LEFT CELLS IN THE AFFINE WEYL GROUP OF TYPE $\widetilde{C}_4$

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Abstract. We find a representative set of left cells of the affine Weyl group W_a of type $\widetilde{C}_4$ as well as its left cell graphs by applying an algorithm. This algorithm was designed in my previous paper [22]. It is reformulated and improved in more efficient form here. These representatives of left cells are presented as the vertices of so called essential graphs so that the generalized τ-invariants of left cells of W_a are actually described explicitly, the latter almost characterize the left cells of W_a. Some comments and conjectures are proposed on cells of affine Weyl groups, mostly for the case of type $\widetilde{C}_\ell$, $\ell \geq 2$.

Since it was designed in [22], an algorithm of finding a representative set of left cells (an l.c.r. set for brevity) of a certain Coxeter group (W, S) has been applied extensively to Weyl groups and affine Weyl groups (see [8; 16; 22; 23; 26; 27]). In the present paper, we shall reformulate and improve this algorithm so that it can be performed more efficiently. We introduce three processes **A**, **B** and **C** instead of only two processes **A** and **B** in the algorithm, by which one can avoid the complicated calculation of Kazhdan-Lusztig polynomials to a great extent. On the other hand, we introduce two important concepts: an essential graph and a left cell graph associated to an element of W. The left cell graphs exhibit the generalized τ-invariants for all

Supported partly by Max-Planck-Institut für Mathematik, Bonn, Germany, partly by Australia Research Fundation, and partly by the National Science Foundation of China and the Science Foundation of the University Doctorial Program of CNEC

the left cells of W. The essential graphs can do more, which provide an l.c.r. set of W in addition. Note that to an element of W, the associated essential graph does not always exist in general. Fortunately, such an obscurity does not occur in the case of the affine Weyl group $W_a(\widetilde{C}_4)$ of type $\widetilde{C}_4$. By the results of Lusztig and Bédard (see [12; 2]), this is still the case for the affine Weyl groups $W_a(\widetilde{C}_2)$ and $W_a(\widetilde{C}_3)$. Moreover, by combining with the results of Lusztig and Bédard (see §6.), we shall see that for $k = 2, 3, 4$, all the left cells of $W_a(\widetilde{C}_k)$ *not* in the lowest two-sided cell (this can be checked easily from the alcove form of any of their elements) can be characterized only by their generalized τ-invariants. Since the left cells of any affine Weyl group in the lowest two-sided cell are characterized by their sign types (see [19; 20] for the definition of a sign type and for this result), all the left cells of $W_a(\widetilde{C}_k)$, $k = 2, 3, 4$, are described completely. I expect that these phenomena should be in common for all the affine Weyl groups $W_a(\widetilde{C}_\ell)$, $\ell \geq 2$.

Some more comments and conjectures are proposed on cells of affine Weyl groups, mostly for the case of type $\widetilde{C}_\ell$, $\ell \geq 2$. This includes the combinatorial description of the Lusztig map relating the two-sided cells of $W_a(\widetilde{C}_\ell)$ to the unipotent conjugacy classes of the corresponding algebraic group, and the group-theoretical interpretation for the number of left cells in some two-sided cells of some affine Weyl groups W_a, which involves both Lusztig map and Bala-Carter correspondence among two-sided cells of an affine Weyl group, unipotent conjugacy classes of the corresponding algebraic group G and the G-classes of pairs $(L, P_{L'})$, where L is a Levi subgroup of G, $P_{L'}$ is a distinguished parabolic subgroup of semisimple part L' of L.

The arrangement of the paper is as follows. In section 1, we collect some results on the cells of a Coxeter group, in particular of an affine Weyl group and even of the group $W_a(\widetilde{C}_4)$. We reformulate and improve the algorithm of finding an l.c.r. set in section 2. Then in section 3, we introduce some results and terminologies needed in performing the algorithm. Sections 4 and 5 are specially concerning the affine Weyl group $W_a(\widetilde{C}_4)$. We obtain an l.c.r. set as well as all the left cell graphs for the group $W_a(\widetilde{C}_4)$ in section 4 and make some comments on the possible generalization of some properties of left cells of $W_a(\widetilde{C}_4)$ to the more general groups $W_a(\widetilde{C}_\ell)$, $\ell \geq 2$, in section

5. Finally, we put an appendix in §6, where we list all the left cell graphs and state some known results for the affine Weyl groups $W_a(\widetilde{C}_2)$ and $W_a(\widetilde{C}_3)$.

§1. Some results on cells.

1.1 Let $W = (W, S)$ be a Coxeter group with S its Coxeter generator set. Let $\leq$ be the Bruhat order on W. For $w \in W$, we denote by $\ell(w)$ the length of w. Let $A = \mathbb{Z}[u]$ be the ring of polynomials in an indeterminate u with integer coefficients. For each ordered pair $y, w \in W$, there exists a unique polynomial $P_{y,w} \in A$, called a Kazhdan-Lusztig polynomial, which satisfies the conditions: $P_{y,w} = 0$ if $y \not\leq w$, $P_{w,w} = 1$, and $\deg P_{y,w} \leq (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$. For $y < w$ in W, let $\mu(w, y) = \mu(y, w)$ be the coefficient of $u^{(1/2)(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$. We denote y—w if $\mu(y, w) \neq 0$.

Checking the relation y—w for $y, w \in W$ usually involves very complicated computation of Kazhdan-Lusztig polynomials. But it becomes easy in some special case: if x, $y \in W$ satisfy $y < x$ and $\ell(y) = \ell(x) - 1$, then we have y—x. Another result concerning this relation will be stated in Proposition 3.3.

1.2 The preorders $\underset{L}{\leq}$, $\underset{R}{\leq}$, $\underset{LR}{\leq}$ and the associated equivalence relations $\underset{\mathrm{L}}{\sim}$, $\underset{\mathrm{R}}{\sim}$, $\underset{\mathrm{LR}}{\sim}$ on W are defined as in [10]. The equivalence classes of W with respect to $\underset{\mathrm{L}}{\sim}$ (resp. $\underset{\mathrm{R}}{\sim}$, $\underset{\mathrm{LR}}{\sim}$) are called left cells (resp. right cells, two-sided cells).

1.3 An affine Weyl group W_a is a Coxeter group which can be realized geometrically as follows. Let G be a connected, adjoint reductive algebraic group over $\mathbb{C}$. We fix a maximal torus T of G. Let X be the group of characters $T \longrightarrow \mathbb{C}$ and let $\Phi \subset X$ be the set of roots with $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ a choice of simple root system. Then $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ is a euclidean space with an inner product $\langle\ ,\ \rangle$ such that the Weyl group (W_0, S_0) of G with respect to T acts naturally on E and preserves its inner product, where S_0 is the set of simple reflections s_i corresponding to the simple roots α_i, $1 \leq i \leq \ell$. We denote by N the group of all translations T_λ ($\lambda \in X$) on E: T_λ sends x to $x + \lambda$,. Then the semidirect product $W_a = W_0 \ltimes N$ is called an affine Weyl group. Let K be the dual of the type of G. Then we define the type of W_a by $\widetilde{K}$. Sometimes we denote W_a by $W_a(\widetilde{K})$ to indicate its type $\widetilde{K}$. There is a canonical homomorphism from W_a to W_0: $w \mapsto \bar{w}$.

Let $-\alpha_0$ be the highest short root in Φ. We define $s_0 = s_{\alpha_0}T_{-\alpha_0}$, where s_{α_0} is the reflection corresponding to α_0. Then the generator set of W_a can be taken as $S = S_0 \cup \{s_0\}$.

1.4 The alcove form of an element $w \in W_a$ is, by definition, a Φ-tuple $(k(w,\alpha))_{\alpha\in\Phi}$ over $\mathbb{Z}$ subject to the following conditions.

(a) $k(w,-\alpha) = -k(w,\alpha)$ for any $\alpha \in \Phi$;

(b) $k(e,\alpha) = 0$ for any $\alpha \in \Phi$, where e is the identity element of W_a;

(c) If $w' = ws_i$ ($0 \le i \le \ell$), then

$$k(w',\alpha) = k(w,(\alpha)\bar{s}_i) + \epsilon(\alpha,i)$$

with

$$\epsilon(\alpha,i) = \begin{cases} 0 & \text{if } \alpha \neq \pm\alpha_i; \\ -1 & \text{if } \alpha = \alpha_i; \\ 1 & \text{if } \alpha = -\alpha_i, \end{cases}$$

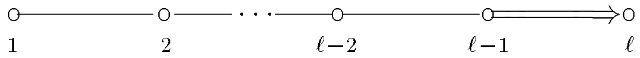
where $\bar{s}_i = s_i$ if $1 \le i \le \ell$, and $\bar{s}_0 = s_{\alpha_0}$ (see [18, Proposition 4.2]).

By condition (a), we can also denote the alcove form of $w \in W_a$ by a Φ^+-tuple $(k(w,\alpha))_{\alpha\in\Phi^+}$.

1.5 Condition 1.4, (c) actually defines a set of operators $\{s_i \mid 0 \le i \le \ell\}$ on the alcove forms of elements of W_a:

$$s_i: \quad (k_\alpha)_{\alpha\in\Phi} \longmapsto (k_{(\alpha)\bar{s}_i} + \epsilon(\alpha,i))_{\alpha\in\Phi}.$$

These operators could be described graphically. For example, assume that W_a has type $\widetilde{C}_\ell$, $\ell \ge 2$, and that the indices of simple roots α_i are compatible with the following Dynkin diagram:



A root $\alpha = \sum_{i=1}^{\ell} a_i\alpha_i$ will be denoted in the form $(a_1, a_2, \cdots, a_\ell)$. Let $\ell = 4$. We arrange the entries of a Φ^+-tuple $(k_\alpha)_{\alpha\in\Phi^+}$ in the following way.

$$\begin{array}{ccccccc} & & & k_{(1,1,1,0)} & & & \\ & & k_{(1,1,0,0)} & & k_{(0,1,1,0)} & & \\ & k_{(1,0,0,0)} & & k_{(0,1,0,0)} & & k_{(0,0,1,0)} & \\ k_{(1,1,1,1)} & & k_{(0,1,1,1)} & & k_{(0,0,1,1)} & & k_{(0,0,0,1)} \\ & k_{(1,2,2,2)} & & k_{(0,1,2,2)} & & k_{(0,0,1,2)} & \\ & & k_{(1,1,2,2)} & & k_{(0,1,1,2)} & & \\ & & & k_{(1,1,1,2)} & & & \end{array} \tag{1.5.1}$$

Then the actions of s_i, $0 \leq i \leq 4$, on a Φ^+-tuple

$$w \;=\; \begin{array}{ccccccc} & & & a & & & \\ & & b & & c & & \\ & d & & e & & f & \\ g & & h & & i & & j \\ & k & & l & & m & \\ & & n & & p & & \\ & & & q & & & \end{array}$$

are listed as in the following table.

s	s_0	s_1	s_2	s_3	s_4
ws	$\begin{array}{ccccccc} & & & -q & & & \\ & & -n & & * & & \\ & -k & & * & & * & \\ -g+1 & & * & & * & & * \\ & -d & & * & & * & \\ & & -b & & * & & \\ & & & -a & & & \end{array}$	$\begin{array}{ccccccc} & & & c & & & \\ & & e & & a & & \\ & -d-1 & & b & & * & \\ h & & g & & * & & * \\ & * & & n & & * & \\ & & l & & q & & \\ & & & p & & & \end{array}$	$\begin{array}{ccccccc} & & & * & & & \\ & & d & & f & & \\ & b & & -e-1 & & c & \\ * & & i & & h & & * \\ & n & & * & & p & \\ & & k & & m & & \\ & & & * & & & \end{array}$	$\begin{array}{ccccccc} & & & b & & & \\ & & a & & e & & \\ & * & & c & & -f-1 & \\ * & & * & & j & & i \\ & * & & p & & * & \\ & & q & & l & & \\ & & & n & & & \end{array}$	$\begin{array}{ccccccc} & & & q & & & \\ & & * & & p & & \\ & * & & * & & m & \\ * & & * & & * & & -j-1 \\ & * & & * & & f & \\ & & * & & c & & \\ & & & a & & & \end{array}$

where the entries in the $*$ positions remain unchanged.

1.6 To each element $x \in W_a$, we associate two subsets of S as below.

$$\mathcal{L}(x) = \{s \in S \mid sx < x\} \quad \text{and} \quad \mathcal{R}(x) = \{s \in S \mid xs < x\}.$$

For w, $w' \in W_a$, we say that w' is *a left extension* of w if $\ell(w') = \ell(w) + \ell(w'w^{-1})$. Then the following results on the alcove form $(k(w,\alpha))_{\alpha\in\Phi}$ of an element $w \in W_a$ are known.

Proposition [18, Propositions 4.1, 4.3]**.** *(1) $\ell(w) = \sum_{\alpha\in\Phi^+} |k(w,\alpha)|$, where the notation $|x|$ stands for the absolute value of x;*

(2) $\mathcal{R}(w) = \{s_i \mid k(w,\alpha_i) < 0\}$.

(3) w' is a left extension of w if and only if the inequalities $k(w',\alpha)k(w,\alpha) \geq 0$ and $|k(w',\alpha)| \geq |k(w,\alpha)|$ hold for any $\alpha \in \Phi$.

1.7 Lusztig defined a function $a: \ W_a \longrightarrow \mathbb{N}$ which satisfies the following properties: (1) $a(z) \leq \nu = |\Phi|/2$, for any $z \in W_a$, where Φ is the root system associated to W_a as in 1.3;

(2) $x \underset{LR}{\leq} y \Longrightarrow a(x) \geq a(y)$. In particular, $x \underset{LR}{\sim} y \Longrightarrow a(x) = a(y)$. So we may define the a-value $a(\Gamma)$ on a (left, right or two-sided) cell Γ of W_a by $a(x)$ for any $x \in \Gamma$.
(3) $a(x) = a(y)$ and $x \underset{L}{\leq} y$ (resp. $x \underset{R}{\leq} y$) $\Longrightarrow x \underset{L}{\sim} y$ (resp. $x \underset{R}{\sim} y$).
(4) For any proper subset I of S, let w_I be the longest element in the subgroup W_I of W_a generated by I. Then $a(w_I) = \ell(w_I)$.

The above properties of function a were shown by Lusztig in his papers [12; 13]. Now we state some more properties of this function, the first two of which are simple consequences of properties (2), (3) and (4).

Let $W_{(i)} = \{w \in W_a \mid a(w) = i\}$ for any non-negative integer i. Then by (2), $W_{(i)}$ is a union of some two-sided cells of W_a.
(5) If $W_{(i)}$ contains an element of the form w_I for some $I \subset S$, then $\{w \in W_{(i)} \mid \mathcal{R}(w) = I\}$ forms a single left cell of W_a.
(6) By the notation $x = y \cdot z$ $(x, y, z \in W_a)$, we mean $x = yz$ and $\ell(x) = \ell(y) + \ell(z)$. In this case, we have $x \underset{L}{\leq} z$, $x \underset{R}{\leq} y$ and hence $a(x) \geq a(y), a(z)$. In particular, if $I = \mathcal{R}(x)$ (resp. $I = \mathcal{L}(x)$), then $a(x) \geq \ell(w_I)$.
(7) $W_{(i)}$ is a single two-sided cell of W_a if $i \in \{0, 1, \nu\}$. As sets, $W_{(i)}$ $(i = 0, 1, \nu)$ can be described as below. $W_{(0)} = \{e\}$, where e is the identity element of W. $W_{(1)}$ consists of all the non-identity elements of W_a each of which has a unique reduced expression (see [11]). $W_{(\nu)}$ consists of all the elements of W_a which have no zero entry in their alcove forms (see 1.4). $W_{(\nu)}$ can also be described as the lowest two-sided cell of W_a with respect to the partial order $\underset{\mathrm{LR}}{\leq}$ (see [19; 20]).
(8) Call an element $s \in S$ special, if the subgroup of W_a generated by $S \setminus \{s\}$ is isomorphic to W_0. Thus the element s_0 is always special. When W_a is of type $\widetilde{C}_\ell$, the element s_ℓ is the unique special element in S other than s_0. It is known that for any two-sided cell $\Omega \neq \{e\}$ of W_a and any special element $s \in S$, the set $Y_s = \{w \in \Omega \mid \mathcal{R}(w) = \{s\}\}$ is non-empty and is a single left cell of W_a (see [15]).

1.8 Let G and W_a be as in 1.3. Then the following result of Lusztig is important to our purpose.

Theorem [14, Theorem 4.8]. *There exists a bijection $\mathbf{u} \mapsto \mathbf{c}(\mathbf{u})$ from the set of unipotent conjugacy classes in G to the set of two-sided cells in W_a. This bijection* *satisfies the equation* $a(\mathbf{c}(\mathbf{u})) = \dim \mathfrak{B}_u$, *where* u *is any element in* $\mathbf{u}$, *and* $\dim \mathfrak{B}_u$ *is the dimension of the variety of Borel subgroups of* G *containing* u.

1.9 Let $W_a = W_a(\widetilde{C}_4)$. Then according to the knowledge of the unipotent classes of the complex simple algebraic group of type B_4, we see from Theorem 1.8 that in W_a, the set $W_{(i)}$ is non-empty if and only if $i \in \Lambda = \{0,1,2,3,4,5,6,8,9,10,16\}$. More precisely, $W_{(i)}$ is a single two-sided cell of W_a if $i \in \{0,1,2,5,6,8,9,10,16\}$, and is a union of two two-sided cells of W_a if $i \in \{3,4\}$.

§2. The algorithm in finding an l.c.r. set.

Here and later, the notation W_a always stands for an affine Weyl group with S its Coxeter generator set. One of the main purposes of the present paper is to describe the left cells of the affine Weyl group W_a of type $\widetilde{C}_4$ by finding its l.c.r. set. We need an algorithm to do so, which was constructed in my paper [22] and is applicable to certain family of crystallographic groups including all the Weyl groups and all the affine Weyl groups. In this section, we shall recall some results of [22]. In particular, we shall reformulate the algorithm in more suitable form, where the concerned Coxeter group is always assumed to be W_a. Some comments on the algorithm are new.

2.1 To each element $x \in W_a$, we associate a set $\Sigma(x)$ of all left cells Γ of W_a satisfying the condition that there is some element $y \in \Gamma$ with $y — x$, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$.

Then the following result is known

Theorem [22, Theorem 2.1]. *If* $x \underset{\mathrm{L}}{\sim} y$ *in* W_a, *then* $\mathcal{R}(x) = \mathcal{R}(y)$ *and* $\Sigma(x) = \Sigma(y)$.

2.2 Say a set Σ of left cells of W_a to be *represented by a set* $K \subset W_a$ if Σ is the set of all left cells Γ of W_a with $\Gamma \cap K \neq \emptyset$. K is called *a representative set* for Σ, if K represents Σ with $|K \cap \Gamma| = 1$ for any $\Gamma \in \Sigma$, where the notation $|X|$ stands for the cardinality of the set X.

The algorithm is based on the following result which is a consequence of Theorem 2.1.

Theorem [22, Theorem 3.1]**.** *Let Ω be a two-sided cell of W_a. Then a non-empty subset $K \subset \Omega$ is a representative set of left cells (an l.c.r. set for short) of W_a in Ω if K satisfies the following conditions.*

(1) $x \underset{L}{\not\sim} y$ for any $x \neq y$ in K;

(2) If an element $y \in W_a$ is such that there is some element $x \in K$ with y—x, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$, then $y \underset{L}{\sim} z$ for some $z \in K$.

2.3 To each element $x \in W_a$, we denote by $M(x)$ the set of all elements y such that there are a sequence of elements $x_0 = x, x_1, \cdots, x_r = y$ in W_a with some $r \geq 0$, where for every i, $1 \leq i \leq r$, the conditions $x_{i-1}^{-1}x_i \in S$ and $\mathcal{R}(x_{i-1}) \overset{\not\supseteq}{\underset{\not\subseteq}{}} \mathcal{R}(x_i)$ are satisfied.

2.4 A subset $K \subset W_a$ is said to be *distinguished* if $K \neq \emptyset$ and $x \underset{L}{\not\sim} y$ for any $x \neq y$ in K.

2.5 By [10, 2.3f], we know that the relations y——x and $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ hold if and only if one of the following cases occurs.

(a) $y^{-1}x \in S$ and $\mathcal{R}(y) \overset{\not\supseteq}{\underset{\not\subseteq}{}} \mathcal{R}(x)$;

(b) $y = x \cdot s$ for some $s \in S$ with $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$;

(c) $y < x$, y——x and $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$.

According to this fact, we design the following three processes on a non-empty set $P \subset W_a$.

(**A**) Find a largest possible subset Q from the set $\bigcup_{x \in P} M(x)$ with Q distinguished.

(**B**) To each $x \in P$, find elements $y \in W_a$ such that $y^{-1}x \in S$, $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$ and $a(y) = a(x)$, add these elements y on the set P to form a set P' and then take a largest possible subset Q from P' with Q distinguished.

(**C**) To each $x \in P$, find elements $y \in W_a$ such that $y < x$, y——x, $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$ and $a(y) = a(x)$, add these elements y on the set P to form a set P' and then take a largest possible subset Q from P' with Q distinguished.

Note that Processes (**B**) and (**C**) put together are amount to Process (**B**) defined in [22].

2.6 A subset P of W_a is called **A**-*saturated* (resp. **B**-*saturated*, resp. **C**-*saturated*), if Process (**A**) (resp. (**B**), resp. (**C**)) on P can't produce any element z satisfying $z \underset{L}{\not\sim} x$ for all $x \in P$.

Clearly, a set of the form $\bigcup_{x\in K} M(x)$ for any $K \subset W_a$ is always **A**-saturated.

It follows from Theorem 2.2 that an l.c.r. set of W_a in a two-sided cell Ω is exactly a distinguished subset of Ω which is **A**-, **B**- and **C**-saturated. So to get such a set, we may use the following

2.7 Algorithm [22, 3.3].

(1) Find a non-empty subset P of Ω (Usually we take P to be distinguished for avoiding unnecessary complication whenever it is possible);

(2) Perform Processes (**A**), (**B**) and (**C**) alternately on P until the resulting distinguished set can't be further enlarged by these processes.

Remark 2.8 (1) An l.c.r. set of W_a in a two-sided cell obtained by the above algorithm is contained in some right cell.

(2) In general, Process (**A**) is easier to be performed than Processes (**B**) and (**C**). The only part of Process (**A**) which may cause some difficulty is to find a largest distinguished subset in the set $\bigcup_{x\in P} M(x)$. On the other hand, Process (**B**) is easier to be performed than Process (**C**). In addition of the difficulty of finding a largest distinguished subset in a given set, the only other difficulty which may occur in Process (**B**) is to check whether or not the condition $a(y) = a(x)$ holds. Process (**C**) is usually quite difficult to be performed, in particular when the lengths of the elements $x \in P$ are getting larger. This is because checking the relation y—x may involve very complicated computation of Kazhdan-Lusztig polynomials. To avoid such kind of troubles to a great extent, we shall give the first priority to Process (**A**) and the second priority to Process (**B**) in applying Algorithm 2.7. In other words, in applying Algorithm 2.7, we always first perform Process (**A**); Process (**B**) is performed only when Process (**A**) alone can not make any further progress; finally Process (**C**) is performed when no progress can be made only by Processes (**A**) and (**B**).

(3) To be simplified, Process (**C**) can be performed in the following way. To an

element $x \in P$, find the set $\Xi(x)$ of all elements y with $y < x$, $\ell(x) \not\equiv \ell(y)(\bmod 2)$, $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$ and $\mathcal{L}(y) \supseteq \mathcal{L}(x)$. Then find the set $\Xi_0(x)$ of all elements y in $\Xi(x)$ such that $a(y) = a(x)$ and $y \underset{L}{\not\sim} z$ for any $z \in P$. If $\Xi_0(x) \neq \emptyset$, then find a maximal distinguished set $\Xi_1(x)$ consisting of all elements $y \in \Xi_0(x)$ with y—x and add it to the set P.

The advantage of the above procedure is that we can reduce the calculation of Kazhdan-Lusztig polynomials to a great extent. Combining this with the convention of the priority in applying Algorithm 2.7, sometimes we can even avoid the calculation of any non-trivial Kazhdan-Lusztig polynomials entirely in practice. The set $\Xi_0(x)$ is empirically always empty in the present case as well as in all the other cases so far we have dealt with [8; 16; 22; 23; 25; 26; 27]. One might think that Process (**C**) is absolutely redundant and should be removed from the algorithm. But I can't rule it out in general.

§3. Some results and terminologies needed in performing Algorithm 2.7.

In applying Algorithm 2.7, we need some results which will be provided in the subsequent discussion. From 1.7, (3) and Theorem 2.1, we have the following result on a set $M(x)$.

Proposition 3.1. *(1) For any $x \in W_a$, the set $M(x)$ is wholly contained in some right cell of W_a.*

(2) If $x \underset{L}{\sim} y$ in W_a, then $M(x)$ and $M(y)$ represent the same set of left cells of W_a.

3.2 In a Coxeter system (W, S), a sequence of elements of the form

(3.2.1) $$\underbrace{ys,\ yst,\ ysts,\ \dots}_{m-1 \text{ terms}}$$

is called *an $\{s,t\}$-string* (or just call it *a string*) if $s, t \in S$ and $y \in W$ satisfy the conditions that the order $o(st)$ of the product st is m and $\mathcal{R}(y) \cap \{s,t\} = \emptyset$.

It is easily seen that a string is wholly contained in some right cell of W. For any $x \in W$, we can re-define $M(x)$ to be the minimal set containing x, subject to the requirement: any string (regarded as a set) meeting $M(x)$ must be wholly contained in $M(x)$. Suppose that we are given two $\{s,t\}$-strings $x_1,\ x_2,\ \dots,\ x_{m-1}$

and $y_1,\ y_2,\ \ldots,\ y_{m-1}$ with $o(st)=m$. We denote the integers $\mu(x_i,y_j)$ (see 1.1) by a_{ij} for $1\le i,j\le m-1$. Then it is known that

Proposition 3.3 [12, 10.4]. *In the above setup, the following assertions hold.*
(a) When $m=3$, we have $a_{12}=a_{21}$, $a_{11}=a_{22}$;
(b) When $m=4$, we have $a_{12}=a_{21}=a_{23}=a_{32}$, $a_{11}=a_{33}$, $a_{13}=a_{31}$ and $a_{22}=a_{11}+a_{13}$.

We have the following result corresponding to this.

Proposition 3.4 [10, Corollary 4.3; 22, Proposition 4.6]. *Keep the setup of 3.2.*
(1) If $m=3$, then

(a) $x_1 \underset{\mathrm{L}}{\sim} y_1 \Longleftrightarrow x_2 \underset{\mathrm{L}}{\sim} y_2$;
(b) $x_1 \underset{\mathrm{L}}{\sim} y_2 \Longleftrightarrow x_2 \underset{\mathrm{L}}{\sim} y_1$.

(2) If $m=4$, then

(a) $x_1 \underset{\mathrm{L}}{\sim} y_2 \Longleftrightarrow x_2 \underset{\mathrm{L}}{\sim} y_1 \Longleftrightarrow x_2 \underset{\mathrm{L}}{\sim} y_3 \Longleftrightarrow x_3 \underset{\mathrm{L}}{\sim} y_2$;
(b) $x_1 \underset{\mathrm{L}}{\sim} y_1 \Longleftrightarrow x_3 \underset{\mathrm{L}}{\sim} y_3$;
(c) $x_1 \underset{\mathrm{L}}{\sim} y_3 \Longleftrightarrow x_3 \underset{\mathrm{L}}{\sim} y_1$;
(d) $x_2 \underset{\mathrm{L}}{\sim} y_2 \Longleftrightarrow$ *either* $x_1 \underset{\mathrm{L}}{\sim} y_1$ *or* $x_1 \underset{\mathrm{L}}{\sim} y_3$

3.5 Two elements $x,y\in W_a$ form *a primitive pair*, if there exist two sequences of elements $x_0=x,x_1,\cdots,x_r$ and $y_0=y,y_1,\cdots,y_r$ in W_a such that the following conditions are satisfied.

(a) x_i—y_i for all i, $0\le i\le r$.

(b) For every i, $1\le i\le r$, there exist some $s_i,t_i\in S$ such that x_{i-1},x_i (and also y_{i-1},y_i)) are two neighboring terms in some $\{s_i,t_i\}$-string.

(c) Either $\mathcal{R}(x)\not\subseteq\mathcal{R}(y)$ and $\mathcal{R}(y_r)\not\subseteq\mathcal{R}(x_r)$, or $\mathcal{R}(y)\not\subseteq\mathcal{R}(x)$ and $\mathcal{R}(x_r)\not\subseteq\mathcal{R}(y_r)$ hold.

In this case, we have $x \underset{\mathrm{R}}{\sim} y$ by Proposition 3.1.

Assume that x,x' (and also y,y') are two neighboring terms in some $\{s,t\}$- string with x—y and that at least one of x, y is a terminal term of the $\{s,t\}$-string containing it. Then by Proposition 3.3, we have x'—y'. In particular, it is always the case when $o(st) = 3$. Thus, if in (b), we have in addition that at least one of x_i, y_i is a terminal term of the $\{s_i, t_i\}$-string containing it for all i, $0 \leq i < r$, then we can replace condition (a) by the following weaker one in the definition of a primitive pair: (a') $x_0 \text{---} y_0$.

3.6 In the present paper, by a graph $\mathfrak{M}$, it always means that a set M of vertices together with a set of edges, where each edge is a two-elements subset of M, and each vertex is labelled by a subset of S.

Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two graphs with their vertex sets M and M'. They are said to be isomorphic (note that we call it *quasi-isomorphic* in [22] owing to the different definition of a graph), written $\mathfrak{M} \cong \mathfrak{M}'$, if there exists a bijective map η from the set M to the set M' satisfying the following conditions.

(1) The labelling of w is the same as that of $\eta(w)$ for any $w \in M$.

(2) For $w, z \in M$, $\{w, z\}$ is an edge of $\mathfrak{M}$ if and only if $\{\eta(w), \eta(z)\}$ is an edge of $\mathfrak{M}'$.

This is an equivalence relation on graphs.

3.7 We define *a graph* $\mathfrak{M}(x)$ associated to an element $x \in W_a$ as follows. Its vertex set is $M(x)$. Its edge set consists of all two-elements subsets $\{y, z\} \subset M(x)$ with y, z two neighboring terms of a string. Each vertex $y \in M(x)$ is labelled by the set $\mathcal{R}(y)$.

Note that unlike the definition given in [22], we do not label the edges of a graph here. Clearly, for any $x \in W_a$, the graph $\mathfrak{M}(x)$ is always connected.

A left cell graph associated to an element $x \in W_a$, written $\mathfrak{M}_L(x)$, is by definition a graph, whose vertex set $M_L(x)$ consists of all left cells Γ of W_a with $\Gamma \cap M(x) \neq \emptyset$. Two vertices $\Gamma, \Gamma' \in M_L(x)$ are joined by an edge, if there are two elements $y \in M(x) \cap \Gamma$ and $y' \in M(x) \cap \Gamma'$ such that $\{y, y'\}$ is an edge of $\mathfrak{M}(x)$. Each vertex Γ of $\mathfrak{M}_L(x)$ is labelled by the common labelling of elements of $M(x) \cap \Gamma$ (This makes sense by [10, Proposition 2.4]). Clearly, the graph $\mathfrak{M}_L(x)$ is always connected.

3.8 A subgraph $\mathfrak{M}$ of $\mathfrak{M}(x)$ ($x \in W_a$) is said to be *essential*, if there exists an isomorphism η from $\mathfrak{M}$ to $\mathfrak{M}_L(x)$ such that each vertex y of $\mathfrak{M}$ is contained in the left cell $\eta(y)$.

It is easily seen that when a subgraph $\mathfrak{M}$ of $\mathfrak{M}(x)$ is essential, its vertex set must be distinguished. In particular, the graph $\mathfrak{M}(x)$ itself is essential if and only if its vertex set $M(x)$ is distinguished. But it should be careful that in general there does not always exist a subgraph of $\mathfrak{M}(x)$ which is essential (Some counter-examples could be found in the two-sided cell $W_{(3)}$ of the affine Weyl group $W_a(\widetilde{D}_4)$ and in $W_{(1)}$ of $W_a(\widetilde{A}_\ell)$, $\ell > 1$). However, we shall see that for any $x \in W_a(\widetilde{C}_4)$, there always exists some essential subgraph of $\mathfrak{M}(x)$ containing x as its vertex.

3.9 By *a path* in the graph $\mathfrak{M}(x)$, we mean a sequence of vertices $z_0, z_1, \ldots, z_t$ in $M(x)$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathfrak{M}(x)$ for any i, $1 \le i \le t$. Two elements $x, x' \in W_a$ have the same *generalized τ-invariant*, if for any path $z_0 = x, z_1, \ldots, z_t$ in the graph $\mathfrak{M}(x)$, there is a path $z'_0 = x', z'_1, \ldots, z'_t$ in $\mathfrak{M}(x')$ with $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for every i, $0 \le i \le t$, and if the same condition holds when interchanging the roles of x with x'.

3.10 It may happen that for two elements $x, y \in W_a$ with $x \underset{\mathrm{L}}{\sim} y$, the graphs $\mathfrak{M}(x)$ and $\mathfrak{M}(y)$ are not isomorphic (take $x = s_0$ and $y = s_1s_0$ in $W_a(\widetilde{C}_4)$ for example). But we have the following result.

Proposition. *(a) The elements in the same left cell of W_a have the same generalized τ-invariant.*

(b) If $x \underset{\mathrm{L}}{\sim} y$ in W_a, then the left cell graphs $\mathfrak{M}_L(x)$ and $\mathfrak{M}_L(y)$ are isomorphic.

The assertion (a) is well-known (see [22, Proposition 4.2]). Then (b) follows from Theorem 2.1 and Proposition 3.1 readily.

The above result allows us to talk about the generalized τ-invariant of a left cell of W_a, which is by definition the generalized τ-invariant of any element in this left cell.

3.11 We state some well-known results concerning the Bruhat order of a Coxeter system (W, S) which will be useful in performing Process (**C**) on a set.

(a) Let $y \le w$ in W. Then for any reduced form $w = s_1s_2\cdots s_r$ with $s_i \in S$, there is a subsequence i_1, i_2, $\cdots$, i_t of $1, 2, \cdots, r$ such that $y = s_{i_1}s_{i_2}\cdots s_{i_t}$ is a reduced expression of y.

(b) Suppose $J = \mathcal{L}(w)$ for $w \in W$. Then there is some $x \in W$ with $w = w_J{\cdot}x$ and

$\ell(w) = \ell(w_J) + \ell(x)$.

Now let $w \in W$ be with $J = \mathcal{L}(w)$. By (b), we can find a reduced expression $w = s_1 s_2 \cdots s_r$, $s_i \in S$, with $w_J = s_1 s_2 \cdots s_t$, where $t = \ell(w_J)$. Denote $w_j = s_1 s_2 \cdots s_j$ for $t \le j \le r$. Let P_j be the set of all elements y with $y \le w_j$ and $\mathcal{L}(y) \supseteq J$. Then $P_t = \{w_J\}$. Suppose that the set P_k has been found for $t \le k < r$. Then by (a), we have

$$P_{k+1} = P_k \bigcup \{\, x s_{k+1} \mid x \in P_k, \quad s_{k+1} \notin \mathcal{R}(x)\}.$$

This provides a recurrence procedure to find all the elements y with $y \le w$ and $\mathcal{L}(y) \supseteq \mathcal{L}(w)$ for any given $w \in W$.

§4. l.c.r. sets and left cell graphs in two-sided cells of $W_a(\widetilde{C}_4)$.

4.1 We shall apply Algorithm 2.7 to find an l.c.r. set, together with the corresponding left cell graphs, in each two-sided cell Ω of $W_a = W_a(\widetilde{C}_4)$. Let us first choose the starting sets P of the algorithm. From the nature of the algorithm, it is preferred (but not necessary) to choose the elements x of the form w_I, $I \subset S$, for the set P whenever it is possible. Let P_i be the set of all the elements of $W_{(i)}$ of the form w_I. Then we have the following table.

i	P_i	i	P_i
0	$\{e\}$	6	$\{w_{123}\}$
1	$\{s_0, s_1, s_2, s_3, s_4\}$	8	$\{w_{0134}\}$
2	$\{w_{02}, w_{03}, w_{04}, w_{13}, w_{14}, w_{24}\}$	9	$\{w_{012}, w_{234}\}$
3	$\{w_{024}, w_{12}, w_{23}\}$	10	$\{w_{0124}, w_{0234}\}$
4	$\{w_{023}, w_{124}, w_{01}, w_{34}\}$	16	$\{w_{0123}, w_{1234}\}$
5	$\{w_{013}, w_{014}, w_{034}, w_{134}\}$		

Each set $W_{(i)}$ ($i \in \{0, 1, 2, 5, 6, 8, 9, 10, 16\}$) consists of a single two-sided cell. For such a set $W_{(i)}$, we shall take P_i as the starting set of the algorithm. The set $W_{(3)}$ (resp. $W_{(4)}$) contains two two-sided cells. We shall take a one-element subset of P_i as the starting set of the algorithm.

For any $z \in W_a$, we denote by $\Omega(z)$ (resp. $\Gamma(z)$) the two-sided cell (resp. the left cell) of W_a containing z.

4.2 The case $W_{(0)}$ is trivial and so we shall always assume $i > 0$ for $W_{(i)}$. In applying the algorithm, we shall first deal with the two-sided cell $W_{(16)}$, then $W_{(10)}$, $W_{(9)}$, $W_{(8)}$, $W_{(6)}$, $W_{(5)}$, $\Omega(w_{023})$, $W_{(4)} \setminus \Omega(w_{023})$, $\Omega(w_{12})$, $W_{(3)} \setminus \Omega(w_{12})$, $W_{(2)}$ and $W_{(1)}$ in turn, where $X \setminus Y = \{x \in X \mid x \notin Y\}$ for any sets X, Y.. The reason for taking such an order is to make it easier in performing Processes (**B**) and (**C**), in particular in the determination of the a-values of the elements occurring in the intermediate steps of these two processes.

Let ψ be the automorphism of $W_a(\widetilde{C}_4)$ which sends s_i to s_{4-i} for $0 \leq i \leq 4$. Then it is clear that ψ stabilizes the sets $W_{(i)}$, $i \geq 0$, and induces a permutation on the set of left (resp. right, resp. two-sided) cells of W_a in each $W_{(i)}$.

We shall use the notation **i** for the simple reflection s_i $(0 \leq i \leq 4)$ in the subsequent discussion.

4.3 $W_{(16)}$ is the lowest two-sided cell of W_a. It is known that an element of W_a is in $W_{(16)}$ if and only if its alcove form has no zero entry (see 1.7,(7)). It is also known that there are totally $|W_0|$ left cells of W_a in $W_{(16)}$ each of which is associated to a sign type (see [20, Theorem 1.1 and Corollary 1.2]). Let $x = \mathbf{3210123} \cdot w_{0124}$ and $y = \mathbf{1234321} \cdot w_{0234}$. Then we see from their alcove forms that the elements x, y, w_{0123} and w_{1234} are all in the set $W_{(16)}$. The graphs $\mathfrak{M}(x)$, $\mathfrak{M}(y)$, $\mathfrak{M}(w_{0123})$ and $\mathfrak{M}(w_{1234})$ are isomorphic to those in Fig.s 16, 17, 18 and 19, respectively (Fig.s mentioned here and later will be displayed at the end of this section). The vertices x, y, w_{0123}, w_{1234} are labelled by $\boxed{\mathbf{0124}}$, $\boxed{\mathbf{0234}}$, $\boxed{\mathbf{0123}}$, $\boxed{\mathbf{1234}}$ respectively in the corresponding graphs. We see that the sign types associating to the elements of the union $M = M(x) \cup M(y) \cup M(w_{0123}) \cup M(w_{1234})$ are all different. This implies that the above four graphs are all essential and that M is an l.c.r. set of $W_{(16)}$ by the fact $|M| = |W_0| = 384$.

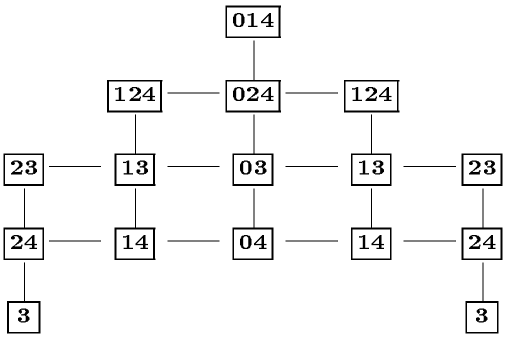
4.4 $W_{(10)}$ is a single two-sided cell of W_a. The graph $\mathfrak{M}(w_{0124})$ is isomorphic to that in Fig. 16, which is essential by Proposition 3.10. The element w_{0234} does not belong to any left cell of W_a in $M_L(w_{024})$ since there is no vertex of $\mathfrak{M}(w_{0124})$ labelled by $\boxed{\mathbf{0234}}$. By the fact $\psi(w_{0124}) = w_{0234}$, we see that the graph $\mathfrak{M}(w_{0234})$ is essential (see Fig. 17.). The left cell set $M_L(w_{0124})$ is disjoint to $M_L(w_{0234})$ again by Proposition

3.10. By applying Algorithm 2.7, we see that the union $M(w_{0124}) \cup M(w_{0234})$ is **A**-, **B**- and **C**-saturated and hence forms an l.c.r. set of the two-sided cell $W_{(10)}$.

4.5 $W_{(9)}$ is a single two-sided cell of W_a. There are two vertices of $\mathfrak{M}(w_{012})$ labelled by $\boxed{\mathbf{012}}$ (resp. $\boxed{\mathbf{0}}$). This fact, together with 1.7, (5),(8) and Propositions 3.4, 3.10, implies that the graph $\mathfrak{M}(w_{012})$ is not essential and that the left cell graph $\mathfrak{M}_L(w_{012})$ should be that in Fig. 14 with the vertex $\Gamma(w_{012})$ labelled by $\boxed{\mathbf{012}}$. Since there is no vertex of $\mathfrak{M}_L(w_{012})$ labelled by $\boxed{\mathbf{234}}$, the element w_{234} does not belong to any left cell of W_a in $M_L(w_{012})$. Since $\psi(w_{012}) = w_{234}$, we see that the left cell graph $\mathfrak{M}_L(w_{234})$ should be displayed as in Fig. 15 with $\Gamma(w_{234})$ the vertex labelled by $\boxed{\mathbf{234}}$. By applying Algorithm 2.7, we can show that the union $M_L(w_{012}) \cup M_L(w_{234})$ is the left cell set of W_a in the two-sided cell $W_{(9)}$.

4.6 $W_{(8)}$ is a single two-sided cell of W_a. The graph $\mathfrak{M}(w_{0134})$ is isomorphic to that in Fig. 13, which is essential by Proposition 3.10. It can be shown by applying Algorithm 2.7 that the set $M(w_{0134})$ forms an l.c.r. set of $W_{(8)}$.

4.7 $W_{(6)}$ is a single two-sided cell of W_a. The graph $\mathfrak{M}(w_{123})$ is infinite. By 1.7, (5),(8), we see that the set $\{w \in M(w_{123}) \mid \mathcal{R}(w) = J\}$ ($J = \{\mathbf{0}\}$, $\{\mathbf{4}\}$ or $\{\mathbf{123}\}$) is contained in some left cell of W_a. On the other hand, let $x = w_{123} \cdot \mathbf{012024}$. Then the graph $\mathfrak{M}(w_{123})$ contains the following subgraph.



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where the vertex labelled by $\boxed{\mathbf{014}}$ is the element x. Let $y = x\mathbf{20}$ and $w = x \cdot \mathbf{21}$. Then y and w are two vertices of this subgraph, both labelled by $\boxed{\mathbf{124}}$. We claim $y \underset{\mathrm{L}}{\sim} w$. For, we have $\mathbf{210} \cdot w = \mathbf{12010} \cdot y$, denote this element by z. Then z is a common left extension of w and y (see 1.6). Let $x_1 = \mathbf{210} \cdot w_{123}$. Then we see that $z \in M(x_1)$ and $x_1^{-1} \in M(w_{123})$. This implies $z \in W_{(6)}$. Hence by 1.7,(3), we have $y \underset{\mathrm{L}}{\sim} z \underset{\mathrm{L}}{\sim} w$. From these facts and by Proposition 3.4, we see that the left cell graph $\mathfrak{M}_L(w_{123})$ should be that in Fig. 12. Applying Algorithm 2.7, we see that $M_L(w_{123})$ is the left cell set of W_a in the two-sided cell $W_{(6)}$.

4.8 $W_{(5)}$ is a single two-sided cell of W_a. The graph $\mathfrak{M}(w_{013})$ is infinite. By 1.7,(5), we know that the set $\{w \in M(w_{013}) \mid \mathcal{R}(w) = J\}$ is contained in some left cell of W_a for $J = \{\mathbf{0}, \mathbf{1}, \mathbf{3}\}$, $\{\mathbf{0}, \mathbf{1}, \mathbf{4}\}$, $\{\mathbf{0}, \mathbf{3}, \mathbf{4}\}$ or $\{\mathbf{1}, \mathbf{3}, \mathbf{4}\}$. By Propositions 3.4 and 3.10, this fact tells us that the left cell graph $\mathfrak{M}_L(w_{013})$ should be that in Fig. 10. Since $\psi(w_{013}) = w_{134}$, we see that the left cell graph $\mathfrak{M}_L(w_{134})$ should be that in Fig. 11. The sets $M_L(w_{013})$ and $M_L(w_{134})$ are disjoint by Proposition 3.10. By applying Algorithm 2.7, we see that $M_L(w_{013}) \cup M_L(w_{134})$ is the left cell set of W_a in the two-sided cell $W_{(5)}$.

4.9 There are two two-sided cells in $W_{(4)}$.

(a) First consider the two-sided cell $\Omega(w_{023})$. The graph $\mathfrak{M}(w_{023})$ is isomorphic to that in Fig. 6, which is essential by Proposition 3.10. Let $y = w_{023} \cdot \mathbf{12}$, $y_0 = y \cdot \mathbf{4}$. Then $y \in M(w_{023})$, and $\{y, y_0\}$ is a primitive pair (see 3.5). So $y_0 \underset{\mathrm{R}}{\sim} w_{023}$. But we have $\mathcal{R}(y_0) = \{\mathbf{1}, \mathbf{2}, \mathbf{4}\}$. This implies $y_0 \underset{\mathrm{L}}{\sim} w_{124}$ by 1.7,(5). Hence $w_{124} \in \Omega(w_{023})$. Since $\psi(w_{023}) = w_{124}$, it implies from the above results that the graph $\mathfrak{M}(w_{124})$ is essential (see Fig. 7). It is seen easily by Proposition 3.10 that the left cell set $M_L(w_{023})$ is disjoint to $M_L(w_{124})$. By applying Algorithm 2.7, we see that the union $M = M(w_{124}) \cup M(w_{023})$ is **A**-, **B**- and **C**-saturated. So M forms an l.c.r. set of the two-sided cell $\Omega(w_{023})$.

(b) We have $w_{01}, w_{34} \notin \Omega(w_{023})$ since there is no vertex of $\mathfrak{M}(w_{023})$ and $\mathfrak{M}(w_{124})$ labelled by $\boxed{\mathbf{01}}$ or $\boxed{\mathbf{34}}$. This implies $\Omega(w_{01}) = W_{(4)} \setminus \Omega(w_{023})$ and $w_{34} \in \Omega(w_{01})$. The graphs $\mathfrak{M}(w_{01})$ and $\mathfrak{M}(w_{34})$ are isomorphic to those in Fig.s 8, 9, respectively. By Proposition 3.10, we see that these two graphs are both essential and that the left cell set $M_L(w_{01})$ is disjoint to $M_L(w_{34})$. By applying Algorithm 2.7, we see that the set $M(w_{01}) \cup M(w_{34})$ forms an l.c.r. set of the two-sided cell $\Omega(w_{01})$.

4.10 There are two two-sided cells in $W_{(3)}$.

(a) First consider the two-sided cell $\Omega(w_{12})$. The graph $\mathfrak{M}(w_{12})$ is essential by Proposition 3.10 (see Fig. 4). By applying Algorithm 2.7, we see that the set $M(w_{12})$

forms an l.c.r. set of $\Omega(w_{12})$.

(b) We have $w_{024} \notin \Omega(w_{12})$ since no vertex of the graph $\mathfrak{M}(w_{12})$ is labelled by $\boxed{024}$. So $\Omega(w_{024}) = W_{(3)} \setminus \Omega(w_{12})$. The graph $\mathfrak{M}(w_{024})$ is essential by Proposition 3.10 (see Fig. 5). The set $M(w_{024})$ forms an l.c.r. set of the two-sided cell $\Omega(w_{024})$ by applying Algorithm 2.7.

4.11 By 1.7,(5) and Propositions 3.4, 3.10, we see that the graph $\mathfrak{M}(w_{02})$ is not essential and that the left cell graph $\mathfrak{M}_L(w_{02})$ is displayed as in Fig. 3. We have by applying Algorithm 2.7 that the set $M_L(w_{02})$ is the left cell set of W_a in the two-sided cell $W_{(2)}$.

4.12 The graph $\mathfrak{M}(s_0)$ is infinite. The left cell graph $\mathfrak{M}_L(s_0)$ can be obtained easily from $\mathfrak{M}(s_0)$ by 1.7, (5) (see Fig.2.).

4.13 We have got all the left cell graphs of the two-sided cells $W_{(i)}$, $i \in \{1,2,5,6,9\}$. By a close observation of the related graphs, we see that for any left cell graph $\mathfrak{M}_L(x)$ of $W_{(i)}$ ($i \in \{1,2,5,6,9\}$), there exists some subgraph of the corresponding graph $\mathfrak{M}(x)$ which is isomorphic to $\mathfrak{M}_L(x)$ and contains x as its vertex. Therefore we have obtained an l.c.r. set for any of these two-sided cells of W_a. On the other hand, we have got an l.c.r. set for each remaining two-sided cell of W_a which is presented as the vertex set of certain essential graphs of the form $\mathfrak{M}(x)$. Thus we have actually got an l.c.r. set and the left cell graphs for any two-sided cell of W_a.

4.14 The following are all the left cell graphs of $W_a(\widetilde{C}_4)$ obtained in the present section.

Fig. 1. $\mathfrak{M}_L(e)$

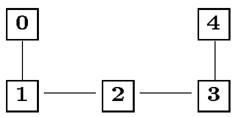


Fig. 2. $\mathfrak{M}_L(s_0)$

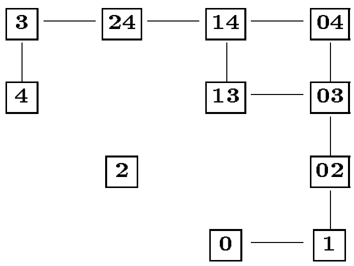


Fig. 3. $\mathfrak{M}_L(w_{04})$

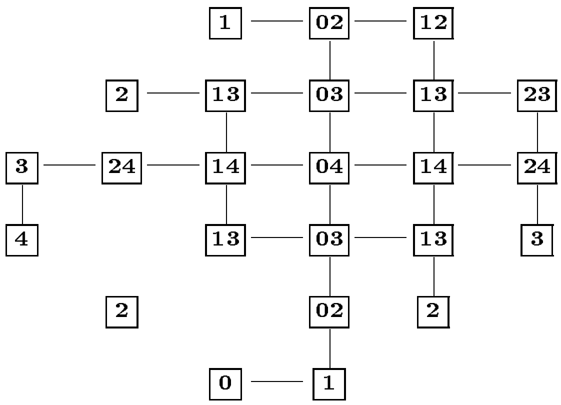


Fig. 4. $\mathfrak{M}_L(w_{12})$

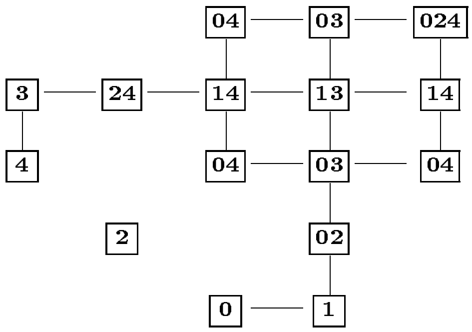


Fig. 5. $\mathfrak{M}_L(w_{024})$

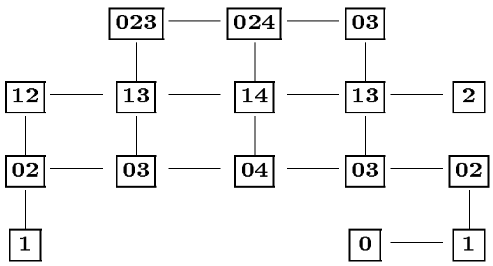


Fig. 6. $\mathfrak{M}_L(w_{023})$

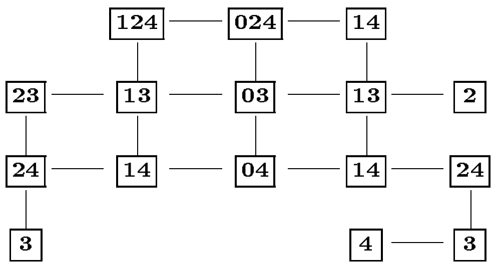


Fig. 7. $\mathfrak{M}_L(w_{124})$

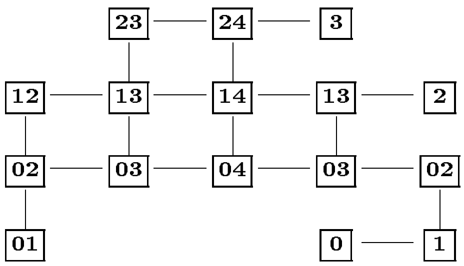


Fig. 8. $\mathfrak{M}_L(w_{01})$

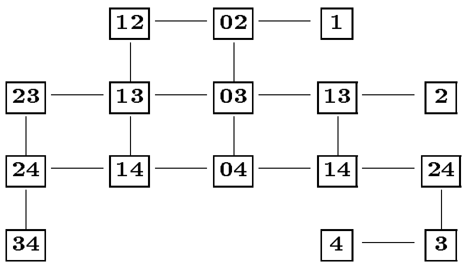


Fig. 9. $\mathfrak{M}_L(w_{34})$

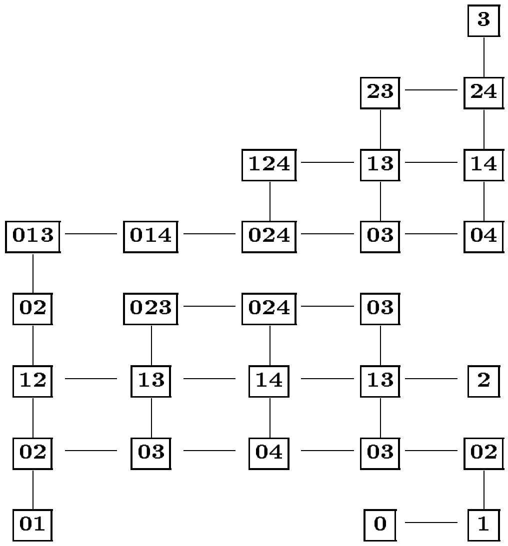


Fig. 10. $\mathfrak{M}_L(w_{013})$

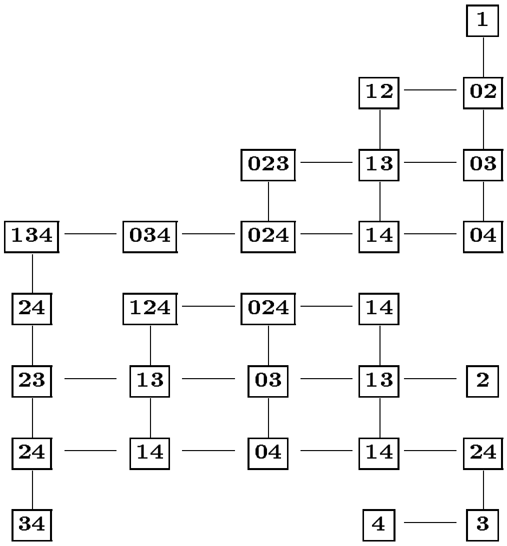


Fig. 11. $\mathfrak{M}_L(w_{134})$

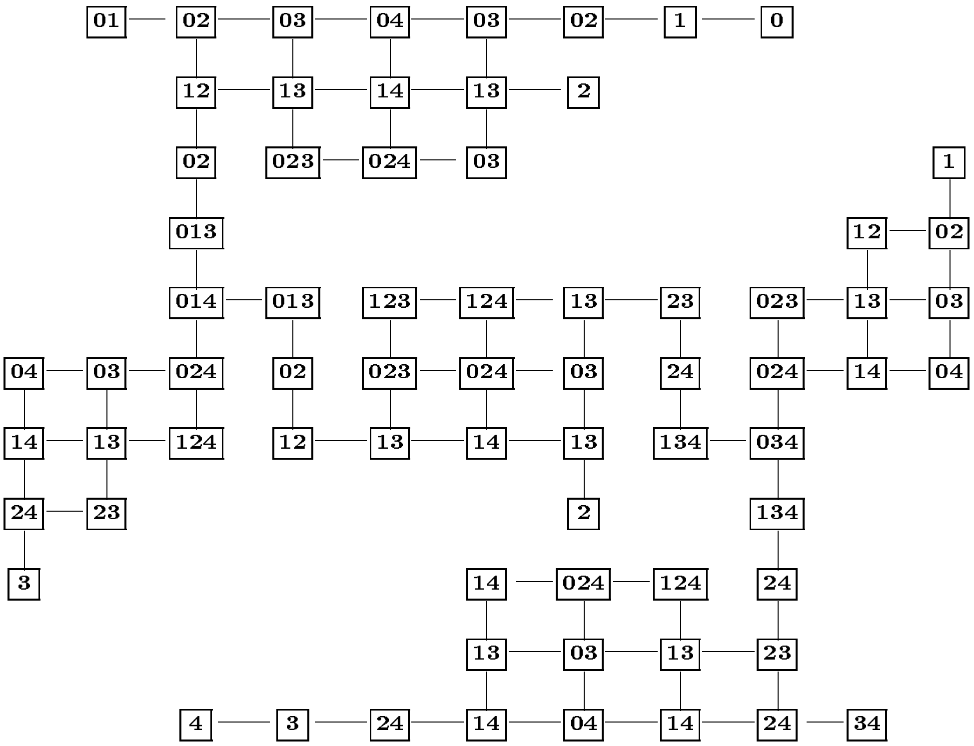


Fig. 12. $\mathfrak{M}_L(w_{123})$

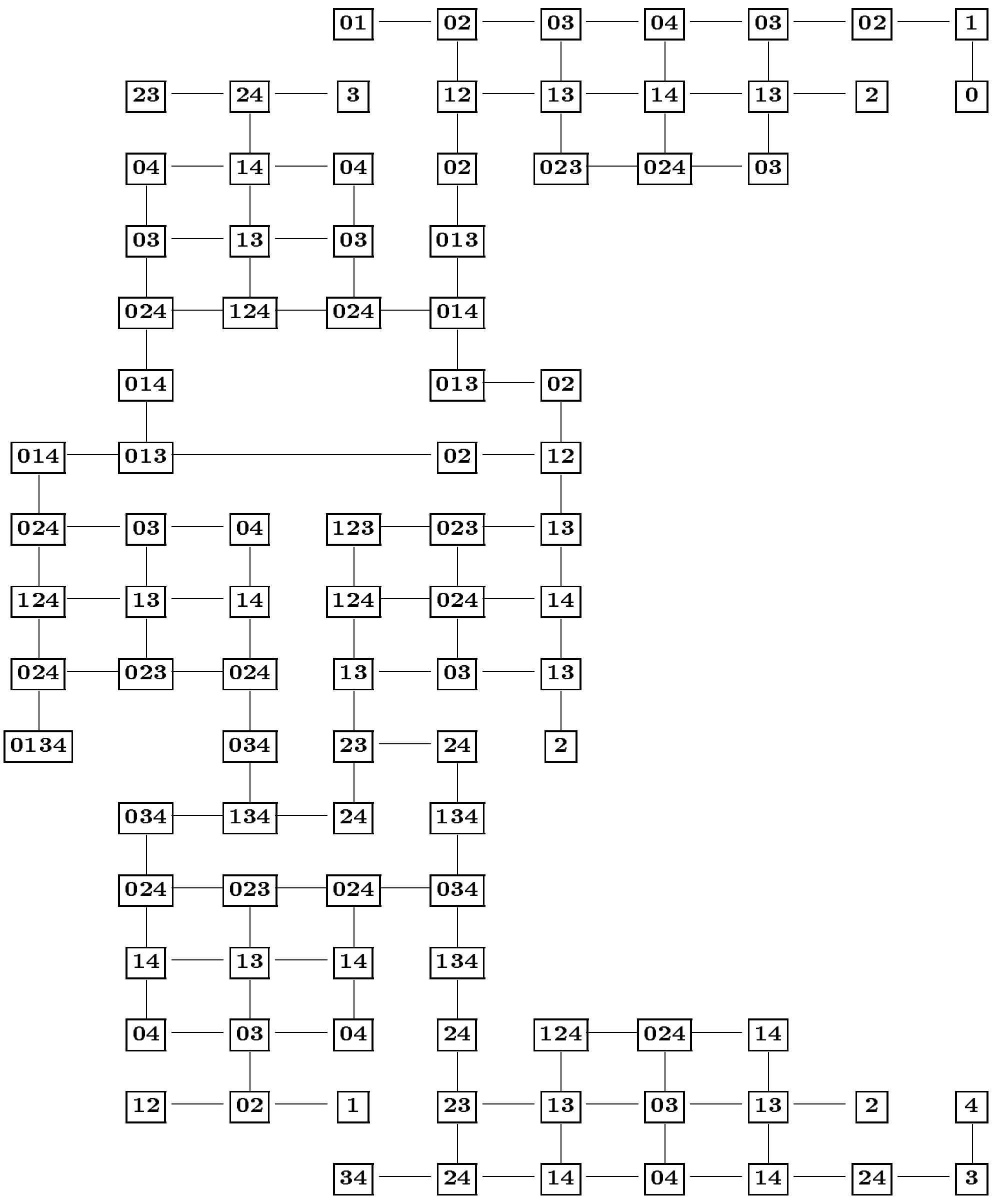


Fig. 13. $\mathfrak{M}_L(w_{0134})$

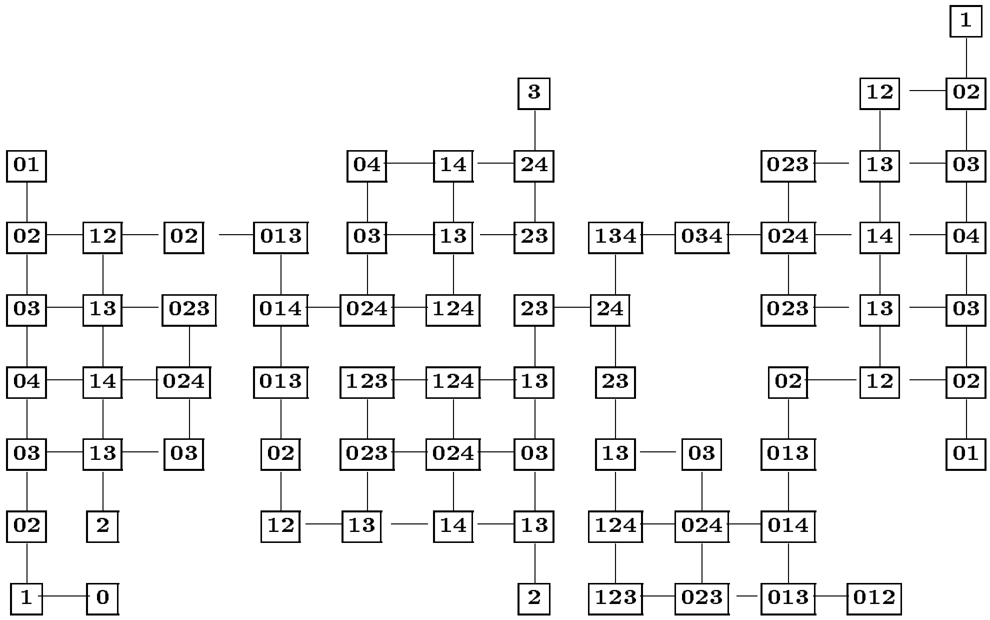


Fig. 14. $\mathfrak{M}_L(w_{012})$

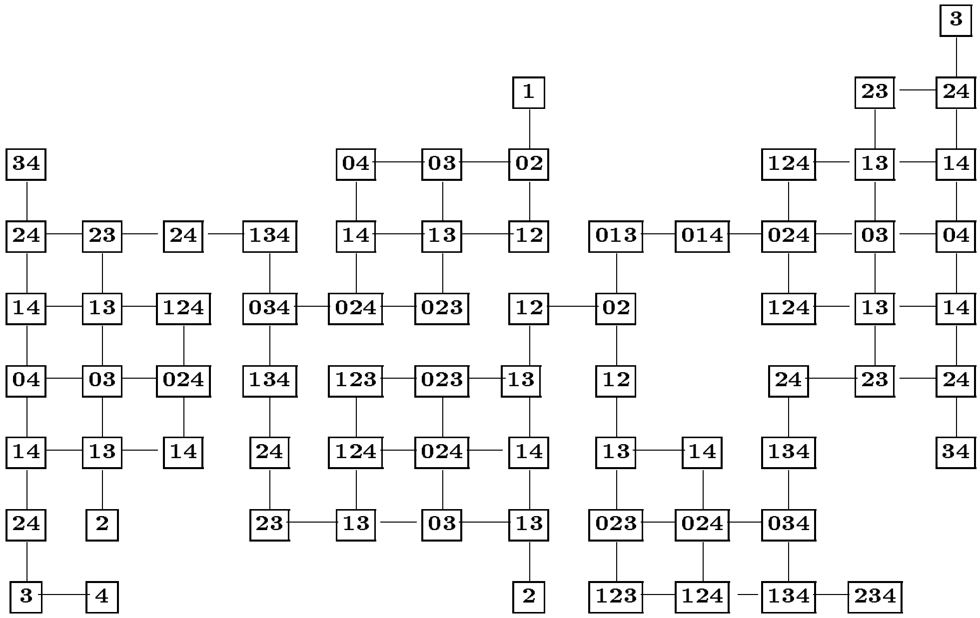


Fig. 15. $\mathfrak{M}_L(w_{234})$

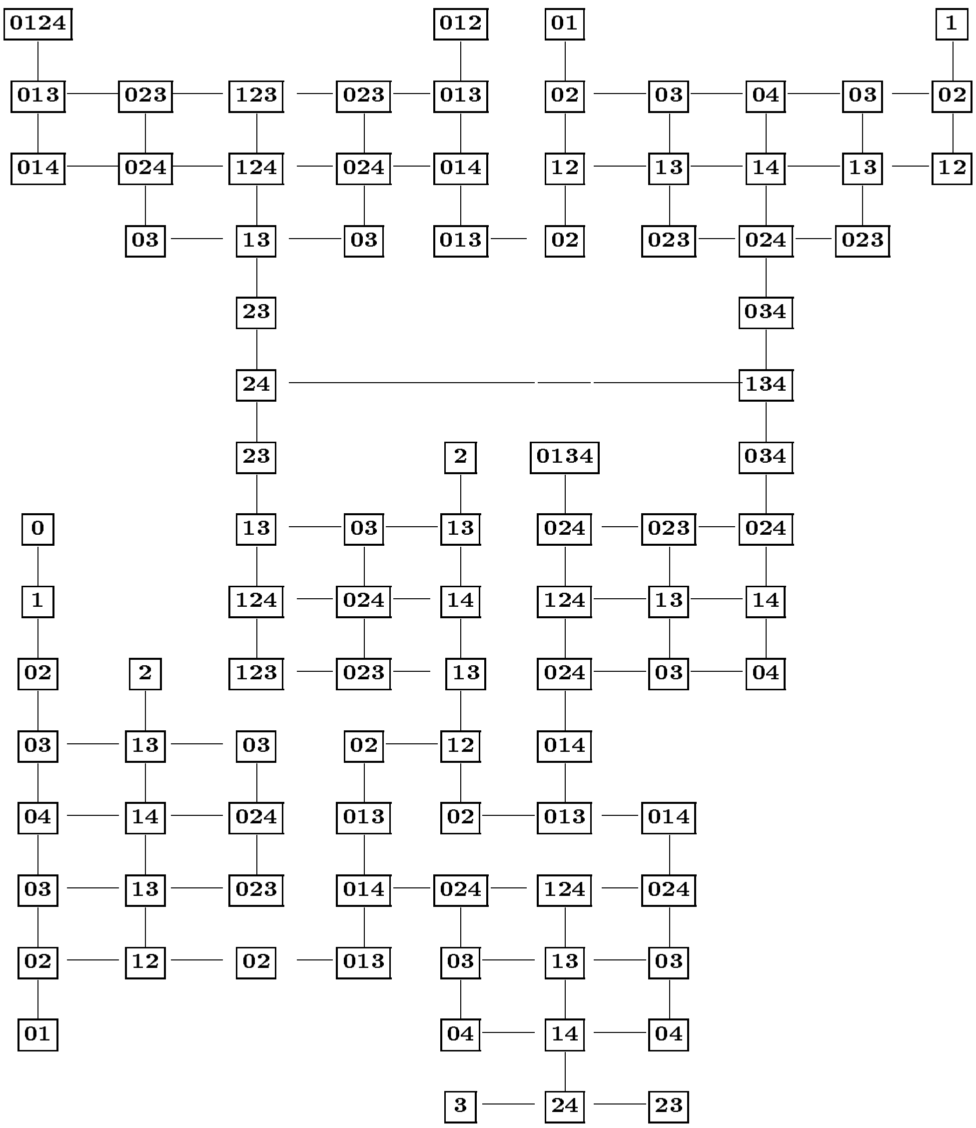


Fig. 16. $\mathfrak{M}_L(w_{0124})$ or $\mathfrak{M}_L(\mathbf{3210123} \cdot w_{0124})$

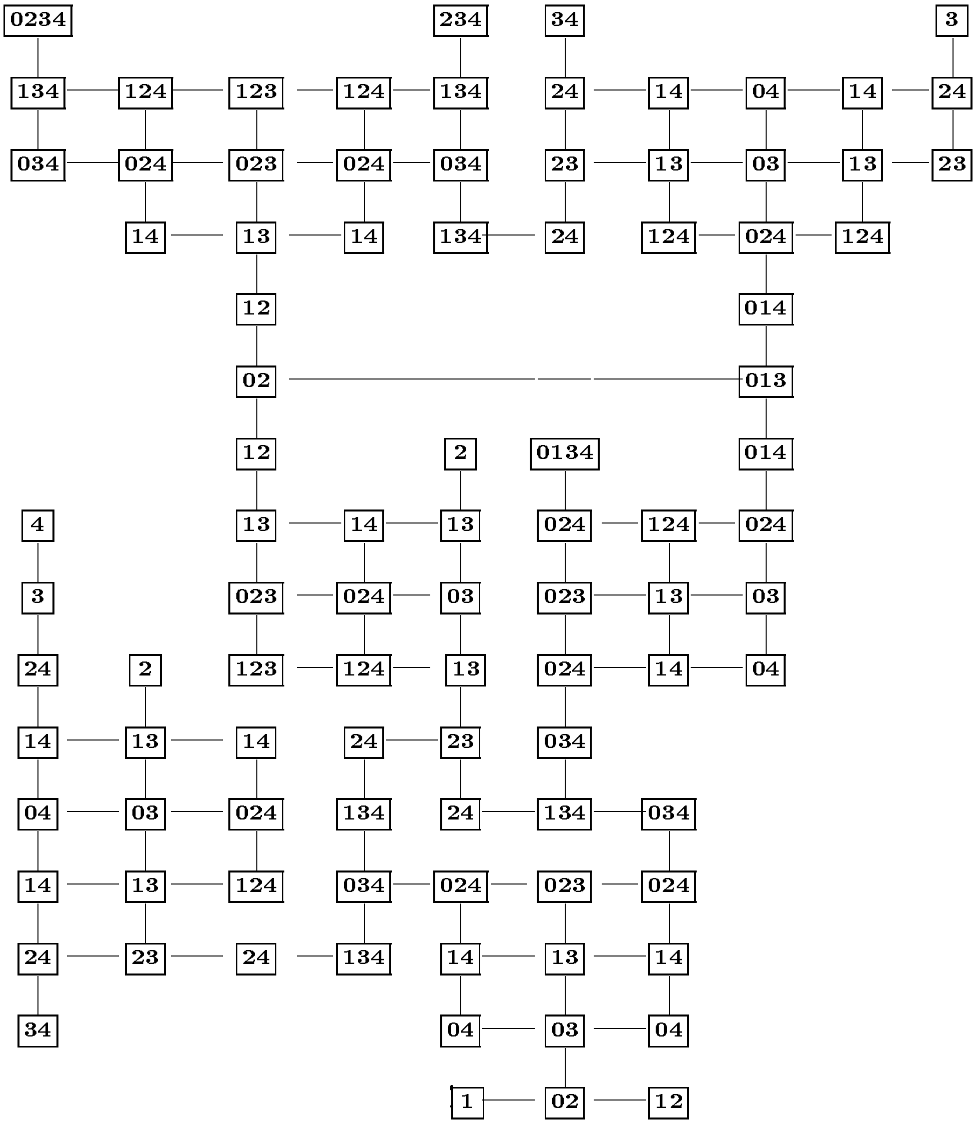


Fig. 17. $\mathfrak{M}_L(w_{0234})$ or $\mathfrak{M}_L(\mathbf{1234321}\cdot w_{0234})$

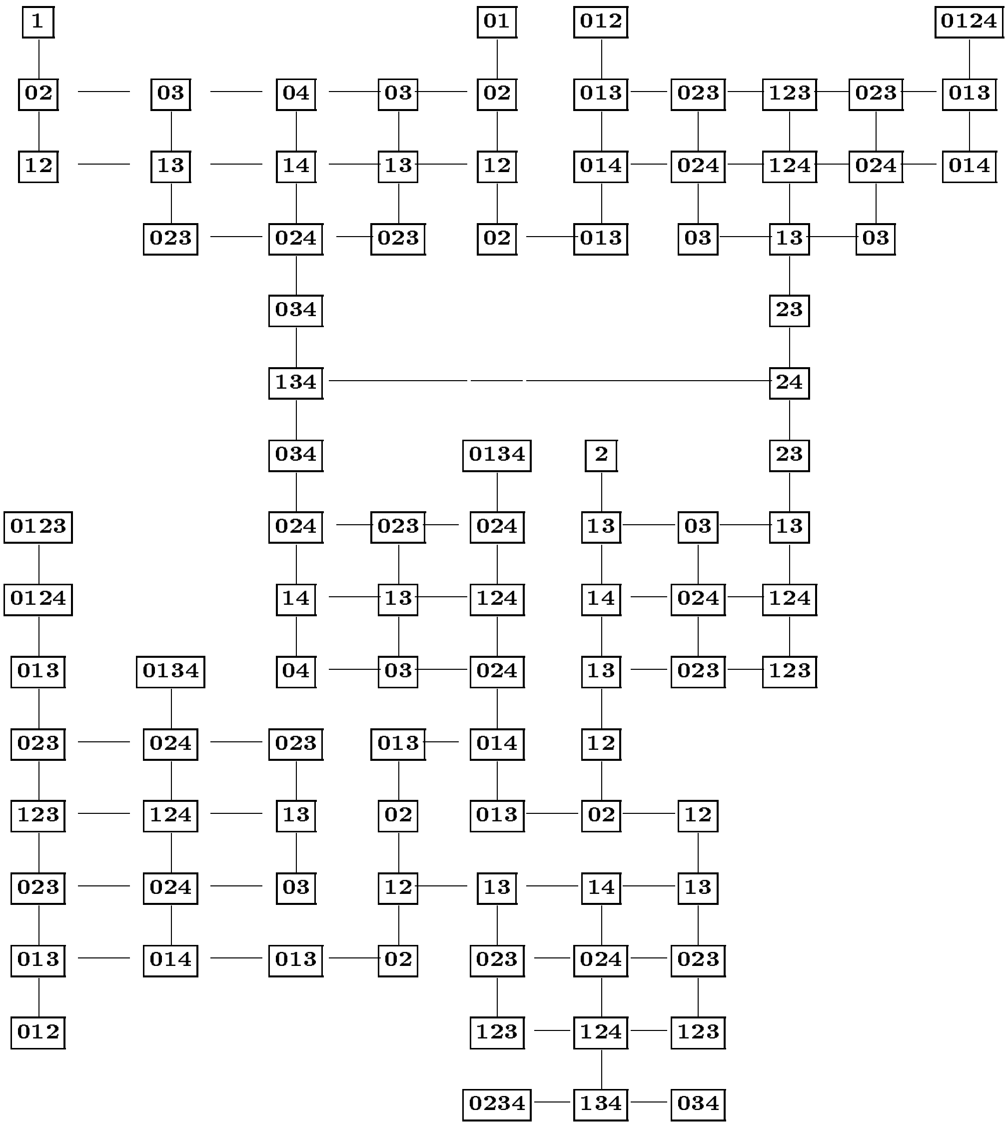


Fig. 18. $\mathfrak{M}_L(w_{0123})$

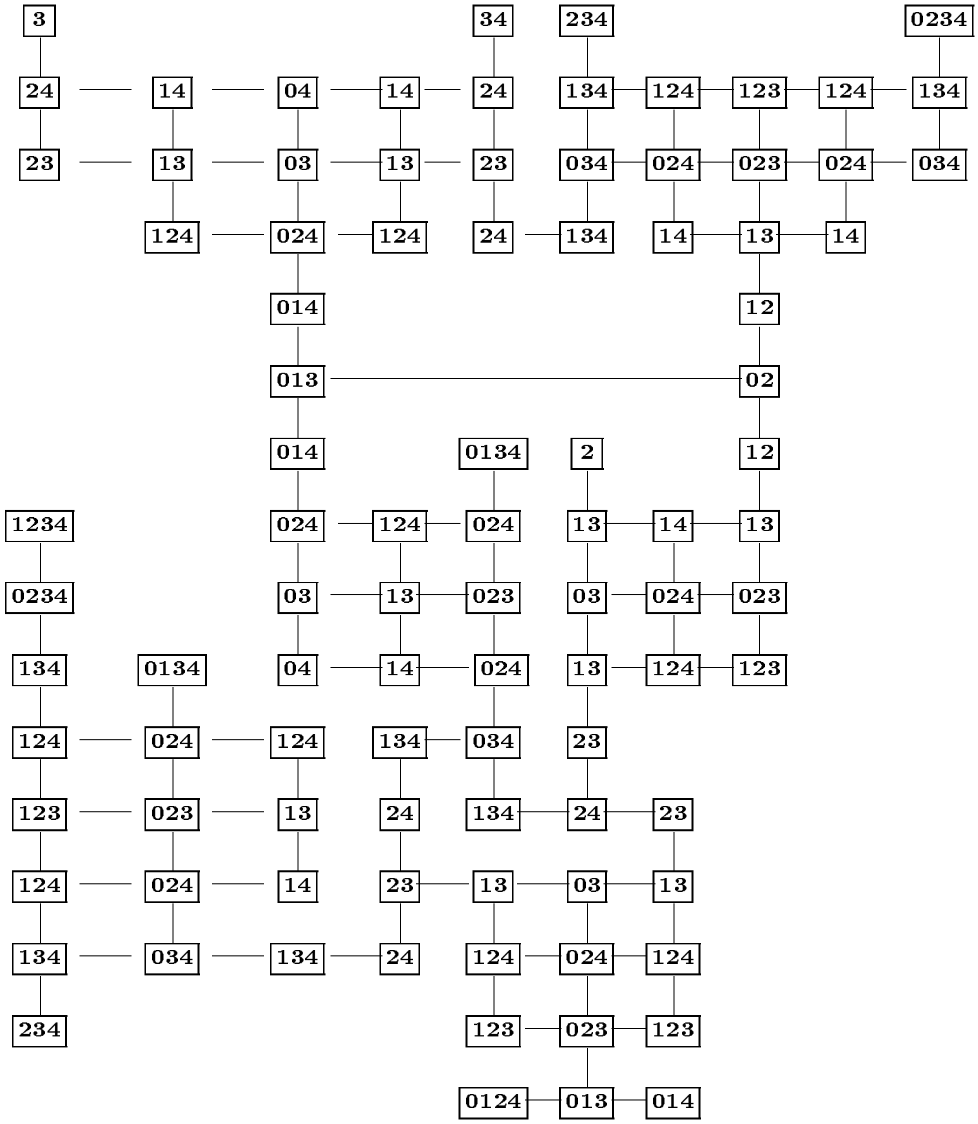


Fig. 19. $\mathfrak{M}_L(w_{1234})$

§5. Some comments.

In this section, we shall make some observations and comments on the left cells of $W_a(\widetilde{C}_k)$, $k = 2, 3, 4$ by the results we have got so far. Based on these, we shall further consider the possible generalization to the more general affine Weyl groups.

5.1 Let us start with recalling some concepts and known results. A partition of $\ell \in \mathbb{N}$ is by definition a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ with $\sum_{i=1}^{r} \lambda_i = \ell$. We shall not distinguish between two such sequences which differ only by a string of zeros at the end. Let $\Lambda_{2\ell+1}$ be the set of all partitions of $2\ell+1$. Let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_r)$, $\mu=(\mu_1,\mu_2,\cdots,\mu_t)$ be in $\Lambda_{2\ell+1}$. We say that μ is dual to λ if for any $i\geq 1$, μ_i is the number of parts λ_j, $1\leq j\leq r$, with $\lambda_j\geq i$. We say that λ dominates μ, written $\lambda\geq\mu$, if $\sum_{i=1}^k\lambda_i\geq\sum_{i=1}^k\mu_i$ for any $k\geq 1$. This defines a partial order, called the natural order, on the set $\Lambda_{2\ell+1}$. Let $\overline{\Lambda}_{2\ell+1}$ be the set of all partitions in $\Lambda_{2\ell+1}$ each of whose even parts occurs with even multiplicity. Let $G(B_\ell)$ be the complex adjoint algebraic group of type B_ℓ, $\ell\geq 2$. Then it is well known that the unipotent conjugacy classes of $G(B_\ell)$ are parametrized by elements of $\overline{\Lambda}_{2\ell+1}$ such that if $\mathbf{u}_\lambda$, $\mathbf{u}_\mu$ are two unipotent classes of $G(B_\ell)$ parametrized by λ, μ, respectively, then $\mathbf{u}_\lambda\subseteq\overline{\mathbf{u}}_\mu$ if and only if $\lambda\leq\mu$, where $\overline{\mathbf{u}}_\mu$ is the closure of $\mathbf{u}_\mu$ in the variety of unipotent elements of $G(B_\ell)$ (see [3, Chapter 13]).

5.2 For $\lambda\in\overline{\Lambda}_9$, let $\mathbf{u}_\lambda$ be the corresponding unipotent conjugacy class of $G(B_4)$. Let $\Omega_\lambda=c(\mathbf{u}_\lambda)$ be the two-sided cell of $W_a=W_a(\widetilde{C}_4)$ associated to $\mathbf{u}_\lambda$ under the Lusztig map in Theorem 1.8 (see [14]). We denote by n_λ the number of left cells of W_a contained in Ω_λ, and by $A(\lambda)=C(u)/C(u)^\circ$ the component group of the centralizer of an element $u\in\mathbf{u}_\lambda$, the latter makes sense since it is independent of the choice of u up to isomorphism. Then by the result of §4 and by Theorem 1.8, we have the following table.

λ	n_λ	Ω_λ	$A(\lambda)$	λ	n_λ	Ω_λ	$A(\lambda)$
(9)	1	$W_{(0)}$	1	(3^21^3)	56	$W_{(5)}$	S_2
(71^2)	5	$W_{(1)}$	S_2	(32^21^2)	72	$W_{(6)}$	S_2
(531)	11	$W_{(2)}$	S_2^2	(2^41)	96	$W_{(8)}$	1
(52^2)	24	$\Omega(w_{12})$	1	(31^6)	144	$W_{(9)}$	S_2
(4^21)	16	$\Omega(w_{024})$	1	(2^21^5)	192	$W_{(10)}$	1
(51^4)	32	$\Omega(w_{01})$	S_2	(1^9)	384	$W_{(16)}$	1
(3^3)	32	$\Omega(w_{023})$	1				

Here the notation (32^21^2) (for example) in the table stands for a partition with five parts 3,2,2,1,1. Thus the total number of left cells in $W_a(\widetilde{C}_4)$ is 1065.

5.3 Let G be a simple algebraic group of adjoint type over $\mathbb{C}$. According to Bala-Carter Theorem, there is a bijective map between unipotent conjugacy classes of

G and G-classes of pairs $(L, P_{L'})$, where L is a Levi subgroup of G and $P_{L'}$ is a distinguished parabolic subgroup of the semisimple part L' of L. The unipotent class corresponding to the pair $(L, P_{L'})$ contains the dense orbit of $P_{L'}$ on its unipotent radical (see [3, Theorem 5.9.6]).

For $\lambda \in \overline{\Lambda}_{2\ell+1}$, let $(L, P_{L'})$ be the pair associated to the unipotent conjugacy class $\mathbf{u}_\lambda$ of $G(B_\ell)$ and let W_L be the Weyl group of L'. Then from the existing datum, we assert that for any $\lambda \in \overline{\Lambda}_{2\ell+1}$ with $\ell = 2, 3, 4$ or $a(\Omega_\lambda) \leq 4$, the number n_λ of left cells of $W_a(\widetilde{C}_\ell)$ in the two-sided cell Ω_λ is equal to $|W_0|/|W_L|$ if and only if $A(\lambda)$ is trivial (see 5.2, §6 and [11; 4; 5; 16]). In fact, the parts of any $\lambda \in \overline{\Lambda}_{2\ell+1}$ can be listed in the following way: $\alpha_1,\ \alpha_1,\ \alpha_2,\ \alpha_2,\ \cdots,\ \alpha_r,\ \alpha_r,\ 2\beta_1+1,\ 2\beta_2+1,\ \cdots,\ 2\beta_t+1$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r > 0$ and $\beta_1 > \beta_2 > \cdots > \beta_t \geq 0$. In the associated pair $(L, P_{L'})$ to $\mathbf{u}_\lambda$, we know that L' (and hence W_L) has type $A_{\alpha_1-1} + \cdots + A_{\alpha_r-1} + B_{\sum \beta_i + [t/2]}$, where $[t/2]$ is the largest integer not exceeding $t/2$. We know that the group $A(\lambda)$ is trivial if and only if λ has exactly one distinct odd part (see [3, Chapter 13]). Thus the above assertion can be checked easily from the existing datum. I conjecture that this result holds also for any two-sided cell of $W_a(\widetilde{C}_\ell)$, $\ell \geq 2$.

Again by the existing datum, we observe that the analogous result also holds for the affine Weyl groups of types $\widetilde{A}_\ell$ $(\ell \geq 1)$, $\widetilde{B}_3$, $\widetilde{B}_4$, $\widetilde{F}_4$ and for all the two-sided cells Ω of the affine Weyl groups of type $\widetilde{B}_\ell$ $(\ell \geq 2)$ with $a(\Omega) \leq 4$. But it is false for the affine Weyl groups of types $\widetilde{D}_\ell$ $(\ell \geq 4)$, $\widetilde{E}_7$, $\widetilde{E}_8$ (see [17; 9; 27; 25; 4; 16; 6; 7; 23]).

Note that the above phenomena were predicted implicitly by Lusztig in his conjecture on the number of left cells in a two-sided cell of an affine Weyl group (see [1]).

5.4 It has been shown in [24] that the Lusztig map $\mathbf{u} \longrightarrow c(\mathbf{u})$ from the set of unipotent conjugacy classes of $G(B_k)$ $(k = 2, 3, 4)$ to the set of two-sided cells of $W_a(\widetilde{C}_k)$ is order-preserving: $\mathbf{u} \subseteq \overline{\mathbf{u}'}$ if and only if $c(\mathbf{u}) \underset{\mathrm{LR}}{\leq} c(\mathbf{u}')$ (see 1.8, the tables in 5.2 and in §6). For a two-sided cell Ω of W_a, let $T(\Omega)$ be the set of all subsets I of S such that $I = \mathcal{L}(w)$ for some $w \in \Omega$. Then we have the following fact in the groups $W_a(\widetilde{C}_k)$ $(k = 2, 3, 4)$: two two-sided cells $\Omega,\ \Omega' \neq \{e\}$ satisfy the relation $\Omega \underset{\mathrm{LR}}{\leq} \Omega'$ if and only if $T(\Omega) \supseteq T(\Omega')$. I conjecture that this result holds in any affine Weyl group. We can say even more for the groups $W_a(\widetilde{C}_k)$, $k = 2, 3, 4$. Let us introduce some more notations.

Let $S = \{s_0, s_1, \cdots, s_\ell\}$ be the Coxeter generator set of the group $W_a(\widetilde{C}_\ell)$ whose indices are compatible with the corresponding extended Dynkin diagram (which is obtained from the Dynkin diagram in 1.5 by adding one more node labelled by 0 to the diagram which joins the node labelled by 1 by a double arrowed edge). For any $J \subset S$, a decomposition $J = J_1 \cup J_2 \cup \cdots \cup J_t$ is called *standard*, if the following conditions are satisfied.

(a) Each J_i, $1 \le i \le t$, is not empty and the corresponding Dynkin subdiagram is connected.

(b) For any pair i, j, $1 \le i < j \le t$, we have $J_i \cap J_j = \emptyset$ and the Dynkin subdiagram corresponding to $J_i \cup J_j$ is not connected.

(c) $|J_1| \ge |J_2| \ge \cdots \ge |J_t|$.

where we stipulate that a standard decomposition of an empty set is the trivial one. Now assume that $J = J_1 \cup J_2 \cup \cdots \cup J_t$ is a standard decomposition of a proper subset $J \subset S$. For any i with $J_i \cap \{s_0, s_\ell\} = \emptyset$, we are given two equal integers $a_{i1} = a_{i2} = |J_i| + 1$. On the other hand, suppose $J_j \cap \{s_0, s_\ell\} \ne \emptyset$ for some j. Then we are given an integer a_j equal to $2|J_j| + 1$ if $J_k \cap \{s_0, s_\ell\} = \emptyset$ for any $k < j$, and equal to $2|J_j|$ if otherwise. We denote by $\zeta(J)$ the dual partition of $(\lambda_1, \lambda_2, \cdots, \lambda_r) \in \Lambda_{2\ell+1}$, where the sequence $\lambda_1, \lambda_2, \cdots, \lambda_r$ is obtained by arranging all the numbers a_{i1}, a_{i2}, a_j, $\forall i, j$, in decreasing order, and adding some parts 1's at the end whenever it is necessary. It is clear that $\zeta(J)$ does not depend on the chosen standard decomposition of J and so it is well defined. Thus we have defined a map ζ from the set of all proper subsets of S to the set $\Lambda_{2\ell+1}$. It can be shown that for any subset K of $\Lambda_{2\ell+1}$, there exists a unique element of $\overline{\Lambda}_{2\ell+1}$ dominated by all the elements in K and being maximum with this property. Then the following result can be checked directly from the existing datum.

Proposition. *Let Ω be a two-sided cell of the group $W_a(\widetilde{C}_k)$, $k = 2, 3, 4$. Let $\lambda(\Omega)$ be the unique maximal element of $\overline{\Lambda}_{2k+1}$ dominated by all the elements in the set $\zeta(T(\Omega)) \subset \Lambda_{2k+1}$. Then the partition $\lambda(\Omega)$ parametrizes the unipotent class of the* *algebraic group G of type B_k corresponding to Ω under the Lusztig map.*

This proposition gives an explicit combinatorial description of the Lusztig map for the groups $W_a(\widetilde{C}_k)$, $k = 2, 3, 4$. I conjecture that it remains valid for all the groups $W_a(\widetilde{C}_\ell)$, $\ell \geq 2$.

5.5 Proposition 5.4 encourages us to propose one more conjecture which is concerned with the relation $\underset{\mathrm{LR}}{\sim}$ on elements of $W_a(\widetilde{C}_\ell)$. For any $w \in W_a = W_a(\widetilde{C}_\ell)$, $\ell \geq 2$, we define a set $T(w) = \{J \subset S \mid w = x \cdot w_J \cdot y, \text{ for some } x, y \in W_a\}$. Let $\lambda(w)$ be the unique maximal element of $\overline{\Lambda}_{2\ell+1}$ dominated by all the elements in the set $\zeta(T(w)) \subset \Lambda_{2\ell+1}$.

Conjecture. *Let $x, y \in W_a(\widetilde{C}_\ell)$. Then $x \underset{\mathrm{LR}}{\sim} y$ if and only if $\lambda(x) = \lambda(y)$.*

5.6 From the left cell graphs displayed in 4.14, we see that any left cell Γ of $W_a(\widetilde{C}_4)$ is determined uniquely by its generalized τ-invariant (see 3.9) except for the cases that there is some (and hence any) element $x \in \Gamma$ such that the graph $\mathfrak{M}_L(x)$ is that in Fig. 16 or 17. But in these exceptional cases, we have $a(\Gamma) \in \{10, 16\}$. Then the alcove form of any element of Γ could tell us the actual value $a(\Gamma)$: it is equal to 16 if no entry of the alcove form is zero, or 10 if otherwise (see 17, (7)), and hence the left cell Γ is determined uniquely. Therefore we have the following

Theorem. *Let Γ be a left cell of the affine Weyl group $W_a(\widetilde{C}_4)$. If Γ is not in the lowest two-sided cell, then it is determined entirely by its generalized τ-invariant. On the other hand, if Γ is in the lowest two-sided cell, then it is determined by the corresponding sign type.*

An analogous result also holds for the groups $W_a(\widetilde{C}_2)$ and $W_a(\widetilde{C}_3)$ (see §6.). One might expect that this also hold for all the affine Weyl groups $W_a(\widetilde{C}_\ell)$, $\ell \geq 2$.

5.7 By closely observing all its left cell graphs, we have the following result for the group $W_a(\widetilde{C}_4)$.

Proposition. *For any $w \in W_a(\widetilde{C}_4)$ not in the lowest two-sided cell, there is a vertex v with $I = \mathcal{R}(v)$ in the graph $\mathfrak{M}(w)$ such that $\ell(w_I) = a(w)$.*

This result, together with the result in 1.7, (7) concerning the lowest two-sided cell, make it easier to determine the a-value of any element of the group $W_a(\widetilde{C}_4)$. Also, by [21, Proposition 5.12], this result enables us to find all the distinguished involutions of $W_a(\widetilde{C}_4)$ only by successively applying star operations on the elements w_J, $J \subset S$ (see [13; 17] for the definitions of a distinguished involution and a star operation). It is known that the distinguished involutions play an important role in the representation theory of Coxeter groups and Hecke algebras and that finding these elements usually involves very complicated computation of Kazhdan-Lusztig polynomials. By the results of Lusztig and Bédard (see §6. or [12; 2]), we see that the above proposition remains valid if we replace the group $W_a(\widetilde{C}_4)$ by $W_a(\widetilde{C}_k)$, $k = 2$ or 3. Thus one might further expect the validity of this result for all the groups $W_a(\widetilde{C}_\ell)$, $\ell \geq 2$. Unfortunately, this is not the case. By Theorem 1.8, we see that for any $\ell \geq 5$, there always exists some element $w \in W_a(\widetilde{C}_\ell)$ such that $a(w) \neq \ell(w_I)$ for any $I \subset S$ and hence for such an element w, the conclusion of the above proposition should be false.

5.8 Let ψ be the unique non-trivial automorphism of the affine Weyl group $W_a(\widetilde{C}_\ell)$ ($\ell \geq 2$) which preserves the Coxeter generator set S (see 4.2). Let $\mathfrak{M}, \mathfrak{N}$ be two graphs with M, N the corresponding vertex sets (in the sense of 3.6). We say that the graph $\mathfrak{N}$ is *opposed* (resp. *dual*) to $\mathfrak{M}$, if there is a bijective map ϕ from the set M to N satisfying that for any $x, y \in M$,

(a) $\mathcal{R}(\phi(x)) = S \setminus \mathcal{R}(x)$ (resp. $\mathcal{R}(\phi(x)) = \psi(\mathcal{R}(x))$).

(b) $\{x, y\}$ is an edge of $\mathfrak{M}$ if and only if $\{\phi(x), \phi(y)\}$ is an edge of $\mathfrak{N}$.

It is easily seen that if $W_{(i)} \subset W_a(\widetilde{C}_\ell)$ consists of a single two-sided cell and has only one left cell graph, say $\mathfrak{M}_L$, then $\mathfrak{M}_L$ must be self-dual. This is the case for the sets $W_{(i)} \subset W_a(\widetilde{C}_4)$ with $i \in \{0, 1, 2, 6, 8\}$ (see Fig.s 1, 2, 3, 12, 13). The other self-dual left cell graphs of $W_a(\widetilde{C}_4)$ are in Fig.s 4 and 5. All the remaining displayed left cell graphs in 4.14 fall into six mutual dual pairs: $\{\mathfrak{M}_L(w_{023}), \mathfrak{M}_L(w_{124})\}$, $\{\mathfrak{M}_L(w_{01}), \mathfrak{M}_L(w_{34})\}$, $\{\mathfrak{M}_L(w_{013}), \mathfrak{M}_L(w_{134})\}$, $\{\mathfrak{M}_L(w_{012}), \mathfrak{M}_L(w_{234})\}$, { Fig. 16, Fig. 17 } and { Fig. 18, Fig. 19 }.

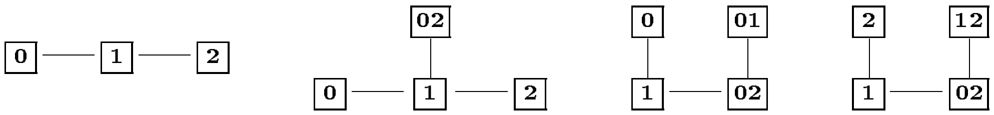
There is no self-opposed left cell graph in $W_a(\widetilde{C}_4)$. But there are just two pairs of mutual opposed left cell graphs: { Fig. 16, Fig. 19} and { Fig. 17, Fig. 18 }, where all the left cell graphs in the lowest two-sided cell of $W_a(\widetilde{C}_4)$ are involved (The last statement should hold for any affine Weyl group).

5.9 We see that the automorphism ψ stabilizes each two-sided cell of $W_a(\widetilde{C}_4)$. This also holds for $W_a(\widetilde{C}_2)$ and $W_a(\widetilde{C}_3)$ (see §6.). As a consequence of the conjectures in 5.4 and 5.5, I expect that this is still the case for the groups $W_a(\widetilde{C}_\ell)$, $\ell \geq 5$.

§6. Appendix.

Here we list all the left cell graphs, and some related results due to Lusztig and Bedard [2; 12] for the affine Weyl groups $W_a(\widetilde{C}_2)$ and $W_a(\widetilde{C}_3)$ (we omit the graph $\mathfrak{M}_L(e)$ in each case since it is too trivial). Keep the notations as before. The indices of the related simple reflections are compatible with the corresponding extended Dynkin diagrams (see 1.5 and 5.4).

6.1 There are four two-sided cells in the group $W_a(\widetilde{C}_2)$, which are $W_{(i)}$, $i = 0, 1, 2, 4$. The non-travial left cell graphs of $W_a(\widetilde{C}_2)$ are as below.



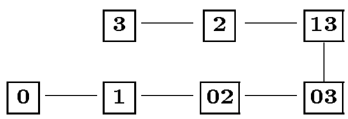
$\mathfrak{M}_L(s_0)$ $\mathfrak{M}_L(w_{02})$ $\mathfrak{M}_L(w_{01})$ $\mathfrak{M}_L(w_{12})$

We have the following table.

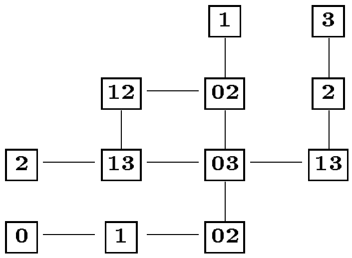
λ	n_λ	Ω_λ	$A(\lambda)$	λ	n_λ	Ω_λ	$A(\lambda)$
(5)	1	$W_{(0)}$	1	$(2^2 1)$	4	$W_{(2)}$	1
(31^2)	3	$W_{(1)}$	S_2	(1^5)	8	$W_{(4)}$	1

So there are sixteen left cells in $W_a(\widetilde{C}_2)$.

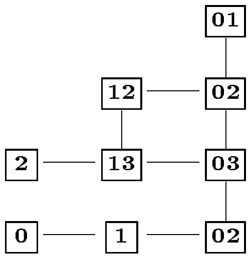
6.2 There are seven two-sided cells in the group $W_a(\widetilde{C}_3)$, which are $W_{(i)}$, $i = 0, 1, 2, 3, 4, 5, 9$. The non-trivial left cell graphs of $W_a(\widetilde{C}_3)$ are listed as follows.



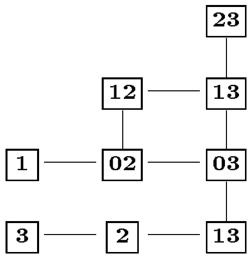
$\mathfrak{M}_L(s_0)$ $\mathfrak{M}_L(w_{02})$



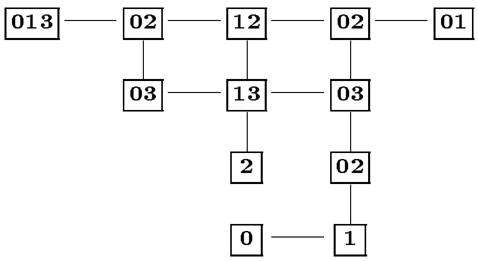
$\mathfrak{M}_L(w_{12})$



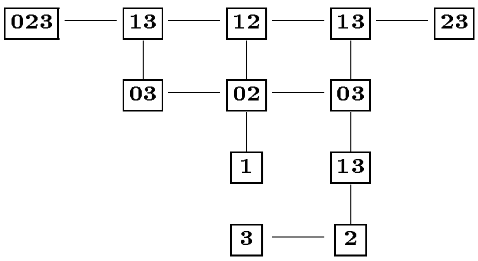
$\mathfrak{M}_L(w_{01})$



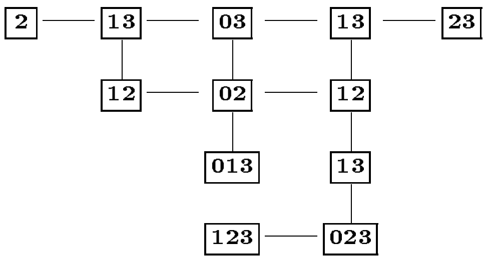
$\mathfrak{M}_L(w_{23})$



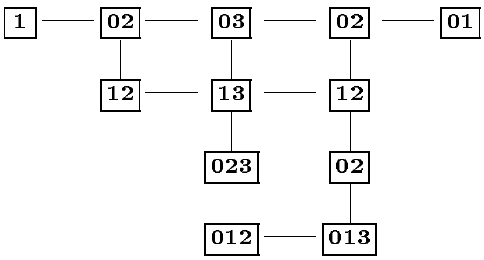
$\mathfrak{M}_L(w_{013})$ or $\mathfrak{M}_L(\mathbf{21012} \cdot w_{013})$



$\mathfrak{M}_L(w_{023})$ or $\mathfrak{M}_L(\mathbf{12321} \cdot w_{023})$



$\mathfrak{M}_L(w_{123})$



$\mathfrak{M}_L(w_{012})$

We have the following table.

λ	n_λ	Ω_λ	$A(\lambda)$	λ	n_λ	Ω_λ	$A(\lambda)$
(7)	1	$W_{(0)}$	1	(31^4)	18	$W_{(4)}$	S_2
(51^2)	4	$W_{(1)}$	S_2	$(2^2 1^3)$	24	$W_{(5)}$	1
$(3^2 1)$	7	$W_{(2)}$	S_2	(1^7)	48	$W_{(9)}$	1
(32^2)	12	$W_{(3)}$	1				

Thus the number of left cells of $W_a(\widetilde{C}_3)$ is 114.

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