hand, we get a new characterization for the alcove form of an affine Weyl group element which has a much simpler form comparing with that in [10]. Our results give an affirmative answer to a conjecture of H. Eriksson and K. Eriksson in [4] concerning the characterization of the inverse table of $w \in W_a$.

Introduction.

Affine Weyl groups, as a family of infinite crystallographic Coxeter groups, play a more and more important role in various fields of mathematics, such as Kac-Moody algebras, algebraic groups and their representation theory, combinatorial and geometric group theory, etc. [2; 3; 5; 7; 9; 14].

Besides the presentations as Coxeter groups (i.e., the ones by generators and relations over the pairs of generators), there are many other presentations for the affine Weyl groups W_a, in particular for those of the classical types, i.e., types $\widetilde{A}_l$ $(l \geqslant 1)$, $\widetilde{B}_m$ $(m \geqslant 3)$, $\widetilde{C}_n$ $(n \geqslant 2)$ and $\widetilde{D}_k$ $(k \geqslant 4)$. Two of their presentations are particularly useful, one is to regard W_a as a certain permutation group over the integer set $\mathbb{Z}$ (only applied for the classical types); the other is to identify W_a with the set of alcoves in a euclidean space E after removing a certain set of hyperphanes (applied for all types) [1; 8; 9; 10; 11; 12; 13].

The relations between these two presentations has been explicitly described for the case of type $\widetilde{A}_l$ (see [9]). but not yet for the other classical types so far. The aim of the

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present paper is to study these relations for the cases of types $\widetilde{B}_m$, $\widetilde{C}_n$ and $\widetilde{D}_k$. Our main results is to obtain the transition formulae between the permutation form and the alcove form of $w \in W_a$ when the type of W_a belongs to these three families (see Theorems 4.1, 5.2, 5.4 and 5.6).

The alcove form $(k(w;\alpha))_\alpha$ of $w \in W_a$ adopted here is essentially the same as that in [10], but with two changes as below. The one is that the relation $k(w;-\alpha) = -k(w;\alpha)$ in [10] is replaced by $k(w;-\alpha) = -k(w;\alpha) - 1$ for all the positive root (and hence for all the root) α. The other is that in dealing with the case of the classical types, we replace the root system Φ by the corresponding coroot system $\Phi^\vee$ as the index set of the alcove form of w. The reason for these changes is to make our transition formulae simpler. The another reason for the second change is that we have got a new characterization for the alcove form of $w \in W_a$, which is equivalent to the original one in [10, Theorem 5.2] but has a much simpler form when stated in terms of coroot system instead of root system (see Theorem 1.3). We shall also show that when an affine Weyl group W_a has the classical type, the entries of the alcove form of $w \in W_a$ exactly comprise the inverse table of w defined by H. Eriksson and K. Eriksson (see [4, §8.] and Theorem 4.1). Thus our new characterization for the alcove form of $w \in W_a$ gives a characterization for the inverse table of w (see Theorem 4.5), the latter was conjectured by H. Eriksson and K. Eriksson in the case of type $\widetilde{C}_l$ in [4, 8.3.].

In general, the permutation form of $w \in W_a$ is not unique. This depends on the way of embedding the group W_a into the permutation group on $\mathbb{Z}$ (see [1; 13]). The permutation form of $w \in W_a$ we take in the present paper has the advantage that when restricting to the corresponding Weyl group, the permutation form of an element is just the usual one.

Thoughout this paper, an affine Weyl group is always assumed irreducible, i.e., its Coxeter graph is connected.

The contents of the paper are organized as follows. In Section 1, we recall some results of [10] concerning the alcove form of an affine Weyl group element and give a new characterization for the alcoves of a euclidean space. Then from Section 2 on, we pay our attention only to the affine Weyl groups W_a of the classical types. We describe the alcove form and the permutation form of $w \in W_a$ in Sections 2 and 3 respectively. Then we show our main results of the paper in Sections 4 and 5, where we obtain the transition formulae between the alcove form and the permutation form of $w \in W_a$ in the cases of the classical types.

§1. The alcove forms.

In this section, we shall recall some results concerning the alcove form of an element w in an affine Weyl group W_a. As mentioned in the Introduction, we make some formal changes in the definition of the alcove form of w. This forces us to reformulate some results in [10]. There is one new result in this section (i.e., Theorem 1.3), which gives a new characterization for an alcove in a euclidean space.

1.1. Let Φ be an irreducible root system of rank $l > 1$. Let E be the euclidean space spanned by Φ with inner product $\langle\ ,\ \rangle$ such that $|\alpha|^2 = \langle\alpha,\alpha\rangle = 1$ for any short root α of Φ. Choose a simple root system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ of Φ and let Φ^+, Φ^- be the corresponding positive and negative root systems. Let $-\alpha_0$ be the highest short root of Φ. Denote by $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ the coroot of $\alpha \in \Phi$ and by $\Phi^\vee$ the coroot system $\{\alpha^\vee \mid \alpha \in \Phi\}$. Note that a coroot system $\Phi^\vee$ itself is also a root system, but with its type dual to that of Φ. This fact will be important later when we make a change for the index set of the alcove form of an affine Weyl group element.

For any $\alpha \in \Phi$ and $k \in \mathbb{Z}$, define a hyperplane

(1.1.1) $$H_{\alpha;k} = \{v \in E \mid \langle v, \alpha^\vee\rangle = k\}$$

and a stripe

(1.1.2) $$H^1_{\alpha;k} = \{v \in E \mid k < \langle v, \alpha^\vee\rangle < k+1\}.$$

Then we have $H_{-\alpha,k} = H_{\alpha;-k}$ and $H^1_{-\alpha;k} = H^1_{\alpha;-k-1}$ for any $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Note that the definition of a stripe $H^1_{\alpha;k}$ for $\alpha \in \Phi^-$ given here slightly differs from that in [10]. According to [10], we would have $H^1_{-\alpha;k} = H^1_{\alpha;-k}$. Thus by this new definition, we have to re-examine some results of [10]. Note that it makes no change for the results only involving positive roots. We call any non-empty connected simplex of

(1.1.3) $$E - \bigcup_{\substack{\alpha\in\Phi^+\\ k\in\mathbb{Z}}} H_{\alpha;k}$$

an alcove of E. Each alcove of E has the form $\bigcap_{\alpha\in\Phi^+} H^1_{\alpha;k_\alpha}$ for some Φ^+-tuple $(k_\alpha)_{\alpha\in\Phi^+}$ over $\mathbb{Z}$. Since $H^1_{-\alpha;k_\alpha} = H^1_{\alpha;-k_\alpha-1}$, sometimes it is more convenient to denote the alcove $\bigcap_{\alpha\in\Phi^+} H^1_{\alpha;k_\alpha}$ by $\bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha}$ with the convention that $k_{-\alpha} = -k_\alpha - 1$ for any $\alpha \in \Phi^+$.

1.2. One should note that not every Φ^+-tuple $(k_\alpha)_{\alpha\in\Phi^+}$ over $\mathbb{Z}$ gives rise to an alcove of E as above. In fact, it is well known that $\bigcap_{\alpha\in\Phi^+} H^1_{\alpha;k_\alpha}$ is an alcove of E if and only if for any $\alpha,\beta \in \Phi^+$ with $\alpha+\beta \in \Phi^+$, the inequality

(1.2.1) $$|\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 \leqslant |\alpha+\beta|^2(k_{\alpha+\beta}+1) \leqslant |\alpha|^2(k_\alpha+1) + |\beta|^2(k_\beta+1) + |\alpha+\beta|^2 - 1$$

holds (see [10, Theorem 5.2]). Now we give a new and simpler form for this result.

Theorem 1.3. $\bigcap_{\alpha\in\Phi^+} H^1_{\alpha;k_\alpha}$ *is an alcove of E if and only if for any $\alpha,\beta\in\Phi^+$ with $\gamma=(\alpha^\vee+\beta^\vee)^\vee\in\Phi^+$, the inequality*

$$k_\alpha+k_\beta\leqslant k_\gamma\leqslant k_\alpha+k_\beta+1. \tag{1.3.1}$$

holds.

Proof. By [10, Theorem 5.2], it is enough to show that the inequality systems (1.2.1) and (1.3.1) are equivalent for any irreducible positive root system Φ^+ of rank 2. Note that when $\alpha,\beta,\alpha+\beta\in\Phi^+$ satisfy $|\alpha|=|\beta|=|\alpha+\beta|$, the inequality (1.2.1) becomes

$$k_\alpha+k_\beta\leqslant k_{\alpha+\beta}\leqslant k_\alpha+k_\beta+1, \tag{1.3.2}$$

and $\gamma=(\alpha^\vee+\beta^\vee)^\vee=\alpha+\beta$. Thus the inequalities (1.2.1) and (1.3.1) are the same in this case. So we need only consider the case that Φ^+ has type B_2 or G_2 and that the lengths of the three roots involved in the inequalities are not all the same. We shall only deal with the case of type B_2 here. The case of type G_2 can be done similarly and hence is left to the reader. Now let $\Phi^+=\{\alpha,\beta,\alpha+\beta,2\alpha+\beta\}$. Restricting on the triples of roots with the lengths not all equal, the inequality system (1.2.1) becomes

$$\begin{cases} k_\alpha+2k_\beta\leqslant k_{\alpha+\beta}\leqslant k_\alpha+2k_\beta+2,\\ k_\alpha+k_{\alpha+\beta}\leqslant 2k_{2\alpha+\beta}+1\leqslant k_\alpha+k_{\alpha+\beta}+2.\end{cases} \tag{1.3.3}$$

and (1.3.1) becomes

$$\begin{cases} k_\alpha+k_\beta\leqslant k_{2\alpha+\beta}\leqslant k_\alpha+k_\beta+1,\\ k_\beta+k_{2\alpha+\beta}\leqslant k_{\alpha+\beta}\leqslant k_\beta+k_{2\alpha+\beta}+1.\end{cases} \tag{1.3.4}$$

We have to show the equivalence of the inequality systems (1.3.3) and (1.3.4). Now assume (1.3.3). Adding two inequalities of (1.3.3) together on the corresponding sides and then subtracting $k_{\alpha+\beta}$ from all sides of the resulting inequality, we get

$$2(k_\alpha+k_\beta)\leqslant 2k_{2\alpha+\beta}+1\leqslant 2(k_\alpha+k_\beta)+4. \tag{1.3.5}$$

Then the first inequality of (1.3.4) follows easily from (1.3.5). Rewrite the second inequality of (1.3.3) into the form

$$2k_{2\alpha+\beta}\leqslant k_\alpha+k_{\alpha+\beta}+1\leqslant 2k_{2\alpha+\beta}+2. \tag{1.3.6}$$

Then adding (1.3.6) to the first inequality of (1.3.3) on the corresponding sides and subtracting k_α from all sides of the resulting inequality, we get

$$2(k_{2\alpha+\beta}+k_\beta)\leqslant 2k_{\alpha+\beta}+1\leqslant 2(k_{2\alpha+\beta}+k_\beta)+4. \tag{1.3.7}$$

The second inequality of (1.3.4) follows easily from (1.3.7). Next assume (1.3.4). Then the first inequality of (1.3.3) is obtained by adding two inequalities of (1.3.4) together on the corresponding sides, followed by subtracting $k_{2\alpha+\beta}$ from all sides of the resulting inequality. Rewrite the second inequality of (1.3.4) into the form

$$k_{\alpha+\beta} \leqslant k_\beta + k_{2\alpha+\beta} + 1 \leqslant k_{\alpha+\beta} + 1. \tag{1.3.8}$$

Then the second inequality of (1.3.3) is obtained by adding (1.3.8) to the first inequality of (1.3.4) on the corresponding sides, followed by subtracting k_β from all sides of the resulting inequality. $\square$

Note that the result still holds if we replace all the notations Φ^+ by Φ in Theorem 1.3.

1.4. Let W be the Weyl group of Φ generated by all the reflections s_α, $\alpha \in \Phi$, on E, where s_α sends x to $x - \langle x, \alpha^\vee \rangle \alpha$. Let Q denote the root lattice $\mathbb{Z}\Phi$, and N the group consisting of all the translations T_λ operating on E for $\lambda \in Q$, where T_λ sends x to $x + \lambda$. We denote by W_a the group of affine transformations of E generated by N and W. It is well known that W_a is the semidirect extension of W by the normal subgroup N on which the action of W is known.

For linear and affine transformations, we shall denote the operation on the right and compose them accordingly. With this convention, we define $s_0 = s_{\alpha_0} T_{-\alpha_0}$ and $s_i = s_{\alpha_i}$, $1 \leqslant i \leqslant l$. It is known that W_a (resp. W) is a Coxeter group on the generators $s_0, s_1, \cdots, s_l$ (resp. $s_1, \cdots, s_l$). We write $\widetilde{S} = \{s_0, s_1, \cdots, s_l\}$ and $S = \{s_1, \cdots, s_l\}$. The group W_a is called an affine Weyl group of type $\widetilde{X}$, where X is the type of the coroot system $\Phi^\vee$.

1.5. It is well known that the right action of W_a on E gives rise to the permutations of the set $\{H_{\alpha;k} \mid \alpha \in \Phi,\ k \in \mathbb{Z}\}$. So it induces the permutations of the set $\mathfrak{U}$ of alcoves of E. It is also well known that $\mathfrak{U}$ is simply transitive under W_a (see [2; 14]).

Let $w = \bar{w} \cdot T_\lambda \in W_a$ with $\bar{w} \in W$ and $\lambda \in Q$. We consider the alcove

$$\bigcap_{\alpha\in\Phi} H^1_{\alpha;h_\alpha} = \left(\bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha}\right) w.$$

For $v \in \bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha}$, we have $(v)w = (v)\bar{w} + \lambda$. This implies that

$$\langle (v)w, \alpha^\vee \rangle = \left\langle v, \left((\alpha)\bar{w}^{-1}\right)^\vee \right\rangle + \langle \lambda, \alpha^\vee \rangle.$$

So we get

$$h_\alpha = k_{(\alpha)\bar{w}^{-1}} + \langle \lambda, \alpha^\vee \rangle \tag{1.5.1}$$

for $\alpha \in \Phi$.

We see that $A_1 = \bigcap_{\alpha\in\Phi} H^1_{\alpha;\epsilon_\alpha}$ is an alcove of E, where $\epsilon_\alpha = 0$ if $\alpha \in \Phi^+$, and $\epsilon_\alpha = -1$ if $\alpha \in \Phi^-$ (comparing with [10, Lemma 1.1]). Denote $A_w = (A_1)w$ for any $w \in W_a$. Thus any alcove of $\mathfrak{U}$ has the form $A_w = \bigcap_{\alpha\in\Phi^+} H^1_{\alpha;k(w;\alpha)}$ or $A_w = \bigcap_{\alpha\in\Phi} H^1_{\alpha;k(w;\alpha)}$ for some $w \in W_a$, with the convention that

$$k(w;-\alpha) = -k(w;\alpha) - 1 \quad \text{for any} \quad \alpha \in \Phi^+ \tag{1.5.2}$$

(comparing with [10, §1.]). Note that (1.5.2) actually holds for any $\alpha \in \Phi$. We shall identify W_a with $\mathfrak{U}$ as sets under the correspondence $w \mapsto A_w$, and call $(k(w;\alpha))_{\alpha\in\Phi^+}$ or $(k(w;\alpha))_{\alpha\in\Phi}$ the alcove form of w. From (1.5.1), we get

$$k(w;\alpha) = \epsilon_{(\alpha)\bar{w}^{-1}} + \langle\lambda, \alpha^\vee\rangle \tag{1.5.3}$$

for $\alpha \in \Phi$, where the decomposition $w = \bar{w}\cdot T_\lambda$ is as above (comparing with [10, Theorem 3.3]). This is also equivalent to a formula given by Iwahori and Matsumoto [6].

1.6. The action of an element $s_j \in \widetilde{S}$ on $\mathfrak{U}$ can be described as follows. For $w \in W_a$, we have by (1.5.1) that

$$k(ws_j,\alpha) = \begin{cases} k(w;(\alpha)s_j), & \text{if } j \neq 0, \\ k(w;(\alpha)s_{\alpha_0}) + \langle -\alpha_0, \alpha^\vee\rangle, & \text{if } j = 0. \end{cases} \tag{1.6.1}$$

for any $\alpha \in \Phi$ (comparing with [10, Proposition 4.2]).

1.7. We can define the left action of W_a on $\mathfrak{U}$ as below. For $y, w \in W_a$, we set $y(A_w) = A_{yw}$. Then for $w \in W_a$ and $s_j \in \widetilde{S}$, we have

$$k(s_jw,\alpha) = \begin{cases} k(w;\alpha), & \text{if } \alpha \neq \pm(\alpha_j)\bar{w}, \\ k(w;\alpha) \mp 1, & \text{if } \alpha = \pm(\alpha_j)\bar{w}, \end{cases} \tag{1.7.1}$$

where $\bar{w}$ is the image of w under the natural map $W_a \longrightarrow W_a/N \cong W$ (comparing with [10, Proposition 4.1]).

1.8. To any $w \in W_a$, we associate two subsets of $\widetilde{S}$: $\mathcal{L}(w) = \{s \in \widetilde{S} \mid sw \leqslant w\}$ and $\mathcal{R}(w) = \{s \in \widetilde{S} \mid ws \leqslant w\}$, where the notation $\leqslant$ stands for the Chevalley-Bruhat order on W_a regarded as a Coxeter group with respect to $\widetilde{S}$. These two subsets of $\widetilde{S}$, together with the length function $\ell(w)$, can be described in terms of the $k(w;\alpha)$'s as below.

Proposition 1.9 (comparing with [10, Propositions 3.4 and 4.3]). *Let $(k(w;\alpha))_{\alpha\in\Phi}$ be the alcove form of $w \in W_a$. Then*

(1) $\ell(w) = \sum_{\alpha\in\Phi^+} |k(w;\alpha)|$.

(2) $\mathcal{R}(w)=\{s_j\in\widetilde{S}\mid$ *either* $j\neq 0, k(w;\alpha_j)<0,$ *or* $j=0, k(w;-\alpha_0)>0\}$.
(3) $\mathcal{L}(w)=\{s_j\in\widetilde{S}\mid k(w;(\alpha_j)\bar{w})>0\}$.

$$(4)\ k(w^{-1},\alpha)=\begin{cases} k(w;-(\alpha)\bar{w}), & \textit{if either } \alpha,-(\alpha)\bar{w}\in\Phi^+ \textit{ or } \alpha,-(\alpha)\bar{w}\in\Phi^-,\\ k(w;-(\alpha)\bar{w})+1, & \textit{if } \alpha,(\alpha)\bar{w}\in\Phi^+,\\ k(w;-(\alpha)\bar{w})-1, & \textit{if } \alpha,(\alpha)\bar{w}\in\Phi^-.\end{cases}$$

§2. The cases of the classical types.

In this section, we shall first give an explicit description for all the irreducible root systems of the classical types. Then we reformulate some results on the alcove forms of $w\in W_a$ in these cases. A significant change is made for the index set of the alcove form of $w\in W_a$, where the root system Φ is replaced by the corresponding coroot system $\Phi^\vee$.

2.1. We set the notations $\Omega_l=\{A_{l-1},B_l,C_l,D_l\}$ and $\Omega_l'=\{B_l,C_l,D_l\}$ for $l\geqslant 2$, which will be used quite often later on.

2.2. We know that there is a natural bijective map $\alpha\mapsto\alpha^\vee$ from a root system Φ to the corresponding coroot system $\Phi^\vee$. By Theorem 1.3, it is more convenient to express the alcove form $(k(w;\alpha))_{\alpha\in\Phi}$ of $w\in W_a$ by $(k(w;\alpha))_{\alpha\in\Phi^\vee}$, where we set $k(w;\alpha)=k(w;\alpha^\vee)$ for any $\alpha\in\Phi^\vee$ (note $(\alpha^\vee)^\vee=\alpha$). We make such a change from now on. Then for an affine Weyl group W_a of type $\widetilde{X}$, $X\in\Omega_l$, the index set of the alcove form of $w\in W_a$ will be the root system of type X. Note that $-\alpha_0^\vee$ is the highest coroot in $\Phi^\vee$.

2.3. Let $e_1, e_2, \cdots, e_l$ be an orthonormal basis of a euclidean space E. Set $e_{-i}=-e_i$, $1\leqslant i\leqslant l$, and $e_0=0$. Define $(i,j)=e_j-e_i$ for $i,j\in[-l,l]$, where the notation $[a,b]$ stands for the interval $\{a,a+1,\cdots,b\}$ for any $a\leqslant b$ in $\mathbb{Z}$. Then the relations

$$(2.3.1)\qquad (i,j)=(i,t)+(t,j)\quad\text{and}\quad (i,j)=(-j,-i)$$

hold for any $i,t,j\in[-l,l]$.

2.4. The root system of type A_{l-1}, $l\geqslant 2$, can be described by

$$\Phi(A_{l-1})=\{(i,j)\mid 1\leqslant i,j\leqslant l; i\neq j\}.$$

The roots $(t-1,t)$, $1<t\leqslant l$, form a simple root system, and $\Phi^+(A_{l-1})=\{(i,j)\mid 1\leqslant i<j\leqslant l\}$ is the corresponding positive root system. The highest root is $(1,l)$.

2.5. The root system of type B_l, $l\geqslant 3$, can be described by

$$\Phi(B_l)=\{(i,j)\mid i,j\in[-l,l]; i\neq\pm j\}.$$

The roots $(t-1,t)$, $1\leqslant t\leqslant l$, form a simple root system, and $\Phi^+(B_l)=\{(i,j)\in\Phi(B_l)\mid i<j\}$ is the corresponding positive root system. The highest root is $(-l+1,l)$.

2.6. The root system of type C_l, $l \geqslant 2$, can be described by

$$\Phi(C_l) = \{(i,j) \mid i,j \in [-l,l] \setminus \{0\}; i \neq j\}.$$

The roots $(t-1,t)$, $1 < t \leqslant l$, and $(-1,1)$ form a simple root system, and $\Phi^+(C_l) = \{(i,j) \in \Phi(C_l) \mid i < j\}$ is the corresponding positive root system. The highest root is $(-l,l)$.

2.7. The root system of type D_l, $l \geqslant 4$, can be described by

$$\Phi(D_l) = \{(i,j) \mid i,j \in [-l,l] \setminus \{0\}; i \neq \pm j\}.$$

The roots $(t-1,t)$, $1 < t \leqslant l$, and $(-1,2)$ form a simple root system, and $\Phi^+(D_l) = \{(i,j) \in \Phi(D_l) \mid i < j\}$ is the corresponding positive root system. The highest root is $(-l+1,l)$.

2.8. Fix $w \in W_a(\widetilde{X})$, $X \in \Omega_l$. Let Φ be the root system of type X. In the alcove form $(k(w;\alpha))_{\alpha\in\Phi}$ of w, we shall denote the entry $k(w;(i,j))$ simply by $k(w;i,j)$ for $(i,j) \in \Phi$. Then the entry $k(w;(i,j)\bar{x})$ becomes $k(w;(i)\bar{x},(j)\bar{x})$ for any element $\bar{x}$ of the corresponding Weyl group, the latter is regarded as a certain permutation group on the set $[-l,l]$. By (1.5.2), (2.3.1), Theorem 1.3 and the notice immediately after the Theorem, we get

Lemma 2.9. *Let $X \in \Omega_l$ for an appropriate integer l and let Φ be the root system of type X. Then for any $w \in W_a(\widetilde{X})$ and $(i,j),(i,t),(t,j) \in \Phi$, we have*

(1) $k(w;j,i) = -k(w;i,j) - 1 = k(w;-i,-j)$,

(2) $k(w;i,t) + k(w;t,j) \leqslant k(w;i,j) \leqslant k(w;i,t) + k(w;t,j) + 1$.

§3. The permutation forms.

3.1. Let $\mathcal{A}_l$ be the affine Weyl group of type $\widetilde{A}_{l-1}$, $l \geqslant 2$. It is well known that $\mathcal{A}_l$ could be regarded as a group of certain permutations on $\mathbb{Z}$:

$$\mathcal{A}_l = \left\{ w : \mathbb{Z} \longrightarrow \mathbb{Z} \,\middle|\, (i+l)w = (i)w + l \text{ for } i \in \mathbb{Z};\ \sum_{i=1}^{l} (i)w = \sum_{i=1}^{l} i \right\}.$$

Its simple reflection set $\widetilde{S} = \{r_0, r_1, \cdots, r_{l-1}\}$ is given by

$$(i)r_t = \begin{cases} i, & \text{if } i \not\equiv t, t+1 \ (\bmod\ l), \\ i+1, & \text{if } i \equiv t \ (\bmod\ l), \\ i-1, & \text{if } i \equiv t+1 \ (\bmod\ l), \end{cases}$$

for $i \in \mathbb{Z}$ and $0 \leqslant t < l$. Each element $w \in \mathcal{A}_l$ is determined entirely by the images $(a_1)w, (a_2)w, \cdots, (a_l)w$ of any given l integers $a_1, \cdots, a_l$ under w, which are pairwisely incongruent modulo l.

3.2. As a Coxeter group, the affine Weyl group $W_a(\widetilde{X})$ of type $X \in \Omega_l'$ has the following presentation:

$$W_a(\widetilde{X}) = \langle s_i \mid 0 \leqslant i \leqslant l, (s_i s_j)^{m_{ij}} = \mathbf{1}, \text{ for } 0 \leqslant i \leqslant j \leqslant l \rangle,$$

where $m_{ij} = 1$ for $i = j$; $m_{ij} = 3$ in one of the following cases:

(i) $2 < i + 1 = j \leqslant l$,

(ii) $(i, j) = (0, l-1)$ and $X = B_l$,

(iii) $(i, j) = (1, 3), (0, l-1)$ and $X = D_l$;

$m_{ij} = 4$ in one of the following cases:

(i) $(i, j) = (1, 2)$ and $X = B_l$,

(ii) $(i, j) = (1, 2), (0, l)$ and $X = C_l$;

$m_{ij} = 2$ in all the other cases. Note that the labelings of the nodes in the corresponding extended Dynkin diagrams slightly differ from the usual ones (see [2; 5] for examples).

3.3. The affine Weyl group $W_a(\widetilde{C}_l)$ $(l \geqslant 2)$ can be embedded into $\mathcal{A}_{2l+2}$ by an injective homomorphism which sends s_t, $1 < t \leqslant l$, to $r_{t-1}r_{2l+2-t}$, s_1 to $r_0 r_{2l+1} r_0$, and s_0 to $r_l r_{l+1} r_l$. Thus by identifying with its image, we can regard $W_a(\widetilde{C}_l)$ as a subgroup of $\mathcal{A}_{2l+2}$. The group $W_a(\widetilde{C}_l)$ can also be regarded as the set of all the fixed points of $\mathcal{A}_{2l+2}$ under the involutive automorphism ϕ which sends r_i to r_{2l+1-i} for all i, $0 \leqslant i \leqslant 2l+1$:

$$\begin{aligned} W_a(\widetilde{C}_l) =& \{w \in \mathcal{A}_{2l+2} \mid \phi(w) = w\} \\ =& \{w \in \mathcal{A}_{2l+2} \mid (-i)w = -(i)w, \forall i \in \mathbb{Z}\}. \end{aligned} \tag{3.3.1}$$

The simple reflections s_t, $0 \leqslant t \leqslant l$, of $W_a(\widetilde{C}_l)$ are the permutations on $\mathbb{Z}$ satisfying that for $i \in \mathbb{Z}$ and $1 < t \leqslant l$, we have

$$(i)s_t = \begin{cases} i, & \text{if } i \not\equiv \pm t, \pm(t-1) \pmod{2l+2}, \\ i+1, & \text{if } i \equiv -t, t-1 \pmod{2l+2}, \\ i-1, & \text{if } i \equiv t, -t+1 \pmod{2l+2}. \end{cases}$$

$$(i)s_1 = \begin{cases} i, & \text{if } i \not\equiv \pm 1 \pmod{2l+2}, \\ i+2, & \text{if } i \equiv -1 \pmod{2l+2}, \\ i-2, & \text{if } i \equiv 1 \pmod{2l+2}. \end{cases}$$

$$(i)s_0 = \begin{cases} i, & \text{if } i \not\equiv \pm l \pmod{2l+2}, \\ i+2, & \text{if } i \equiv l \pmod{2l+2}, \\ i-2, & \text{if } i \equiv -l \pmod{2l+2}. \end{cases}$$

3.4. The group $W_a(\widetilde{B}_l)$ $(l \geqslant 3)$ can be embedded into $\mathcal{A}_{2l+2}$ by an injective homomorphism which sends s_t, $1 < t \leqslant l$, to $r_{t-1}r_{2l+2-t}$, s_1 to $r_0r_{2l+1}r_0$ and s_0 to $r_lr_{l+1}r_lr_{l-1}r_{l+2}r_lr_{l+1}r_l$. Thus we can regard $W_a(\widetilde{B}_l)$ as a subgroup of $\mathcal{A}_{2l+2}$ by identifying with its image. We can also regard $W_a(\widetilde{B}_l)$ as a certain fixed point set of $\mathcal{A}_{2l+2}$ under the automorphism ϕ (see 3.3):

(3.4.1)
$$\begin{aligned} W_a(\widetilde{B}_l) =&\{w \in \mathcal{A}_{2l+2} \mid \phi(w) = w; N_l(w) \equiv 0 \pmod 2\}, \\ =&\{w \in \mathcal{A}_{2l+2} \mid (-i)w = -(i)w, \forall i \in \mathbb{Z}; N_l(w) \equiv 0 \pmod 2\}, \end{aligned}$$

where $N_l(w)$ is the number of the integers j with $j < l+1$ and $(j)w > l+1$. The simple reflections s_t, $0 \leqslant t \leqslant l$, of $W_a(\widetilde{B}_l)$ are the permutations on $\mathbb{Z}$ as below. For $i \in \mathbb{Z}$ and $1 < t \leqslant l$, we have

$$(i)s_t = \begin{cases} i, & \text{if } i \not\equiv \pm t, \pm(t-1) \pmod{2l+2}, \\ i+1, & \text{if } i \equiv -t, t-1 \pmod{2l+2}, \\ i-1, & \text{if } i \equiv t, -t+1 \pmod{2l+2}. \end{cases}$$

$$(i)s_1 = \begin{cases} i, & \text{if } i \not\equiv \pm 1 \pmod{2l+2}, \\ i+2, & \text{if } i \equiv -1 \pmod{2l+2}, \\ i-2, & \text{if } i \equiv 1 \pmod{2l+2}. \end{cases}$$

$$(i)s_0 = \begin{cases} i, & \text{if } i \not\equiv \pm(l-1), \pm l \pmod{2l+2}, \\ i+3, & \text{if } i \equiv l-1, l \pmod{2l+2}, \\ i-3, & \text{if } i \equiv -l, -l+1 \pmod{2l+2}. \end{cases}$$

3.5. The affine Weyl group $W_a(\widetilde{D}_l)$ $(l \geqslant 4)$ can be regarded as a subgroup of $\mathcal{A}_{2l+2}$ by an injective homomorphism which sends s_t, $1 < t \leqslant l$, to $r_{t-1}r_{2l+2-t}$, s_1 to $r_0r_{2l+1}r_0r_1r_{2l}r_0r_{2l+1}r_0$ and s_0 to $r_lr_{l+1}r_lr_{l-1}r_{l+2}r_lr_{l+1}r_l$, or equivalently, regarded as a certain fixed point set of $\mathcal{A}_{2l+2}$ under the automorphism ϕ (see 3.3):

(3.5.1)
$$\begin{aligned} W_a(\widetilde{D}_l) &= \{w \in \mathcal{A}_{2l+2} \mid \phi(w) = w; N_0(w) \equiv N_l(w) \equiv 0 \pmod 2\}, \\ &= \{w \in \mathcal{A}_{2l+2} \mid (-i)w = -(i)w, \forall i \in \mathbb{Z}; N_0(w) \equiv N_l(w) \equiv 0 \pmod 2\}, \end{aligned}$$

where $N_0(w)$ is the number of the integers j with $j < 0$ and $(j)w > 0$. The simple reflections s_t, $0 \leqslant t \leqslant l$, of $W_a(\widetilde{D}_l)$ are the permutations on $\mathbb{Z}$ as below. For $1 < t \leqslant l$ and $i \in \mathbb{Z}$, we have

$$(i)s_t = \begin{cases} i, & \text{if } i \not\equiv \pm t, \pm(t-1) \pmod{2l+2}, \\ i+1, & \text{if } i \equiv -t, t-1 \pmod{2l+2}, \\ i-1, & \text{if } i \equiv t, -t+1 \pmod{2l+2}. \end{cases}$$

$$(i)s_1 = \begin{cases} i, & \text{if } i \not\equiv \pm 1, \pm 2 \pmod{2l+2}, \\ i+3, & \text{if } i \equiv -1, -2 \pmod{2l+2}, \\ i-3, & \text{if } i \equiv 1, 2 \pmod{2l+2}. \end{cases}$$

$$(i)s_0 = \begin{cases} i, & \text{if } i \not\equiv \pm l, \pm(l-1) \pmod{2l+2}, \\ i+3, & \text{if } i \equiv l-1, l \pmod{2l+2}, \\ i-3, & \text{if } i \equiv -l, -l+1 \pmod{2l+2}. \end{cases}$$

From the above discussion, we get the following result easily.

Lemma 3.6. *Let $X \in \Omega_l$ and $w \in W_a(\widetilde{X})$ for an appropriate integer l. Define an integer m_X to be l if $X = A_{l-1}$, and $2l+2$ if $X \in \Omega'_l$. Then regarding w as an element of $\mathcal{A}_{m_X}$, we have*

(1) $(i)w \not\equiv (j)w \pmod{m_X}$ for any $i, j \in \mathbb{Z}$ with $i \not\equiv j \pmod{m_X}$.

In the case of $X \in \Omega'_l$, we have

(2) $(-i)w = -(i)w$ for any $i \in \mathbb{Z}$.

(3) $(h(l+1))w = h(l+1)$ for any $h \in \mathbb{Z}$.

(4) $(i)w \not\equiv 0, l+1 \pmod{2l+2}$ for any $i \in \mathbb{Z}$ with $i \not\equiv 0, l+1 \pmod{2l+2}$.

Remark 3.7. (i) Our definition of $W_a(\widetilde{C}_l)$ as a group of permutations on $\mathbb{Z}$ is slightly different from that given by R. Bédard (see [1]). According to R. Bédard, the group $W_a(\widetilde{C}_l)$ was embedded into the group $\mathcal{A}_{2l+1}$ instead of $\mathcal{A}_{2l+2}$. The advantage of our definition is that the symmetry between the generators s_0 and s_1 in $W_a(\widetilde{C}_l)$ could be exposed more explicitly in form.

(ii) The descriptions (3.3.1), (3.4.1), (3.5.1) of the groups $W_a(\widetilde{X})$, $X \in \Omega'_l$, can be shown easily by applying induction on $\ell(w) \geqslant 0$ and by the fact that the number of inversions of w is increasing when $\ell(w)$ is getting larger.

(iii) Let $X \in \Omega'_l$. Then regarded as a permutation on $\mathbb{Z}$, an element $w \in W_a(\widetilde{X})$ is determined uniquely by the images $(a_1)w, \cdots, (a_l)w$ of any given l integers $a_1, \cdots, a_l$ under w, provided that they are pairwisely incongruent modulo $2l+2$, none of which is divisible by $l+1$, and none pair of which have the sum divisible by 2l+2.

§4. Transition from the permutation forms to the alcove forms.

Let $X \in \Omega_l$ for an appropriate integer l (see 2.1.). In this section, we shall show the transition formulae from the permutation form to the alcove form of $w \in W_a(\widetilde{X})$. Let $\Phi = \Phi(X)$ be the root system of type X. Let m_X be the integer defined as in Lemma 3.6.

Theorem 4.1. *For any $(i, j) \in \Phi$ and $w \in W_a(\widetilde{X})$, we have*

$$k(w; i, j) = \left[\frac{(j)w^{-1} - (i)w^{-1}}{m_X} \right], \tag{4.1.1}$$

where $[x]$ denotes the largest integer not greater than x for a rational number x.

Proof. For any $i, j \in \mathbb{Z}$ with $i \not\equiv j \pmod{m_X}$, we have the relation

$$\left[\frac{(i)w^{-1}-(j)w^{-1}}{m_X}\right] = -\left[\frac{(j)w^{-1}-(i)w^{-1}}{m_X}\right] - 1 \tag{4.1.2}$$

by Lemma 3.6, (1). Also, we have $(i,j) \in \Phi^+ \Longleftrightarrow (j,i) \in \Phi^-$. Thus by Lemma 2.9, (1), we need only show our result in the case when (i,j) is a positive root. The result is known for the case of $X = A_{l-1}$ (see [9, Proposition 6.2.1]). Now assume $X \neq A_{l-1}$ and hence $m_X = 2l+2$. We see that the set

$$\Phi^+ = \{(i,j) \in \Phi \mid i < j; |i| \leqslant |j|\}.$$

forms a positive root system of Φ. Apply induction on $\ell(w) \geqslant 0$. When $\ell(w) = 0$, i.e., $w = \mathbf{1}$, we have $((1)\mathbf{1}^{-1}, \cdots, (l)\mathbf{1}^{-1}) = (1, 2, \cdots, l)$ and $k(w; i, j) = 0$ for all the positive root (i,j). The result is obviously true. Now assume $\ell(w) > 0$. Take $s_t \in \mathcal{R}(w)$ (see 1.8). By inductive hypothesis, the result is true for ws_t. First assume $1 < t \leqslant l$. When $(i,j) \neq (t-1,t), (-t+1,t)$ with $(i,j) \in \Phi^+$, we have $(i)s_t < (j)s_t$ and $|(i)s_t| \leqslant |(j)s_t|$. Thus in this case, we get from (1.6.1) that

$$\begin{aligned} k(w; i, j) = k(ws_t; (i)s_t, (j)s_t) &= \left[\frac{(j)s_t(ws_t)^{-1} - (i)s_t(ws_t)^{-1}}{2l+2}\right] \\ &= \left[\frac{(j)w^{-1} - (i)w^{-1}}{2l+2}\right]. \end{aligned}$$

On the other hand, by (1.6.1), (2.3.1), (4.1.2), Lemmas 2.9 and 3.6, we have

$$\begin{aligned} k(w; t-1, t) = -k(ws_t; t-1, t) - 1 &= -\left[\frac{(t)(ws_t)^{-1} - (t-1)(ws_t)^{-1}}{2l+2}\right] - 1 \\ &= -\left[\frac{(t-1)w^{-1} - (t)w^{-1}}{2l+2}\right] - 1 = \left[\frac{(t)w^{-1} - (t-1)w^{-1}}{2l+2}\right] \end{aligned}$$

and

$$\begin{aligned} k(w; -t+1, t) &= k(ws_t; -t, t-1) = k(ws_t; -t+1, t) \\ &= \left[\frac{(t)(ws_t)^{-1} - (-t+1)(ws_t)^{-1}}{2l+2}\right] \\ &= \left[\frac{(t-1)w^{-1} - (-t)w^{-1}}{2l+2}\right] = \left[\frac{(t)w^{-1} - (-t+1)w^{-1}}{2l+2}\right]. \end{aligned}$$

Next consider the cases $t = 0, 1$. Here we only deal with the case of $X = B_l$. The proofs for the cases of $X = C_l$, D_l are similar and hence are left to the reader. In the following calculation, we shall repeatedly apply (1.6.1), (2.3.1), (4.1.2), Lemmas 2.9 and 3.6.

Now assume $t=1$. The result can be checked easily when $i,j\neq\pm1$. On the other hand, for $1<j\leqslant l$, we have

$$k(w;\pm1,j)=k(ws_1;\mp1,j)=\left[\frac{(j)(ws_1)^{-1}-(\mp1)(ws_1)^{-1}}{2l+2}\right]$$
$$=\left[\frac{(j)w^{-1}-(\pm1)w^{-1}}{2l+2}\right].$$

We also have

$$k(w;0,1)=k(ws_1;0,-1)=-k(ws_1;0,1)-1$$
$$=-\left[\frac{(1)(ws_0)^{-1}-(0)(ws_0)^{-1}}{2l+2}\right]-1=-\left[\frac{(-1)w^{-1}}{2l+2}\right]-1$$
$$=\left[\frac{(1)w^{-1}-(0)w^{-1}}{2l+2}\right].$$

Finally assume $t=0$. We need only check the cases when $\{i,j\}\cap\{l-1,l\}\neq\emptyset$ for otherwise the result is obvious. For $i\in[-l+2,l-2]$, we have

$$k(w;i,l-1)=k(ws_0;i,-l)+1=-k(ws_0;-i,l)$$
$$=-\left[\frac{(l)(ws_0)^{-1}-(-i)(ws_0)^{-1}}{2l+2}\right]$$
$$=-\left[\frac{(l+3)w^{-1}-(-i)w^{-1}}{2l+2}\right]$$
$$=-\left[1-\frac{(l-1)w^{-1}-(i)w^{-1}}{2l+2}\right]$$
$$=\left[\frac{(l-1)w^{-1}-(i)w^{-1}}{2l+2}\right].$$

The equation $k(w;i,l)=\left[\frac{(l)w^{-1}-(i)w^{-1}}{2l+2}\right]$ can be shown similarly. Also, we have

$$k(w;l-1,l)=k(ws_0;-l,-l+1)=k(ws_0;l-1,l)$$
$$=\left[\frac{(l)(ws_0)^{-1}-(l-1)(ws_0)^{-1}}{2l+2}\right]=\left[\frac{(l+3)w^{-1}-(l+2)w^{-1}}{2l+2}\right]$$
$$=\left[\frac{(-l+1)w^{-1}-(-l)w^{-1}}{2l+2}\right]=\left[\frac{(l)w^{-1}-(l-1)w^{-1}}{2l+2}\right]$$

and

$$k(w;-l+1,l)=k(ws_0;l,-l+1)+2=-k(ws_0;-l+1,l)+1$$
$$=-\left[\frac{(l)(ws_0)^{-1}-(-l+1)(ws_0)^{-1}}{2l+2}\right]+1$$
$$=-\left[\frac{(l+3)w^{-1}-(-l-2)w^{-1}}{2l+2}\right]+1$$
$$=-\left[\frac{(-l+1)w^{-1}-(l)w^{-1}}{2l+2}\right]-1=\left[\frac{(l)w^{-1}-(-l+1)w^{-1}}{2l+2}\right].$$

Therefore the equation (4.1.1) holds in all the cases. $\square$

4.2. By Theorem 4.1, it is natural to extend the alcove form of $w \in W_a(\widetilde{X})$ to $(k(w;i,j))_{i,j\in\mathbb{Z}}$ by setting

$$k(w;i,j) = \left\lceil \frac{(j)w^{-1} - (i)w^{-1}}{m_X} \right\rceil \tag{4.2.1}$$

for any $X \in \Omega_l$ and $i,j \in \mathbb{Z}$. Call $(k(w;i,j))_{i,j\in\mathbb{Z}}$ the extended alcove form of w. The relations in Lemma 2.9 remain valid for any $i,t,j \in \mathbb{Z}$ with $i \not\equiv j \pmod{m_X}$. Clearly, we have

$$k(w;i-m_X,j) = k(w;i,j+m_X) = k(w;i,j)+1 \tag{4.2.2}$$

for any $i,j \in \mathbb{Z}$. We call $k(w;i,j)$ a basic entry of the alcove form of w if $(i,j) \in \Phi(X)$. It is straightforward to deduce the following

Corollary 4.3. *Let $X \in \Omega_l$ for an appropriate integer l. Then we have*

$$k(wy;i,j) = k(w;(i)y^{-1},(j)y^{-1}) \tag{4.3.1}$$

for any $i,j \in \mathbb{Z}$ and $w,y \in W_a(\widetilde{X})$.

Note that in the above corollary, the element y is regarded as a permutation on $\mathbb{Z}$.

4.4. The right-hand side of (4.1.1) was named the inverse table of w by H. Eriksson and K. Eriksson in [4, §8.] as (i,j) ranges over a certain subset of $\mathbb{Z}\times\mathbb{Z}$. They proposed a conjecture in the case of $X = C_l$ in order to characterize the inverse table of $w \in W_a(\widetilde{C}_l)$ as a family of integers. Then they claimed that they would be able to formulate a similar conjecture in the cases of $X = B_l, D_l$. Now Theorems 1.3, 4.1 and Lemma 2.9 together give the following characterization for the inverse table of $w \in W_a(\widetilde{X})$, $X \in \Omega_l$.

Theorem 4.5. *Let $X \in \Omega_l$ for an appropriate integer l and let Φ be the root system of type X. Then a family of integers $(k_{ij})_{(i,j)\in\Phi}$ is the inverse table of some element in $W_a(\widetilde{X})$ if and only if the following conditions are satisfied:*

(1) $k_{ji} = -k_{ij} - 1$ for any $(i,j) \in \Phi$;

(2) $k_{it} + k_{tj} \leqslant k_{ij} \leqslant k_{it} + k_{tj} + 1$ for any i,t,j, with $(i,j),(i,t),(t,j) \in \Phi$.

This theorem gives an affirmative answer to the conjecture in [4, 8.3].

§5. Transition from the alcove forms to the permutation forms.

Keep the notations in 4.2. Denote the integer $m = m_X$ (see Lemma 3.6) and the root system $\Phi = \Phi(X)$ (see 2.4-2.7) for $X \in \Omega_l$. Fix $w \in W_a(\widetilde{X})$. The aim of the present section is to express the numbers $(t)w^{-1}$, $1 \leqslant t \leqslant l$, in terms of the basic entries of the alcove form of w. We actually go further by expressing these numbers in terms of $k(w;i,j)$'s with (i,j) the positive roots.

5.1. By (4.1.1), we can write

$$(j)w^{-1} - (i)w^{-1} = m \cdot k(w;i,j) + r(w;i,j) \tag{5.1.1}$$

for any $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, where the integer $r(w;i,j)$ is taken from the interval $[0, m-1]$. Fix t, $1 \leqslant t \leqslant l$. By taking the sums over $j = 1, 2, \cdots, m$ on both sides of (5.1.1), we get

$$\sum_{j=1}^{m} \left((j)w^{-1} - (t)w^{-1}\right) = m \cdot \sum_{j=1}^{m} k(w;t,j) + \sum_{j=1}^{m} r(w;t,j). \tag{5.1.2}$$

By the fact

$$m + \sum_{j=1}^{m} r(w;t,j) = \sum_{j=1}^{m} j = \sum_{j=1}^{m} (j)w^{-1}, \tag{5.1.3}$$

we get from (5.1.2) that

(5.1.4)
$$\begin{aligned}
(t)w^{-1} =& 1 - \sum_{j=1}^{m} k(w;t,j) \\
=& 1 - \sum_{j=t+1}^{l} k(w;t,j) + \left(t - 1 + \sum_{j=1}^{t-1} k(w;j,t) \right) - \sum_{h=0}^{m-l-1} k(w;t,m-h) \\
=& t + \sum_{j=1}^{t-1} k(w;j,t) - \sum_{j=t+1}^{l} k(w;t,j) + \sum_{h=0}^{m-l-1} k(w;-h,t).
\end{aligned}$$

The last equality follows by the fact $k(w;t,m-h) = k(w;t,-h) + 1 = -k(w;-h,t)$ for any $t \not\equiv h \pmod{m}$ in $\mathbb{Z}$. When $X = A_{l-1}$, we have $m = l$ and then the last sum of the right-hand side of (5.1.4) vanishes. We get the formula in [9, Corollary 6.2.2, (i)]. Now assume $X \in \Omega_l'$. Then $m = 2l+2$ and (5.1.4) becomes

$$(t)w^{-1} = t + \sum_{j=1}^{t-1} k(w;j,t) - \sum_{j=t+1}^{l} k(w;t,j) + \sum_{\substack{1 \leqslant h \leqslant l \\ h \neq t}} k(w;-h,t) + 2k(w;-t,t), \tag{5.1.5}$$

where we use the relation

$$k(w;0,t)+k(w;-l-1,t)=k(w;-t,t). \tag{5.1.6}$$

which can be deduced easily from (4.2.1). On the other hand, we have

$$(t)w^{-1}=k(w;0,t)(2l+2)+r(w;0,t) \tag{5.1.7}$$

by (5.1.1) and Lemma 3.6, (3). Note that $k(w;0,t)$ is a basic entry of the alcove form of w only when $X=B_l$.

We see that all the $k(w;a,b)$'s occurring on the right-hand side of (5.1.5) are the basic entries of the alcove form of w except for $k(w;-t,t)$. On the other hand, $k(w;-t,t)$ is a basic entry of the alcove form of w only when $X=C_l$. So we get

Theorem 5.2. *Let $w\in W_a(\widetilde{X})$, $l\geqslant 2$ and $t\in[1,l]$. Then we have the following transition formulae on w.*

(1) When $X=A_{l-1}$, we have

$$(t)w^{-1}=t+\sum_{j=1}^{t-1}k(w;j,t)-\sum_{j=t+1}^{l}k(w;t,j), \tag{5.2.1}$$

(2) When $X=C_l$, we have

$$(t)w^{-1}=t+\sum_{j=1}^{t-1}k(w;j,t)-\sum_{j=t+1}^{l}k(w;t,j)+\sum_{h=1}^{l}k(w;-h,t)+k(w;-t,t). \tag{5.2.2}$$

Since $k(w;-t,t)$ is not a basic entry of the alcove form of w when $X=B_l,D_l$. So in these cases, we have to calculate $k(w;-t,t)$ via the basic entries of the alcove form of w in order to get the required transition formulae.

5.3. Assume $X=B_l$, $l\geqslant 3$. By (5.1.7), we need only determine $r(w;0,t)$ in order to calculate $(t)w^{-1}$. By (4.2.1), we have

$$k(w;-t,t)=2k(w;0,t)+\epsilon(w,t), \tag{5.3.1}$$

where $\epsilon(w,t)=0$ or 1 according to $r(w;0,t)<l+1$ or $>l+1$ respectively (note that $r(w;0,t)\neq l+1$ by Lemma 3.6, (3), (4)). Let

$$\Delta(w,t)=t+\sum_{j=1}^{t-1}k(w;j,t)-\sum_{j=t+1}^{l}k(w;t,j)+\sum_{\substack{1\leqslant h\leqslant l\\ h\neq t}}k(w;-h,t)+4k(w;0,t). \tag{5.3.2}$$

Then by (5.1.5) and (5.3.1), we have

$$(t)w^{-1} = \Delta(w,t) + 2\epsilon(w,t), \tag{5.3.3}$$

where $\Delta(w,t)$ is determined entirely by the basic entries of the alcove form of w. Let $\delta(w,t)$ be the remainder of $\Delta(w,t)$ divided by $2l+2$. Then

$$r(w;0,t) = \delta(w,t) + 2\epsilon(w,t) \tag{5.3.4}$$

by the definition of $\epsilon(w,t)$ and by the fact $1 \leqslant r(w;0,t) \leqslant 2l+1$ (see Lemma 3.6, (3), (4)). So $\epsilon(w,t)$ can also be determined uniquely by $\delta(w,t)$ except for the case of $\delta(w,t) = l$. When $\delta(w,t) = l$, $r(w;0,t)$ could be either l or $l+2$ and hence $\epsilon(w,t)$ could be either 0 or 1. But we see from Lemma 3.6, (1), (2) that there is exactly one integer t with $1 \leqslant t \leqslant l$ satisfying $r(w;0,t) \in \{l, l+2\}$ for a given $w \in W_a(\widetilde{B}_l)$. This tells us that all the $(t)w^{-1}$, $1 \leqslant t \leqslant l$, but one are determined uniquely by $\Delta(w,t)$. For this exceptional t, we have $\delta(w,t) = l$, $r(w;0,t) \in \{l, l+2\}$ and $(t)w^{-1} = \Delta(w,t)$ or $\Delta(w,t)+2$. But the evenness of the value $N_l(w^{-1})$ (see 3.4) can determine which value we should take for this exceptional $(t)w^{-1}$.

Note that we have $1 \leqslant \delta(w,t) \leqslant 2l-1$ for any $t \in [1,l]$ by (5.3.4). So by (5.1.7) and (5.3.4), the above results can be summarized as follows.

Theorem 5.4. *In the above setup, we have*

$$(t)w^{-1} = \begin{cases} k(w;0,t)(2l+2) + \delta(w,t), & \text{if } 1 \leqslant \delta(w,t) < l, \\ k(w;0,t)(2l+2) + \delta(w,t) + 2, & \text{if } l < \delta(w,t) \leqslant 2l-1, \\ k(w;0,t)(2l+2) + l + 1 \pm 1, & \text{if } \delta(w,t) = l. \end{cases} \tag{5.4.1}$$

for any $w \in W_a(\widetilde{B}_l)$, $l \geqslant 3$. In the last case, the value $(t)w^{-1}$ is determined eventually by the evenness of the number $N_l(w^{-1})$.

5.5. Next assume $X = D_l$, $l \geqslant 4$. Let

$$\Delta'(w,t) = t + \sum_{j=1}^{t-1} k(w;j,t) - \sum_{j=t+1}^{l} k(w;t,j) + \sum_{\substack{1 \leqslant i \leqslant l \\ i \neq t}} k(w;-i,t). \tag{5.5.1}$$

Then $\Delta'(w,t)$ is determined entirely by the basic entries of the alcove form of w. By (5.1.5) and (4.2.1), we have

$$(t)w^{-1} - 2\left[\frac{2 \cdot (t)w^{-1}}{2l+2}\right] = \Delta'(w,t). \tag{5.5.2}$$

This implies by (5.1.7) that

$$k(w;0,t)(2l-2)+r(w;0,t)-2\epsilon'(w,t)=\Delta'(w,t), \tag{5.5.3}$$

where $\epsilon'(w,t)=0$ or 1 according to $r(w;0,t)<l+1$ or $>l+1$ respectively. Note that $k(w;0,t)$ is not a basic entry of the alcove form of w. We see from Lemma 3.6, (1) that $r(w;0,t)-2\epsilon'(w,t)$ is in $[1,2l-1]$. Let $\delta'(w,t)$ be the remainder of $\Delta'(w,t)$ divided by $2l-2$. Then we see from (5.5.3) that when the remainder $\delta'(w,t)\neq 1$, the numbers $k(w;0,t)$ and $r(w;0,t)-2\epsilon'(w,t)$ are determined uniquely by $\Delta'(w,t)$. On the other hand, when $\delta'(w,t)\neq 1,l$, the number $r(w;0,t)$ is also determined uniquely by $\Delta'(w,t)$. If $\delta'(w,t)=1$, then $r(w;0,t)=1$ or $2l+1$; also, if $\delta'(w,t)=l$, then $r(w;0,t)=l$ or $l+2$. By Lemma 3.6, (1), (2), we see that there are exactly two integers $t',t''\in[1,l]$ satisfying $r(w;0,t')\in\{1,2l+1\}$ and $r(w;0,t'')\in\{l,l+2\}$ respectively. Thus all the $(t)w^{-1}$ $(1\leqslant t\leqslant l)$ but two are determined entirely by $\Delta'(w,t)$ and hence by the basic entries of the alcove form of w. Then by (5.5.3) and (5.1.7), we see that the evenness of the values $N_0(w^{-1})$ and $N_l(w^{-1})$ (see 3.5 and 3.4) can determine the values of these two exceptional $(t')w^{-1}$ and $(t'')w^{-1}$.

Note that we have $\delta'(w,t)\in[0,2l-3]$ for any $t\in[1,l]$. Let $\lambda(w,t)=\left[\frac{\Delta'(w,t)}{2l-2}\right]$. Then by (5.1.7) and (5.5.3), we can summarize our results as follows.

Theorem 5.6. *In the above setup, we have*

$$(t)w^{-1}=\begin{cases}\lambda(w,t)(2l+2)+\delta'(w,t), & \textit{if } 1<\delta'(w,t)<l,\\ \lambda(w,t)(2l+2)+\delta'(w,t)+2, & \textit{if } l<\delta'(w,t)\leqslant 2l-3,\\ \lambda(w,t)(2l+2)-2, & \textit{if } \delta'(w,t)=0,\\ \lambda(w,t)(2l+2)\pm 1, & \textit{if } \delta'(w,t)=1,\\ \lambda(w,t)(2l+2)+l+1\pm 1, & \textit{if } \delta'(w,t)=l.\end{cases} \tag{5.6.1}$$

for any $w\in W_a(\widetilde{D}_l)$, $l\geqslant 4$. In the last two cases, the values $(t)w^{-1}$ are determined by the evenness of the numbers $N_0(w^{-1})$ and $N_l(w^{-1})$ respectively.

5.7. Finally we make some discussion on the functions $N_0(w)$ and $N_l(w)$ of $W_a(\widetilde{X})$ $(X\in\Omega_l')$. Write, for $1\leqslant t\leqslant l$,

$$(t)w=k_t(2l+2)+r_t=k_t'(2l+2)-(l+1)+r_t' \tag{5.7.1}$$

with $k_t,k_t',r_t,r_t'\in\mathbb{Z}$ and $0\leqslant r_t,r_t'<2l+2$ (By Lemma 3.6 (4), we have $1\leqslant r_i,r_i'<2l+2$). Then

$$N_0(w)=\sum_{i=1}^{l}|k_t| \quad\text{and}\quad N_l(w)=\sum_{i=1}^{l}|k_t'|. \tag{5.7.2}$$

By (5.7.1), we get the inequality

$$N_0(w) - l \leqslant N_l(w) \leqslant N_0(w) + l. \tag{5.7.3}$$

By the relations $(-i)w = -(i)w$ and $(i)w + (2l+2-i)w = 2l+2$ $(i \in \mathbb{Z})$, we also get $N_0(w) = N_0(w^{-1})$ and $N_l(w) = N_l(w^{-1})$ for any $w \in W_a(\widetilde{X})$.

When $X = C_l$, $N_0(w)$ (resp. $N_l(w)$) is equal to the multiplicity of s_1 (resp. s_0) occurring as the factors in any reduced expression of w. The same holds for $N_0(w)$ in the case of $X = B_l$. When $X = D_l$, let $b_1, b_2, \cdots, b_l$ (resp. $b'_1, b'_2, \cdots, b'_l$) be the rearrangement of $|k_1|, |k_2|, \cdots, |k_l|$ (resp. $|k'_1|, |k'_2|, \cdots, |k'_l|$) in (weakly) decreasing order. Denote $c_0(w) = b_1$, $c'_0(w) = b'_1$, $c_1(w) = \sum_{i=2}^{l} b_i$ and $c'_1(w) = \sum_{i=2}^{l} b'_i$. Let $c(w) = \max\{0, c_0(w) - c_1(w)\}$ and $c'(w) = \max\{0, c'_0(w) - c'_1(w)\}$. Then $c(w), c'(w) \in 2\mathbb{Z}$. Let $\pi_1(w)$ (resp. $\pi_0(w)$) be the minimal possible multiplicity of s_1 (resp. s_0) occurring as the factors in a reduced expression of w. Then we can deduce the following formulae by applying induction on $\ell(w) \geqslant 0$.

$$2\pi_1(w) = N_0(w) + c(w) \quad \text{and} \quad 2\pi_0(w) = N_l(w) + c'(w). \tag{5.7.4}$$

The same holds for $N_l(w)$ in the case of $X = B_l$.

By (5.7.2) and (5.7.4), we get

$$\pi_1(w) = \begin{cases} \frac{1}{2}N_0(w), & \text{if } c(w) = 0, \\ c_0(w), & \text{if } c(w) > 0. \end{cases} \tag{5.7.5}$$

or equivalently,

$$\pi_1(w) = \max\{\frac{1}{2}N_0(w), c_0(w)\}. \tag{5.7.6}$$

Similarly, we have

$$\pi_0(w) = \max\{\frac{1}{2}N_l(w), c'_0(w)\}. \tag{5.7.7}$$

This, together with (5.7.1) and (5.7.3), implies

$$\pi_1(w) - \frac{l}{2} \leqslant \pi_0(w) \leqslant \pi_1(w) + \frac{l}{2}. \tag{5.7.8}$$

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