EXPLICIT IRREDUCIBLE REPRESENTATIONS OF THE IWAHORI-HECKE ALGEBRA OF TYPE F_4

ARUN RAM AND D. E. TAYLOR

Abstract. A general method for computing irreducible representations of Weyl groups and Iwahori-Hecke algebras was introduced by the first author in [8]. In that paper the representations of the algebras of types A_n, B_n, D_n and G_2 were computed and it is the purpose of this paper to extend these computations to F_4. The main goal here is to compute irreducible representations of the Iwahori-Hecke algebra of type F_4 by only using information in the character table of the Weyl group.

1. Introduction

In his thesis [6] P.N. Hoefsmit wrote down explicit irreducible representations of the Iwahori-Hecke algebras HA_{n-1}, HB_n, and HD_n, of types A_{n-1}, B_n, and D_n, respectively. Hoefsmit's thesis was never published and H. Wenzl [9] independently discovered these representations in the type A_{n-1} case. The irreducible representations of Hoefsmit are analogues of the "seminormal" representations of the Weyl groups of types A_{n-1}, B_n and D_n which were written down by A. Young [10]. The Iwahori-Hecke algebras depend on parameters p and q and one can recover the representations of Young by setting p and q equal to 1 in Hoefsmit's representations.

In this paper we shall extend Hoefsmit's result and determine explicit realizations of all the irreducible representations of the Iwahori-Hecke algebra HF_4. The matrix entries of these representations are well defined when $p = q = 1$ and, when one sets $p = q = 1$, our representations specialize to give explicit realizations of all the irreducible representations of the Weyl group of type F_4. The final results are tabulated in the last section of this paper. Our numbering scheme for the irreducible characters of HF_4 follows Geck [5].

In analogy with Hoefsmit, our representations of HF_4 are in "seminormal" form with respect to the chain of subalgebras

$$HF_4 \supseteq HB_3 \supseteq HA_2 \supseteq HA_1,$$

which means that we choose the irreducible representations φ^k of HF_4 so that, for all $h \in HB_3$, the matrices $\varphi^k(h)$ are block diagonal matrices where the blocks $\varphi^\mu(h)$ are determined by the irreducible representations φ^μ of HB_3. Similarly, for all $h \in HA_2$, the matrices $\varphi^k(h)$ are block diagonal where the blocks are determined by the irreducible

1991 *Mathematics Subject Classification.* 20F55, 20F28.

The research of the first author was supported in part by an Australian Research Council Research Fellowship, National Science Foundation grant DMS-9622985, and the Mathematical Sciences Research Institute.

representations $\varphi^\lambda(h)$ of HA_2. In this way we construct the irreducible representations of HF_4 inductively, by using the branching rules for restricting representations from HF_4 to HB_3 and from HB_3 to HA_2. These branching rules can be calculated easily from the character tables of the corresponding Weyl groups.

The matrix entries of our representations are rational functions in the variables p and q. These rational functions are quotients of polynomials in $\mathbb{Z}[p,q,p^{-1},q^{-1}]$ and the denominators contain only the polynomials

$$[2]_p,\ \ [2]_q,\ \ [2]_{pq},\ \ [2]_{pq^{-1}},\ \ [2]_{p^2q},\ \ [2]_{p^2q^{-1}},\ \ [3]_p,\quad \text{and}\quad [3]_q, \tag{1.1}$$

where $[2]_x = x + x^{-1}$ and $[3]_x = x^2+1+x^{-2}$. This means that our representations are well defined over any field F such that $p, q \in F$ and none of the polynomials in (1.1) are equal to 0.

The research in this paper was begun while the first author was visiting Sydney University on an Australian Research Council Research fellowship. Arun Ram thanks Sydney University and especially G. Lehrer for their wonderful hospitality during the year that he spent in Sydney. The first author is also grateful for support from a postdoctoral fellowship at the Mathematical Sciences Research Institute in Berkeley where the writing of this paper was completed.

2. Preliminaries

Let p and q be indeterminates. The Iwahori-Hecke algebra HF_4 is the associative algebra with 1 over the field $\mathbb{C}(p,q)$ generated by T_1, T_2, T_3, T_4 with relations

$$\begin{aligned}
T_1T_2T_1 &= T_2T_1T_2\\
T_3T_4T_3 &= T_4T_3T_4\\
T_2T_3T_2T_3 &= T_3T_2T_3T_2\\
T_iT_j &= T_jT_i, \qquad \text{if } j \neq i \pm 1,\\
T_i^2 &= (p-p^{-1})T_i + 1, \quad \text{for } i = 1,2,\\
T_i^2 &= (q-q^{-1})T_i + 1, \quad \text{for } i = 3,4.
\end{aligned}$$

This is the Iwahori-Hecke algebra corresponding to the Weyl group WF_4. The Weyl group WF_4 is generated by s_1, s_2, s_3, s_4 which satisfy the same relations as the T_i except with $p = q = 1$. Let HA_1, HA_2, and HB_3 be the subalgebras of HF_4 such that

$$\begin{aligned}
&HA_1 \quad \text{is generated by } T_1,\\
&HA_2 \quad \text{is generated by } T_1 \text{ and } T_2, \text{ and}\\
&HB_3 \quad \text{is generated by } T_1,\ T_2 \text{ and } T_3.
\end{aligned}$$

These are the Iwahori-Hecke algebras corresponding to the Weyl groups $WA_1 =< s_1 >$, $WA_2 =< s_1, s_2 >$ and $WB_3 =< s_1, s_2, s_3 >$, respectively.

Our goal in this paper is to compute explicit representations of HF_4 using only the information in the character tables of the Weyl groups WA_1, WA_2, WB_3 and WF_4. We shall use the following notations.

(a) d_λ will denote the dimension of the irreducible representation indexed by λ;
(b) χ^λ will denote the character of the irreducible representation of the Weyl group W indexed by λ;
(c) Id_λ will denote the $d_\lambda \times d_\lambda$ identity matrix;
(d) T_w, $w \in W$, will denote the usual basis of the Iwahori-Hecke algebra H given by $T_w = T_{i_1} \cdots T_{i_k}$ if $w = s_{i_1} \cdots s_{i_k}$ is a reduced word for w.
(e) If A and B are matrices then $A \oplus B$ and $A \otimes B$ will denote the standard operations of direct sum and tensor product of matrices.

We shall need the following well known facts:

FACT 1. The irreducible representations of the Iwahori-Hecke algebra are indexed in the same way as the corresponding Weyl group. Thus,

(a) The irreducible representations of HF_4 are indexed by $k \in \{1, 2, \ldots, 25\}$ (in the same manner as in [5] and in the same order as in the table on p. 412 of [2]).
(b) The irreducible representations of HB_3 are indexed by pairs of partitions (α, β) such that $|\alpha| + |\beta| = 3$.
(c) The irreducible representations of HA_2 are indexed by partitions λ of 3.
(d) The irreducible representations of HA_1 are indexed by partitions γ of 2.

FACT 2. [2] §10.11. The dimension of an irreducible Iwahori-Hecke algebra representation is the same as that of the corresponding representation of the Weyl group and the branching rules for Iwahori-Hecke algebras are the same as for the corresponding Weyl groups. Thus the branching rules for the inclusions $HF_4 \supseteq HB_3 \supseteq HA_2$ can be calculated directly from the character tables of the corresponding Weyl groups. We have tabulated these branching rules in Tables 4.2 and 4.3.

FACT 3. [2] §10.9 and [4] (9.21). Let H be an Iwahori-Hecke algebra and let W be the corresponding Weyl group. If λ is an index for an irreducible representation of the Iwahori-Hecke algebra H then the minimal central idempotent corresponding to λ can be written in the form

$$z_\lambda = \sum_{w \in W} z_w^\lambda T_w,$$

where $z_w^\lambda \in \mathbb{C}(p, q)$ are elements which are well defined when $p = q = 1$. Furthermore, at $p = q = 1$,

$$z_\lambda \Big|_{p=q=1} = \frac{\chi^\lambda(1)}{|W|} \sum_{w \in W} \chi^\lambda(w^{-1}) w, \tag{2.1}$$

where χ^λ is the character of the irreducible representation of W indexed by λ.

FACT 4. Let H be an Iwahori-Hecke algebra and let W be the corresponding Weyl group. Let R be the root system corresponding to W and let

$$\begin{aligned} r_s &= \text{a reflection in a short root,} \\ r_l &= \text{a reflection in a long root,} \\ N_s &= \text{the number of positive short roots in } R, \text{ and} \\ N_l &= \text{the number of positive long roots in } R. \end{aligned}$$

If there is only one root length then we declare all roots to be short. For each λ indexing an irreducible representation of H let χ^λ be the character of the corresponding irreducible representation of the Weyl group and define

$$c(\lambda) = \chi^\lambda(w_0) p^{c(\lambda,s)} q^{c(\lambda,\ell)} \tag{2.2}$$

where

$$c(\lambda, s) = \frac{N_s \chi^\lambda(r_s)}{\chi^\lambda(1)} \quad \text{and} \quad c(\lambda, l) = \frac{N_l \chi^\lambda(r_l)}{\chi^\lambda(1)}.$$

Let φ^λ be a realization of the irreducible representation indexed by λ and let Id_λ be the $d_\lambda \times d_\lambda$ identity matrix, where d_λ is the dimension of φ^λ. Then we have the following result [7], [5], [8]:

(a) If w_0 is central in W then $\varphi^\lambda(T_{w_0}) = c(\lambda)\,\mathrm{Id}_\lambda$,
(b) If w_0 is not central in W then $\varphi^\lambda(T_{w_0}^2) = c(\lambda)^2\,\mathrm{Id}_\lambda$.

3. Seminormal representations

We shall compute the irreducible representations of HF_4 inductively: the representations of HA_1 are one dimensional and one can immediately write them down, then we compute irreducible representations of HA_2, then HB_3, and finally HF_4. At each step we use the information from the previous cases since we construct the representations such that upon restriction to any of these subalgebras they are in block diagonal form with diagonal blocks determined by the previous calculations. The irreducible representations of HA_2 are easy to derive and the irreducible representations of HB_3 can be derived in a similar fashion to the way that we complete the calculations for HF_4 below. Thus, in our description below we shall *assume that the irreducible representations of* HA_1, HA_2, *and* HB_3 *are already known* and we shall describe how to obtain the irreducible representations of HF_4. The irreducible "seminormal" representations of HA_1, HA_2, and HB_3 are tabulated in Section 4 below.

Let k be an index for an irreducible representation of HF_4. The branching rule

$$\varphi^k \downarrow_{HB_3} \cong \varphi^{\mu^{(1)}} \oplus \varphi^{\mu^{(2)}} \oplus \cdots \oplus \varphi^{\mu^{(\ell)}}$$

describing the restriction of representations of HF_4 to HB_3 can be computed from the character table of the corresponding Weyl groups. We shall say that the irreducible representation φ^k of HF_4 is in *seminormal form* if

$$\varphi^k(h) = \varphi^{\mu^{(1)}}(h) \oplus \varphi^{\mu^{(2)}}(h) \oplus \cdots \oplus \varphi^{\mu^{(\ell)}}(h), \qquad \text{for all } h \in HB_3. \tag{3.1}$$

We require the two sides of (3.1) to be equal as matrices.

We shall compute irreducible representations of HF_4 which are in seminormal form. Assuming that the irreducible representations of HB_3 are known, the seminormal condition implies that to determine the irreducible representations of HF_4 it is only necessary to determine the matrices $\varphi^k(T_4)$ for each k.

Suppose that φ^k and ψ^k are two solutions to this problem, i.e. φ^k and ψ^k are both realizations of the irreducible representation of HF_4 indexed by k and we have

$$\varphi^k(h) = \psi^k(h) = \varphi^{\mu^{(1)}}(h) \oplus \varphi^{\mu^{(2)}}(h) \oplus \cdots \oplus \varphi^{\mu^{(\ell)}}(h),$$

for all $h \in HB_3$. Then there is a matrix $P \in GL(d_k)$, where d_k is the dimension of φ^k, such that $P\varphi^k(h)P^{-1} = \psi^k(h)$, for all $h \in HF_4$. By Schur's lemma this matrix is unique up to constant multiples. On the other hand we have

$$\begin{aligned} P(\varphi^{\mu^{(1)}}(h) \oplus \varphi^{\mu^{(2)}}(h) \oplus \cdots \oplus \varphi^{\mu^{(\ell)}}(h))P^{-1} &= P\varphi^k(h)P^{-1} = \psi^k(h) \\ &= \varphi^{\mu^{(1)}}(h) \oplus \varphi^{\mu^{(2)}}(h) \oplus \cdots \oplus \varphi^{\mu^{(\ell)}}(h), \end{aligned}$$

for all $h \in HB_3$. By inspection of the table of branching rules from HF_4 to HB_3 one sees that the summands $\varphi^{\mu^{(i)}}$ are all distinct irreducible representations of HB_3. Hence, Schur's lemma implies that

$$P = p_1 \,\mathrm{Id}_{\mu^{(1)}} \oplus p_2 \,\mathrm{Id}_{\mu^{(2)}} \oplus \cdots \oplus p_\ell \,\mathrm{Id}_{\mu^{(\ell)}}, \tag{3.2}$$

where the p_i are nonzero constants. Replacing P by $p_1^{-1}P$ we may suppose that $p_1 = 1$. Conversely, any choice of $p_i \neq 0$, $p_1 = 1$, in the equation (3.2) defines a matrix P such that $P\varphi^k P^{-1}$ is a seminormal representation. Thus we have the following result.

Proposition 3.3. *If φ^k is in seminormal form then the matrix $\varphi^k(T_4)$ is determined up to the choice of $\ell - 1$ free parameters where ℓ is the number of irreducible summands in φ^k on restriction to HB_3.*

Let $w_{0,1}$, $w_{0,2}$, $w_{0,3}$, and $w_{0,4}$ be the longest elements in the Weyl groups WA_1, WA_2, WB_3, and WF_4, respectively. Define elements

$$\begin{aligned} D_1 &= T_{w_{0,1}} = T_1, \\ D_2 &= T_{w_{0,2}}^2 = (T_1T_2T_1)^2, \\ D_3 &= T_{w_{0,3}} = (T_3T_2T_1)^3, \\ D_4 &= T_{w_{0,4}} = (T_4T_{w_{0,3}})^3 T_{w_{0,2}}^{-2}. \end{aligned} \tag{3.4}$$

in HF_4.

Lemma 3.5. *If φ^k is in seminormal form then the matrices $\varphi^k(D_j)$ are uniquely determined, for all $1 \le k \le 25$, $1 \le j \le 4$.*

Proof. This follows from Fact 4. $\square$

The matrices $\varphi^k(D_j)$ are tabulated in 5.4.

Let σ be a permutation matrix such that

$$\begin{aligned}\sigma\varphi^k(h)\sigma^{-1} &= \varphi^{(1^3)}(h) \oplus \cdots \oplus \varphi^{(1^3)}(h) \\ &\qquad\qquad \oplus \quad \varphi^{(21)}(h) \oplus \cdots \oplus \varphi^{(21)}(h) \\ &\qquad\qquad\qquad\qquad \oplus \quad \varphi^{(3)}(h) \oplus \cdots \oplus \varphi^{(3)}(h) \\ &= \bigoplus_{\lambda \vdash 3} \varphi^\lambda(h)^{\oplus m_\lambda}\end{aligned} \tag{3.6}$$

for all $h \in HA_2$. The constant m_λ is the number of times the matrix $\varphi^\lambda(h)$ appears. Since $\sigma\varphi^k(T_4)\sigma^{-1}$ commutes with all of the matrices in (3.6), it follows from Schur's lemma that

$$\begin{aligned}\sigma\varphi^k(T_4)\sigma^{-1} &= (T^k_{(3)} \otimes \mathrm{Id}_{(3)}) \oplus (T^k_{(21)} \otimes \mathrm{Id}_{(21)}) \oplus (T^k_{(1^3)} \otimes \mathrm{Id}_{(1^3)}) \\ &= \bigoplus_{\lambda \vdash 3} T^k_\lambda \otimes \mathrm{Id}_\lambda,\end{aligned}$$

where, for each λ, T^k_λ is an $m_\lambda \times m_\lambda$ matrix and Id_λ is the $d_\lambda \times d_\lambda$ identity matrix. Note that

$$T^k_\lambda \otimes \mathrm{Id}_\lambda = \sigma\varphi^k(z_\lambda T_4)\sigma^{-1}, \tag{3.7}$$

where z_λ is the minimal central idempotent in HA_2 corresponding to λ. We can use the same method to write

$$\sigma\varphi^k(D_3)\sigma^{-1} = \bigoplus_{\lambda \vdash 3} D^k_\lambda \otimes \mathrm{Id}_\lambda,$$

where $D_3 = T_{w_{0,3}}$, as given in (3.4).

To determine the matrices $\varphi^k(T_4)$ it is sufficient to determine the matrices T^k_λ. The matrices D^k_λ are completely determined by Lemma 3.5 and can easily be determined from the tables in 5.4. The relations $(T_4D_3)^3 = D_4D_2^2$ and the relation $T_4^2 = (q - q^{-1})T_4 + 1$ imply that

$$(T^k_\lambda D^k_\lambda)^3 = c(k)c(\lambda)^2\,\mathrm{Id}_{m_\lambda} \quad \text{and} \quad (T^k_\lambda)^2 = (q - q^{-1})T^k_\lambda + \mathrm{Id}_{m_\lambda}, \tag{3.8}$$

where $c(k)$ and $c(\lambda)$ are the constants given in equation (2.2).

3.1. **Determining the diagonal entries of $\varphi^k(T_4)$.** We shall determine the diagonal entries of the matrices the matrices T^k_λ by determining the traces of the matrices

$$T^k_\lambda(D^k_\lambda)^{-2}, \quad T^k_\lambda(D^k_\lambda)^{-1}, \quad T^k_\lambda, \quad T^k_\lambda D^k_\lambda, \quad \text{and} \quad T^k_\lambda(D^k_\lambda)^2.$$

Proposition 3.9. *Fix an index k for an irreducible representation of HF_4 and let λ be an index for an irreducible representation of HA_2. Let T_λ^k, D_λ^k and z_λ be as above and let χ^k and χ^λ be the irreducible characters of the Weyl groups WF_4 and WA_2 which correspond to k and λ, respectively. Let $c(k)$ and $c(\lambda)$ be the constants defined in (2.2). Then*

(a) $$\mathrm{Tr}(T_\lambda^k) = \frac{1}{12}\sum_{w\in WA_2}\chi^\lambda(w^{-1})\left((q-q^{-1})\chi^k(w)+(q+q^{-1})\chi^k(ws_4)\right),$$

(b) $$\mathrm{Tr}(T_\lambda^k D_\lambda^k) = \frac{\chi^k(w_{0,4})c(k)^{\frac{1}{3}}c(\lambda)^{\frac{2}{3}}}{6}\sum_{w\in WA_2}\chi^\lambda(w^{-1})\chi^k(ws_4w_{0,3}),$$

(c) $$\mathrm{Tr}((T_\lambda^k D_\lambda^k)^2) = \frac{c(k)^{\frac{2}{3}}c(\lambda)^{\frac{4}{3}}}{6}\sum_{w\in WA_2}\chi^\lambda(w^{-1})\chi^k(w(s_4w_{0,3})^2).$$

Proof. (a) ¿From the second equation in (3.8) we have that each eigenvalue of T_λ^k is either q or $-q^{-1}$ and consequently $\mathrm{Tr}(T_\lambda^k) = t_1q - t_2q^{-1}$ for some positive integers t_1 and t_2. These constants are determined as follows. Using (3.7) we get that

$$\begin{aligned} t_1 - t_2 = \mathrm{Tr}(T_\lambda^k)\big|_{p=q=1} &= \frac{1}{\chi^\lambda(1)}\mathrm{Tr}(T_\lambda^k\otimes \mathrm{Id}_\lambda)\big|_{p=q=1} \\ &= \frac{1}{\chi^\lambda(1)}\mathrm{Tr}(\sigma\varphi^k(z_\lambda T_4)\sigma^{-1})\big|_{p=q=1} = \frac{1}{\chi^\lambda(1)}\mathrm{Tr}(\varphi^k(z_\lambda T_4))\big|_{p=q=1}. \end{aligned}$$

Then we use (2.1) to obtain

$$t_1 - t_2 = \frac{1}{\chi^\lambda(1)}\chi^k\left(\frac{\chi^\lambda(1)}{6}\sum_{w\in WA_2}\chi^\lambda(w^{-1})ws_4\right) = \frac{1}{6}\sum_{w\in WA_2}\chi^\lambda(w^{-1})\chi^k(ws_4).$$

If Id_λ^k is the identity matrix of the same dimension as T_λ^k then

$$\begin{aligned} t_1 + t_2 = \mathrm{Tr}(\mathrm{Id}_\lambda^k)\big|_{p=q=1} &= \frac{1}{\chi^\lambda(1)}\mathrm{Tr}(\mathrm{Id}_\lambda^k\otimes \mathrm{Id}_\lambda)\big|_{p=q=1} \\ &= \frac{1}{\chi^\lambda(1)}\mathrm{Tr}(\varphi^k(z_\lambda))\big|_{p=q=1} = \frac{1}{6}\sum_{w\in WA_2}\chi^\lambda(w^{-1})\chi^k(w). \end{aligned}$$

These two equations determine t_1 and t_2 and thus $\mathrm{Tr}(T_\lambda^k)$ is determined.

(b) It follows from Fact 4 and the first equation in (3.8) that the eigenvalues of $T_\lambda^k D_\lambda^k$ are all of the form $\omega^i c(\lambda)^{\frac{2}{3}}c(k)^{\frac{1}{3}}$ where ω is a primitive cube root of unity. Hence

$$\mathrm{Tr}(T_\lambda^k D_\lambda^k) = \eta c(\lambda)^{\frac{2}{3}}c(k)^{\frac{1}{3}}$$

for some constant $\eta\in\mathbb{C}$. By setting p and q equal to 1 we have

$$\begin{aligned}\eta\chi^k(w_{0,4}) &= \mathrm{Tr}(T^k_\lambda D^k_\lambda)\big|_{p=q=1} = \frac{1}{\chi^\lambda(1)}\,\mathrm{Tr}(T^k_\lambda D^k_\lambda\otimes \mathrm{Id}_\lambda)\big|_{p=q=1}\\ &= \frac{1}{\chi^\lambda(1)}\,\mathrm{Tr}(\varphi^k(z_\lambda T_4 T_{w_{0,3}}))\big|_{p=q=1}\end{aligned}$$

as in (3.1). Using (2.1) we get

$$\eta\chi^k(w_{0,4}) = \frac{1}{6}\sum_{w\in WA_2}\chi^\lambda(w^{-1})\chi^k(ws_4w_{0,3}).$$

The proof of (c) is similar to that of (b) once one notes that Fact 4 and the first equation in (3.8) imply that the eigenvalues of the matrix $(T^k_\lambda D^k_\lambda)^2$ are all of the form $\omega^{2i}c(\lambda)^{\frac{4}{3}}c(k)^{\frac{2}{3}}$. □

Lemma 3.10. *Given matrices T and D such that $T^2=(q-q^{-1})T+\mathrm{Id}$ and $(TD)^3=c\,\mathrm{Id}$ where c is a constant, we have*

(a) $\mathrm{Tr}(TD^{-1}) = (q-q^{-1})\,\mathrm{Tr}(D^{-1}) + c^{-1}\,\mathrm{Tr}((TD)^2)$.
(b) $\mathrm{Tr}(TD^2) = c\,\mathrm{Tr}(D^{-1}) - (q-q^{-1})\,\mathrm{Tr}((TD)^2)$.
(c) $\mathrm{Tr}(TD^{-2}) = (q-q^{-1})\,\mathrm{Tr}(D^{-2}) + c^{-1}(q-q^{-1})\,\mathrm{Tr}(TD) + c^{-1}\,\mathrm{Tr}(D)$.

Proof. (a) Writing the given equations in the form $T=(q-q^{-1})\,\mathrm{Id}+T^{-1}$ and $(TD)^{-1} = c^{-1}(TD)^2$, we have

$$\begin{aligned}\mathrm{Tr}(TD^{-1}) &= (q-q^{-1})\,\mathrm{Tr}(D^{-1}) + \mathrm{Tr}(T^{-1}D^{-1})\\ &= (q-q^{-1})\,\mathrm{Tr}(D^{-1}) + c^{-1}\,\mathrm{Tr}(TDTD).\end{aligned}$$

(b) Similarly, from the fact that $T^{-2}=\mathrm{Id}-(q-q^{-1})T^{-1}$,

$$\begin{aligned}\mathrm{Tr}(TD^2) = \mathrm{Tr}(DTD) &= c\,\mathrm{Tr}(T^{-1}D^{-1}T^{-1}) = c\,\mathrm{Tr}(T^{-2}D^{-1})\\ &= c\,\mathrm{Tr}(D^{-1}) - c(q-q^{-1})\,\mathrm{Tr}(T^{-1}D^{-1})\\ &= c\,\mathrm{Tr}(D^{-1}) - (q-q^{-1})\,\mathrm{Tr}(TDTD).\end{aligned}$$

(c)
$$\begin{aligned}\mathrm{Tr}(TD^{-2}) &= (q-q^{-1})\,\mathrm{Tr}(D^{-2}) + \mathrm{Tr}(T^{-1}D^{-2})\\ &= (q-q^{-1})\,\mathrm{Tr}(D^{-2}) + \mathrm{Tr}(D^{-1}T^{-1}D^{-1})\\ &= (q-q^{-1})\,\mathrm{Tr}(D^{-2}) + c^{-1}\,\mathrm{Tr}(TDT)\\ &= (q-q^{-1})\,\mathrm{Tr}(D^{-2}) + c^{-1}\,\mathrm{Tr}(T^2D)\\ &= (q-q^{-1})\,\mathrm{Tr}(D^{-2}) + c^{-1}(q-q^{-1})\,\mathrm{Tr}(TD) + c^{-1}\,\mathrm{Tr}(D).\end{aligned}$$

□

Assume that T^k_λ has dimension at most 5 and write $D^k_\lambda = \mathrm{diag}(d_1,d_2,\ldots,d_r)$. The diagonal entries of D^k_λ are determined by Proposition 3.5 and one can check directly that these diagonal entries are always all distinct. Let S be a subset of $\{1,2,\ldots,r\}\backslash\{i\}$ such that S and its complement have at most 2 elements. Then the diagonal entries of T^k_λ are given by

$$(T^k_\lambda)_{ii} = \mathrm{Tr}(TE_{ii}) \quad \text{where,} \quad \text{for each } 1 \le i \le r, \tag{3.11}$$
$$E_{ii} = \Bigg(\prod_{\substack{j\in S\\ j\neq i}} \frac{D^k_\lambda - d_j}{d_i - d_j}\Bigg)\Bigg(\prod_{\substack{j\notin S\\ j\neq i}} \frac{(D^k_\lambda)^{-1} - d_j^{-1}}{d_i^{-1} - d_j^{-1}}\Bigg).$$

These values can be evaluated explicitly by expanding E_{ii} in terms of $(D^k_\lambda)^j$ and using Lemma 3.10 and Proposition 3.9 to evaluate the traces $\mathrm{Tr}(T^k_\lambda(D^k_\lambda)^j)$.

Formula (3.11) suffices for computing the diagonal entries of the matrices T^k_λ, and thus of the matrices $\varphi^k(T_4)$, for all k except $k = 25$. The matrix $T^{25}_{(21)}$ has dimension 6 and formula (3.11) is not applicable. The diagonal entries of the matrix $\varphi^{25}(T_4)$ are computed as follows. Since the matrices $T^{25}_{(1^3)}$ and $T^{25}_{(3)}$ are each of dimension two we use formula (3.11) to determine their diagonal entries. By Lemma 3.10 and Proposition 3.9 we can determine the traces of the matrices

$$T^{25}_{(21)}(D^{25}_{(21)})^{-2}, \quad T^{25}_{(21)}(D^{25}_{(21)})^{-1}, \quad T^{25}_{(21)}, \quad T^{25}_{(21)}D^{25}_{(21)}, \quad T^{25}_{(21)}(D^{25}_{(21)})^{2},$$

and these traces give five linear relations that the diagonal entries of $T^{25}_{(21)}$ must satisfy. Finally, we use the formula

$$0 = \mathrm{Tr}(\varphi^{25}(T_4T_3T_2T_1)) = \sum_i \varphi^{25}(T_4)_{ii}\varphi^{25}(T_3)_{ii}\varphi^{25}(T_2)_{ii}\varphi^{25}(T_1)_{ii}$$

to determine the diagonal entries of $\varphi^{25}(T_4)$ completely. This last formula is a consequence of the following lemma.

Lemma 3.12. (a) $\mathrm{Tr}(\varphi^{25}(T_4T_3T_2T_1)) = 0$.
(b) *The diagonal entries of the matrix* $\varphi^k(T_4T_3T_2T_1)$ *satisfy*

$$\varphi^k(T_4T_3T_2T_1)_{ii} = \varphi^k(T_4)_{ii}\varphi^{25}(T_3)_{ii}\varphi^k(T_2)_{ii}\varphi^{25}(T_1)_{ii}.$$

Proof. (a) Since the Coxeter number for the Weyl group WF_4 is 12 (see [1]) we have that $(T_4T_3T_2T_1)^6 = T_{w_{0,4}}$. Then it follows from Fact 4 that the eigenvalues of the matrix $\varphi^{25}(T_4T_3T_2T_1)$ must be of the form $\omega^j c(25)$ where ω is a primitive 6th root of unity and $c(25)$ is the constant given in (2.2). It follows that $\mathrm{Tr}(\varphi^{25}(T_4T_3T_2T_1)) = \eta c(25)$ for some constant η. Then, from the character table of the Weyl group WF_4, we have

$$\eta = \eta c(25)\big|_{p=q=1} = \mathrm{Tr}(\varphi^{25}(T_4T_3T_2T_1))\big|_{p=q=1} = \chi^{25}(s_4s_3s_2s_1) = 0.$$

(b) Let $h \in HB_3$. Let z_λ, $\lambda \vdash 3$, be the minimal central idempotents in HA_2. Since φ^k is in seminormal form the matrices $\varphi^k(z_\lambda)$ are diagonal matrices with 1's and 0's on the diagonal, their sum is the identity matrix and they are mutually orthogonal. It follows that there is a single partition λ such that

$$\varphi^k(T_4h)_{ii} = \varphi^k(z_\lambda T_4h)_{ii} = \big(\varphi^k(z_\lambda T_4)\varphi^k(z_\lambda h)\big)_{ii}$$

It follows from the seminormal condition and the fact that the branching rules for restricting representations from HF_4 to HB_3 are multiplicity free that the matrix $\varphi^k(z_\lambda h)$ is a diagonal matrix. Thus the diagonal entries of the matrix $\varphi^k(T_4 h)$ satisfy

$$\varphi^k(T_4h)_{ii} = \varphi^k(T_4)_{ii}\varphi^k(h)_{ii},$$

for all $h \in HB_3$. Since the irreducible representations of HB_3 and HA_2 that we are using are also chosen to be in seminormal form, their representations also satisfy a similar identity. The result then follows by induction. □

3.2. Computing the off-diagonal entries of $\varphi^k(T_4)$.

Proposition 3.13. *Let $T = (t_{ij})$ and $D = \operatorname{diag}(d_1, d_2, \ldots, d_r)$ be $r \times r$ matrices such that the diagonal entries of D are distinct, none of the entries t_{ij} of T are 0, and*

$$T^2 = (q - q^{-1})T + \mathrm{Id} \quad \textit{and} \quad (TD)^3 = c\,\mathrm{Id},$$

for some constant c. For distinct indices i, j, k define $u_{ij} = u_{ji} = t_{ij}t_{ji}$ and $v_{ijk} = t_{ij}t_{jk}/t_{ik}$. Then

(a) $$\sum_{j\neq i} u_{ij} = -t_{ii}^2 + (q - q^{-1})t_{ii} + 1,$$

(b) $$\sum_{j\neq i} u_{ij}d_j = -t_{ii}^2 d_i + ct_{ii}d_i^{-2} - c(q - q^{-1})d_i^{-2},$$

(c) $$\sum_{j\neq i,k} v_{ijk} = -t_{ii} - t_{kk} + (q - q^{-1}), \quad \textit{for } i \neq k,$$

(d) $$\sum_{j\neq i,k} v_{ijk}d_j = -t_{ii}d_i - t_{kk}d_k + cd_i^{-1}d_k^{-1}, \quad \textit{for } i \neq k.$$

Proof. Equations (a) and (b) are obtained by comparing the (i, i) entries on each side of the matrix equations $T^2 = (q - q^{-1})T + q$ and $TDT = cD^{-1}T^{-1}D^{-1}$. Equations (c) and (d) are obtained by comparing the (i, k) entries. □

The equations

$$v_{ijk} = \frac{v_{1ij}v_{1jk}v_{1ki}}{u_{ik}}, \quad v_{1ji} = \frac{u_{ij}}{v_{1ij}}, \quad t_{ij} = \frac{t_{1j}v_{1ij}}{t_{1i}}, \tag{3.14}$$

imply that all of the values in Proposition 3.13 are determined once we know u_{ij} and v_{1ij} for $i < j$ and t_{1i} for $1 < i \leq r$.

In view of the relations (3.8) we may apply Proposition 3.13 to the matrices T_λ^k and D_λ^k. Equations (a) and (b) of Proposition 3.13 give

4 equations in the single variable u_{12} when $\dim(T_\lambda^k) = 2$,

6 equations in the 3 variables u_{ij} when $\dim(T_\lambda^k) = 3$,

8 equations in the 6 variables u_{ij} when $\dim(T_\lambda^k) = 4$,

12 equations in the 15 variables u_{ij} when $\dim(T_\lambda^k) = 6$.

These equations are sufficient to determine the products $u_{ij} = t_{ij}t_{ji}$ for all T^k_λ except $T^{25}_{(21)}$ where we have $\dim(T^{25}_{(21)}) = 6$.

If $\dim(T^k_\lambda) = 2, 3$, or 4 we use the linear equations in (a) and (b) of Proposition 3.13 to solve for the u_{ij}. Then we use the equations in (3.14) to write the equations in (c) and (d) of Proposition 3.13 in terms of the variables t_{1i}, and v_{1ij}, $i < j$. After doing this we are able to use the subset of the resulting equations which are linear in the v_{1ij} to uniquely determine the values of the v_{1ij}, $i < j$. This determines the T^k_λ up to the choice of the t_{1i}. Finally, the equations resulting from the condition $T_3T_4T_3 = T_4T_3T_4$ force certain relations between the T^k_λ for fixed k and different λ. For each fixed k we picked out a few nice equations resulting from this condition to determine the T^k_λ completely for all λ. This completely determined the representations φ^k for all k except $k = 25$.

The case of φ^{25} is slightly more complex. We used the same methods as above to determine the matrices T^{25}_λ in terms of the variables t_{1i} for each λ except $T^{25}_{(21)}$. In the case of $T^{25}_{(21)}$ we have $\dim(T^{25}_{(21)}) = 6$ and the system of 12 equations obtained from (a) and (b) of Proposition 3.13 is a rank 11 system in the 15 unknowns u_{ij}. These linear equations can be used to write 11 of the u_{ij} variables in terms of the other 4. Next we chose the nicest equations resulting from (c) and (d) of Proposition 3.13 *and* the condition $T_4T_3T_4 = T_3T_4T_3$ and used Maple [3] to solve these equations. These equations are quite nontrivial and we found that we needed to choose these equations carefully in order to stay within the bounds of the capability of Maple. In this way we determined the matrices T^{25}_λ, for all λ, and thus determined φ^{25} completely.

4. Branching rules

4.1. The branching rules from HA_2 to HA_1.

φ^λ	dim	Restriction to HA_1
$\varphi^{(3)}$	1	$\varphi^{(2)}$
$\varphi^{(2,1)}$	2	$\varphi^{(2)} \oplus \varphi^{(1^2)}$
$\varphi^{(1^3)}$	1	$\varphi^{(1^2)}$

4.2. The branching rules from HB_3 to HA_2.

φ^μ	dim	Restriction to HA_2
$\varphi^{(3),\emptyset}$	1	$\varphi^{(3)}$
$\varphi^{(1^3),\emptyset}$	1	$\varphi^{(1^3)}$
$\varphi^{\emptyset,(3)}$	1	$\varphi^{(3)}$
$\varphi^{\emptyset,(1^3)}$	1	$\varphi^{(1^3)}$
$\varphi^{(21),\emptyset}$	2	$\varphi^{(21)}$
$\varphi^{\emptyset,(21)}$	2	$\varphi^{(21)}$
$\varphi^{(2),(1)}$	3	$\varphi^{(3)} \oplus \varphi^{(21)}$
$\varphi^{(1^2),(1)}$	3	$\varphi^{(21)} \oplus \varphi^{(1^3)}$
$\varphi^{(1),(2)}$	3	$\varphi^{(3)} \oplus \varphi^{(21)}$
$\varphi^{(1),(1^2)}$	3	$\varphi^{(21)} \oplus \varphi^{(1^3)}$

4.3. **The branching rules from HF_4 to HB_3.** The bands in this table separate the orbits of the group of field automorphisms $\langle\, \alpha_p, \alpha_q \,\rangle$, see 5.4.

φ^k	dim	Restriction to HB_3
φ^1	1	$\varphi^{(3),\emptyset}$
φ^2	1	$\varphi^{(1^3),\emptyset}$
φ^3	1	$\varphi^{\emptyset,(3)}$
φ^4	1	$\varphi^{\emptyset,(1^3)}$
φ^5	2	$\varphi^{(21),\emptyset}$
φ^6	2	$\varphi^{\emptyset,(21)}$
φ^7	2	$\varphi^{(3),\emptyset} \oplus \varphi^{\emptyset,(3)}$
φ^8	2	$\varphi^{(1^3),\emptyset} \oplus \varphi^{\emptyset,(1^3)}$
φ^9	4	$\varphi^{(21),\emptyset} \oplus \varphi^{\emptyset,(21)}$
φ^{10}	9	$\varphi^{(3),\emptyset} \oplus \varphi^{(21),\emptyset} \oplus \varphi^{(2),(1)} \oplus \varphi^{(1),(2)}$
φ^{11}	9	$\varphi^{(21),\emptyset} \oplus \varphi^{(1^3),\emptyset} \oplus \varphi^{(1^2),(1)} \oplus \varphi^{(1),(1^2)}$
φ^{12}	9	$\varphi^{(2),(1)} \oplus \varphi^{(1),(2)} \oplus \varphi^{\emptyset,(3)} \oplus \varphi^{\emptyset,(21)}$
φ^{13}	9	$\varphi^{(1^2),(1)} \oplus \varphi^{(1),(1^2)} \oplus \varphi^{\emptyset,(21)} \oplus \varphi^{\emptyset,(1^3)}$
φ^{14}	6	$\varphi^{(1^2),(1)} \oplus \varphi^{(1),(2)}$
φ^{15}	6	$\varphi^{(2),(1)} \oplus \varphi^{(1),(1^2)}$
φ^{16}	12	$\varphi^{(2),(1)} \oplus \varphi^{(1^2),(1)} \oplus \varphi^{(1),(2)} \oplus \varphi^{(1),(1^2)}$
φ^{17}	4	$\varphi^{(3),\emptyset} \oplus \varphi^{(2),(1)}$
φ^{18}	4	$\varphi^{(1^3),\emptyset} \oplus \varphi^{(1^2),(1)}$
φ^{19}	4	$\varphi^{(1),(2)} \oplus \varphi^{\emptyset,(3)}$
φ^{20}	4	$\varphi^{(1),(1^2)} \oplus \varphi^{\emptyset,(1^3)}$
φ^{21}	8	$\varphi^{(21),\emptyset} \oplus \varphi^{(2),(1)} \oplus \varphi^{(1^2),(1)}$
φ^{22}	8	$\varphi^{(1),(2)} \oplus \varphi^{(1),(1^2)} \oplus \varphi^{\emptyset,(21)}$
φ^{23}	8	$\varphi^{(3),\emptyset} \oplus \varphi^{(2),(1)} \oplus \varphi^{(1),(2)} \oplus \varphi^{\emptyset,(3)}$
φ^{24}	8	$\varphi^{(1^3),\emptyset} \oplus \varphi^{(1^2),(1)} \oplus \varphi^{(1),(1^2)} \oplus \varphi^{\emptyset,(1^3)}$
φ^{25}	16	$\varphi^{(21),\emptyset} \oplus \varphi^{(2),(1)} \oplus \varphi^{(1^2),(1)} \oplus \varphi^{(1),(2)} \oplus \varphi^{(1),(1^2)} \oplus \varphi^{\emptyset,(21)}$

5. Seminormal Representations for HA_1, HA_2, HB_3, and HF_4

5.1. **The Iwahori-Hecke algebra HA_1.** The irreducible representations φ^λ of HA_1 are indexed by the partitions λ of 2 and we have

$$\varphi^{(2)}(T_1) = (p) \quad \text{and} \quad \varphi^{(1^2)}(T_1) = (-p^{-1}).$$

5.2. **The Iwahori-Hecke algebra HA_2.** The irreducible representations φ^λ of HA_2 are indexed by the partitions of 3 and can be given explicitly by

$$\begin{aligned}
\varphi^{(3)}(T_1) &= (p), & \varphi^{(3)}(T_2) &= (p), \\
\varphi^{(21)}(T_1) &= \mathrm{diag}(p, -p^{-1}), & \varphi^{(21)}(T_2) &= M_2(p,\alpha), \\
\varphi^{(1^3)}(T_1) &= (-p^{-1}), & \varphi^{(1^3)}(T_2) &= (-p^{-1}),
\end{aligned}$$

where

$$M_2(p,\alpha) = -\frac{1}{[2]_p}\begin{pmatrix} p^{-2} & \alpha([2]_p - 1) \\ \frac{1}{\alpha}([2]_p + 1) & -p^2 \end{pmatrix}$$

and $[2]_p = p + p^{-1}$. The variable α is a free parameter, see Lemma 3.3.

5.3. **The Iwahori-Hecke algebra HB_3.** The irreducible representations $\varphi^\mu = \varphi^{\alpha,\beta}$ of $HB_3(p^2, q^2)$ are indexed by pairs of partitions $\mu = (\alpha, \beta)$ such that $|\alpha| + |\beta| = 3$. Let $\mathrm{diag}(A, B, \ldots, C)$ denote the block diagonal matrix with the matrices $A, B, \ldots, C$ in order on the diagonal. Then, using the notation

$$[2]_x = x + x^{-1}, \qquad [3]_x = x^2 + 1 + x^{-2}, \qquad \text{and} \qquad [0]_x = x - x^{-1},$$

irreducible seminormal representations of the Iwahori-Hecke algebra HB_3 can be given explicitly as follows:

$$\begin{aligned}
\varphi^{(3),\emptyset}(T_1) &= (p), & \varphi^{(3),\emptyset}(T_2) &= (p), & \varphi^{(3),\emptyset}(T_3) &= (q), \\
\varphi^{(1^3),\emptyset}(T_1) &= (-p^{-1}), & \varphi^{(1^3),\emptyset}(T_2) &= (-p^{-1}), & \varphi^{(1^3),\emptyset}(T_3) &= (q), \\
\varphi^{\emptyset,(3)}(T_1) &= (p), & \varphi^{\emptyset,(3)}(T_2) &= (p), & \varphi^{\emptyset,(3)}(T_3) &= (-q^{-1}), \\
\varphi^{\emptyset,(1^3)}(T_1) &= (-p^{-1}), & \varphi^{\emptyset,(1^3)}(T_2) &= (-p^{-1}), & \varphi^{\emptyset,(1^3)}(T_3) &= (-q^{-1}), \\
\varphi^{(21),\emptyset}(T_1) &= \mathrm{diag}(p, -p^{-1}), & \varphi^{(21),\emptyset}(T_2) &= M_2(p,1), & \varphi^{(21),\emptyset}(T_3) &= \mathrm{diag}(q, q), \\
\varphi^{\emptyset,(21)}(T_1) &= \mathrm{diag}(p, -p^{-1}), & \varphi^{\emptyset,(21)}(T_2) &= M_2(p,1), & \varphi^{\emptyset,(21)}(T_3) &= \mathrm{diag}(-q^{-1}, -q^{-1}).
\end{aligned}$$

$$\begin{aligned}
\varphi^{(2),(1)}(T_1) &= \mathrm{diag}(p, p, -p^{-1}), & \varphi^{(2),(1)}(T_2) &= \mathrm{diag}(p, M_2(p,1)), \\
\varphi^{(1^2),(1)}(T_1) &= \mathrm{diag}(p, -p^{-1}, -p^{-1}), & \varphi^{(1^2),(1)}(T_2) &= \mathrm{diag}(M_2(p,1), -p^{-1}), \\
\varphi^{(1),(2)}(T_1) &= \mathrm{diag}(p, p, -p^{-1}), & \varphi^{(1),(2)}(T_2) &= \mathrm{diag}(p, M_2(p,1)), \\
\varphi^{(1),(1^2)}(T_1) &= \mathrm{diag}(p, -p^{-1}, -p^{-1}), & \varphi^{(1),(1^2)}(T_2) &= \mathrm{diag}(M_2(p,1), -p^{-1}).
\end{aligned}$$

$$\begin{aligned}
\varphi^{(2),(1)}(T_3) &= \mathrm{diag}(M_{(2),(1)}, q),\\
\varphi^{(1^2),(1)}(T_3) &= \mathrm{diag}(q, M_{(1^2),(1)}),\\
\varphi^{(1),(2)}(T_3) &= \mathrm{diag}(M_{(1),(2)}, -q^{-1}),\\
\varphi^{(1),(1^2)}(T_3) &= \mathrm{diag}(-q^{-1}, M_{(1),(1^2)}),
\end{aligned}$$

where

$$M_{(2),(1)} = \frac{1}{[3]_p}\begin{pmatrix} q + p^{-2}[0]_q & -[2]_p[2]_{p/q} \\ -[2]_{p^2q} & -q^{-1} + p^2[0]_q \end{pmatrix},$$

$$M_{(1^2),(1)} = \frac{1}{[3]_p}\begin{pmatrix} -q^{-1} + p^{-2}[0]_q & -[2]_{p^2/q} \\ -[2]_p[2]_{p^2q} & q + p^2[0]_q \end{pmatrix},$$

$$M_{(1),(2)} = \frac{1}{[3]_p}\begin{pmatrix} -q^{-1} + p^{-2}[0]_q & [2]_p[2]_{pq} \\ [2]_{p^2/q} & q + p^2[0]_q \end{pmatrix},$$

$$M_{(1),(1^2)} = \frac{1}{[3]_p}\begin{pmatrix} q + p^{-2}[0]_q & [2]_{p^2q} \\ [2]_p[2]_{p/q} & -q^{-1} + p^2[0]_q \end{pmatrix}.$$

5.4. **The Iwahori-Hecke algebra HF_4.** Let α_p be the automorphism of $\mathbb{Q}(p,q)$ which fixes q and sends p to $-p^{-1}$. Similarly, let α_q be the automorphism which fixes p and sends q to $-q^{-1}$. These field automorphisms act on the entries of the matrices $\varphi^\lambda(T_i)$ and thereby permute the representations φ^λ. The representation resulting from the application of a field automorphism to a representation in seminormal form may no longer be seminormal. In order to bring the representation back to seminormal form it may be necessary to conjugate by a permutation matrix π. The orbits of the irreducible representations under the action of α_p and α_q and the permutations π for conjugating back to seminormal form are given in the following table. If φ is a representation of HF_4 and π is a permutation then we shall let $\pi \circ \varphi$ denote the representation determined by $(\pi \circ \varphi)(h) = \pi\varphi(h)\pi^{-1}$, for all $h \in HF_4$.

φ^k	Orbit of $\langle\, \alpha_p, \alpha_q \,\rangle$
φ^1	$\varphi^2 = \alpha_p\varphi^1$, $\varphi^3 = \alpha_q\varphi^1$, and $\varphi^4 = \alpha_p\alpha_q\varphi^1$
φ^5	$\varphi^6 = \alpha_q\varphi^5$
φ^7	$\varphi^8 = \alpha_p\varphi^7$
φ^{10}	$\varphi^{11} = \pi_{11}\circ(\alpha_p\varphi^{10})$, where $\pi_{11} = (1,3)(4,6)(7,9)$ $\varphi^{12} = \pi_{12}\circ(\alpha_q\varphi^{10})$, where $\pi_{12} = (1,7)(2,8)(3,9)$ $\varphi^{13} = \pi_{13}\circ(\alpha_p\alpha_q\varphi^{10})$, where $\pi_{13} = (1,9)(2,8)(3,7)(4,6)$
φ^{14}	$\varphi^{15} = \pi_{15}\circ(\alpha_p\varphi^{14})$, where $\pi_{15} = (1,3)(4,6)$
φ^{17}	$\varphi^{18} = \pi_{18}\circ(\alpha_p\varphi^{17})$, where $\pi_{18} = (2,4)$ $\varphi^{19} = \pi_{19}\circ(\alpha_q\varphi^{17})$, where $\pi_{19} = (1,4,3,2)$ $\varphi^{20} = \pi_{20}\circ(\alpha_p\alpha_q\varphi^{17})$, where $\pi_{20} = (1,4)(2,3)$
φ^{21}	$\varphi^{22} = \pi_{22}\circ(\alpha_q\varphi^{21})$, where $\pi_{22} = (1,7,5,3)(2,8,6,4)$
φ^{23}	$\varphi^{24} = \pi_{24}\circ(\alpha_p\varphi^{23})$, where $\pi_{24} = (2,4)(5,7)$

Let $w_{0,1}$, $w_{0,2}$, $w_{0,3}$ and $w_{0,4}$ be the longest elements in the Weyl groups WA_1, WA_2, WB_3 and WF_4, respectively. Let

$$\begin{aligned}
D_1 &= T_{w_{0,1}} = T_1, \\
D_2 &= T^2_{w_{0,2}} = (T_1T_2T_1)^2, \\
D_3 &= T_{w_{0,3}} = (T_3T_2T_1)^3, \\
D_4 &= T_{w_{0,4}} = (T_4T_{w_{0,3}})^3T^{-2}_{w_{0,2}}
\end{aligned}$$

in HF_4. The following tables give the values of $\varphi^k(D_j)$, for one representative from each equivalence class of representations. The rest of the matrices $\varphi^k(D_j)$ are easily obtained by applying the automorphisms α_p and α_q and conjugating by a permutation π as indicated in 5.4 above.

$$\begin{aligned}
\varphi^1(T_1) &= (p), \\
\varphi^1(T^2_{w_{0,2}}) &= (p^6), \\
\varphi^1(T_{w_{0,3}}) &= (p^6q^3), \\
\varphi^1(T_{w_{0,4}}) &= (p^{12}q^{12}),
\end{aligned}
\qquad
\begin{aligned}
\varphi^5(T_1) &= \mathrm{diag}(p, -p^{-1}), \\
\varphi^5(T^2_{w_{0,2}}) &= \mathrm{Id}, \\
\varphi^5(T_{w_{0,3}}) &= q^3\ \mathrm{Id}, \\
\varphi^5(T_{w_{0,4}}) &= q^{12}\ \mathrm{Id},
\end{aligned}$$

$$\begin{aligned} \varphi^7(T_1) &= p \ \mathrm{Id}, \\ \varphi^7(T^2_{w_{0,2}}) &= p^6 \ \mathrm{Id}, \\ \varphi^7(T_{w_{0,3}}) &= \mathrm{diag}(p^6q^3, -p^6q^{-3}), \\ \varphi^7(T_{w_{0,4}}) &= p^{12} \ \mathrm{Id}, \end{aligned}$$

$$\begin{aligned} \varphi^9(T_1) &= \mathrm{diag}(p, -p^{-1}, p, -p^{-1}), \\ \varphi^9(T^2_{w_{0,2}}) &= \mathrm{Id}, \\ \varphi^9(T_{w_{0,3}}) &= \mathrm{diag}(q^3, q^3, -q^{-3}, -q^{-3}), \\ \varphi^9(T_{w_{0,4}}) &= \mathrm{Id}, \end{aligned}$$

$$\begin{aligned} \varphi^{10}(T_1) &= \mathrm{diag}(p, p, -p^{-1}, p, p, -p^{-1}, p, p, -p^{-1},), \\ \varphi^{10}(T^2_{w_{0,2}}) &= \mathrm{diag}(p^6, 1, 1, p^6, 1, 1, p^6, 1, 1), \\ \varphi^{10}(T_{w_{0,3}}) &= \mathrm{diag}(p^6q^3, q^3, q^3, -p^2q, -p^2q, -p^2q, p^2q^{-1}, p^2q^{-1}, p^2q^{-1}), \\ \varphi^{10}(T_{w_{0,4}}) &= p^4q^4 \ \mathrm{Id}, \end{aligned}$$

$$\begin{aligned} \varphi^{14}(T_1) &= \mathrm{diag}(p, -p^{-1}, -p^{-1}, p, p, -p^{-1}), \\ \varphi^{14}(T^2_{w_{0,2}}) &= \mathrm{diag}(1, 1, p^{-6}, p^6, 1, 1), \\ \varphi^{14}(T_{w_{0,3}}) &= \mathrm{diag}(-p^{-2}q, -p^{-2}q, -p^{-2}q, p^2q^{-1}, p^2q^{-1}, p^2q^{-1}), \\ \varphi^{14}(T_{w_{0,4}}) &= \mathrm{Id}, \end{aligned}$$

$$\begin{aligned} \varphi^{16}(T_1) &= \mathrm{diag}(p, p, -p^{-1}, p, -p^{-1}, -p^{-1}, p, p, -p^{-1}, p, -p^{-1}, -p^{-1}), \\ \varphi^{16}(T^2_{w_{0,2}}) &= \mathrm{diag}(p^6, 1, 1, 1, 1, p^{-6}, p^6, 1, 1, 1, 1, p^{-6}), \\ \varphi^{16}(T_{w_{0,3}}) &= \mathrm{diag}(-p^2q, -p^2q, -p^2q, -p^{-2}q, -p^{-2}q, -p^{-2}q, p^2q^{-1}, p^2q^{-1}, p^2q^{-1}, \\ &\qquad p^{-2}q^{-1}, p^{-2}q^{-1}, p^{-2}q^{-1}), \\ \varphi^{16}(T_{w_{0,4}}) &= \mathrm{Id}, \end{aligned}$$

$$\begin{aligned} \varphi^{17}(T_1) &= \mathrm{diag}(p, p, p, -p^{-1}), \\ \varphi^{17}(T^2_{w_{0,2}}) &= \mathrm{diag}(p^6, p^6, 1, 1), \\ \varphi^{17}(T_{w_{0,3}}) &= \mathrm{diag}(p^6q^3, -p^2q, -p^2q, -p^2q), \\ \varphi^{17}(T_{w_{0,4}}) &= -p^6q^6 \ \mathrm{Id}, \end{aligned}$$

$$\begin{aligned} \varphi^{21}(T_1) &= \mathrm{diag}(p, -p^{-1}, p, p, -p^{-1}, p, -p^{-1}, -p^{-1}), \\ \varphi^{21}(T^2_{w_{0,2}}) &= \mathrm{diag}(1, 1, p^6, 1, 1, 1, 1, p^{-6}), \\ \varphi^{21}(T_{w_{0,3}}) &= \mathrm{diag}(q^3, q^3, -p^2q, -p^2q, -p^2q, -p^{-2}q, -p^{-2}q, -p^{-2}q), \\ \varphi^{21}(T_{w_{0,4}}) &= -q^6 \ \mathrm{Id}, \end{aligned}$$

$$
\begin{aligned}
\varphi^{23}(T_1) &= \operatorname{diag}(p,p,p,-p^{-1},p,p,-p^{-1},p),\\
\varphi^{23}(T_{w_{0,2}}^2) &= \operatorname{diag}(p^6,p^6,1,1,p^6,1,1,p^6),\\
\varphi^{23}(T_{w_{0,3}}) &= \operatorname{diag}(p^6q^3,-p^2q,-p^2q,-p^2q,p^2q^{-1},p^2q^{-1},p^2q^{-1},-p^6q^{-3}),\\
\varphi^{23}(T_{w_{0,4}}) &= -p^6 \operatorname{Id},
\end{aligned}
$$

$$
\begin{aligned}
\varphi^{25}(T_1) &= \operatorname{diag}(p,-p^{-1},p,p,-p^{-1},p,-p^{-1},-p^{-1},p,p,-p^{-1},p,-p^{-1},-p^{-1},p,-p^{-1}),\\
\varphi^{25}(T_{w_{0,2}}^2) &= \operatorname{diag}(1,1,p^6,1,1,1,1,p^{-6},p^6,1,1,1,1,p^{-6},1,1),\\
\varphi^{25}(T_{w_{0,3}}) &= \operatorname{diag}(q^3,q^3,-p^2q,-p^2q,-p^2q,-p^{-2}q,-p^{-2}q,-p^{-2}q,p^2q^{-1},p^2q^{-1},p^2q^{-1},\\
&\qquad p^{-2}q^{-1},p^{-2}q^{-1},p^{-2}q^{-1},-q^{-3},q^{-3}),\\
\varphi^{25}(T_{w_{0,4}}) &= -\operatorname{Id}.
\end{aligned}
$$

Using the methods described in the previous sections we have produced matrices $\varphi^k(T_i)$ giving the 25 irreducible representations φ^k of HF_4. The following tables give the values of $\varphi^k(T_i)$, for one representative from each equivalence class of representations. The rest of the matrices $\varphi^k(T_i)$ are obtained by applying the automorphisms α_p and α_q and conjugating by a permutation π as indicated in 5.4 above.

We shall use the notations

$$[2]_x = x + x^{-1}, \qquad [3]_x = x^2 + 1 + x^{-2}, \qquad \text{and} \qquad [0]_x = x - x^{-1},$$

and the notation

$$\varphi^k(T_i)^{[a_1,a_2,\dots,a_r]}$$

will denote the $r \times r$ submatrix of $\varphi^k(T_i)$ which is formed by the intersection of the $a_1,\dots,a_r$th rows and columns. The notation $\operatorname{diag}(A,B,\dots,C)$ will denote the block diagonal matrix with the matrices $A,B,\dots,C$ in order along the diagonal. The matrix $M_2(x,y)$ will be as given in 5.2, the matrices $M_{\alpha,\beta}$ are as given in 5.3 and the variables α, β, ξ, θ, and η are free parameters, see Lemma 3.3. *Any entries of the matrices $\varphi^k(T_i)$ which are not given explicitly below are taken to be 0.*

The representations φ^1 and φ^5.

$$
\begin{aligned}
\varphi^1(T_1) &= (p), & \varphi^5(T_1) &= \operatorname{diag}(p,-p^{-1}),\\
\varphi^1(T_2) &= (p), & \varphi^5(T_2) &= M_2(p,1),\\
\varphi^1(T_3) &= (q), & \varphi^5(T_3) &= \operatorname{diag}(q,q),\\
\varphi^1(T_4) &= (q), & \varphi^5(T_4) &= \operatorname{diag}(q,q).
\end{aligned}
$$

The representations φ^7 and φ^9.

$$\begin{aligned}
\varphi^7(T_1) &= \operatorname{diag}(p,p), & \varphi^9(T_1) &= \operatorname{diag}(p,-p^{-1},p,-p^{-1}),\\
\varphi^7(T_2) &= \operatorname{diag}(p,p), & \varphi^9(T_2) &= \operatorname{diag}(M_2(p,1),M_2(p,1)),\\
\varphi^7(T_3) &= \operatorname{diag}(q,-q^{-1}), & \varphi^9(T_3) &= \operatorname{diag}(q,q,-q^{-1},q^{-1}),\\
\varphi^7(T_4) &= M_2(q,\alpha), & \varphi^9(T_4)^{[1,3]} = \varphi^9(T_4)^{[2,4]} &= M_2(q,\alpha).
\end{aligned}$$

The representation φ^{10}.

$$\begin{aligned}
\varphi^{10}(T_1) &= \operatorname{diag}(p,p,-p^{-1},p,p,-p^{-1},p,p,-p^{-1}),\\
\varphi^{10}(T_2) &= \operatorname{diag}(p,M_2(p,1),p,M_2(p,1),p,M_2(p,1)),\\
\varphi^{10}(T_3) &= \operatorname{diag}(q,q,q,M_{(2,1)},q,M_{(1,2)},q),\\
\varphi^{10}(T_4)^{[1,4,7]} &= M_{10},\\
\varphi^{10}(T_4)^{[2,5,8]} &= \varphi^{10}(T_4)^{[3,6,9]} = N_{10},
\end{aligned}$$

where

$$M_{10} = \frac{1}{[2]_q[2]_{p^2q}}\begin{pmatrix} p^{-2}q^{-1}[2]_q[0]_q & -[2]_q[2]_{p^2q^2}\xi\eta^{-1} & -[2]_q[2]_{p^2q^2}\xi \\ -[2]_{p^2q^{-1}}\eta\xi^{-1} & [2]_{p^2q} + p^2q[2]_q[0]_q & -[2]_{p^2q^{-1}}\eta \\ -[2]_{p^2q}\xi^{-1} & -[2]_{p^2q}\eta^{-1} & q^2[2]_{p^2q} \end{pmatrix}$$

and

$$N_{10} = \frac{1}{[2]_q[2]_{pq^{-1}}}\begin{pmatrix} pq^{-1}[2]_q[0]_q & -[2]_q[2]_{pq^{-2}}\theta\eta^{-1} & -[2]_q[2]_{pq^{-2}}\theta \\ -[2]_{pq}\eta\theta^{-1} & [2]_{pq^{-1}} + p^{-1}q[2]_q[0]_q & -[2]_{pq}\eta \\ -[2]_{pq^{-1}}\theta^{-1} & -[2]_{pq^{-1}}\theta^{-1} & q^2[2]_{pq^{-1}} \end{pmatrix}.$$

The representation φ^{14}.

$$\begin{aligned}
\varphi^{14}(T_1) &= \mathrm{diag}(p, -p^{-1}, -p^{-1}, p, p, -p^{-1}),\\
\varphi^{14}(T_2) &= \mathrm{diag}(M_2(p,1), -p^{-1}, p, M_2(p,1)),\\
\varphi^{14}(T_3) &= \mathrm{diag}(q, M_{(1^2,1)}, M_{(1,2)}, -q^{-1}),\\
\varphi^{14}(T_4)^{[3]} &= -q^{-1},\\
\varphi^{14}(T_4)^{[4]} &= q,\\
\varphi^{14}(T_4)^{[1,5]} &= M_{14},\\
\varphi^{14}(T_4)^{[2,6]} &= M_{14},
\end{aligned}$$

where

$$M_{14} = \frac{1}{[2]_{p^2q^{-1}}} \begin{pmatrix} 1 + p^2q^{-1}[0]_q & -[3]_p\alpha \\ (1-[2]_{p^2q^{-2}})\alpha^{-1} & -1 + p^{-2}q[0]_q \end{pmatrix}.$$

The representation φ^{16}.

$$\begin{aligned}
\varphi^{16}(T_1) &= \operatorname{diag}(p, p, -p^{-1}, p, -p^{-1}, -p^{-1}, p, p, -p^{-1}, p, -p^{-1}, -p^{-1}),\\
\varphi^{16}(T_2) &= \operatorname{diag}(p, M_2(p,1), M_2(p,1), -p^{-1}, p, M_2(p,1), M_2(p,1) - p^{-1}),\\
\varphi^{16}(T_3) &= \operatorname{diag}(M_{(2,1)}, q, q, M_{(1^2,1)}, M_{(1,2)}, -q^{-1}, -q^{-1}, M_{(1,1^2)}),\\
\varphi^{16}(T_4)^{[1,7]} &= M_{16}(\xi),\\
\varphi^{16}(T_4)^{[6,12]} &= M_{16}(\eta),\\
\varphi^{16}(T_4)^{[2,4,8,10]} &= \varphi^{16}(T_4)^{[3,5,9,12]} = N_{16},
\end{aligned}$$

where

$$M_{16}(\alpha) = \frac{1}{[2]_q}\begin{pmatrix} 1 + q^{-1}[0]_q & -3\alpha \\ -[3]_{q^2}/\alpha[3]_q & -1 + q[0]_q \end{pmatrix},$$

$$N_{16} = \frac{1}{[2]_p[2]_q}\begin{pmatrix}
f_{16}(p,q) & \dfrac{3[2]_{pq}\xi\theta}{[2]_{p^2/q}\eta} & \dfrac{3[2]_{pq}\xi}{[2]_{p^2/q}} & \dfrac{3[2]_{pq}\xi\theta}{[2]_{p^2q}} \\
\dfrac{[3]_{p^2}[2]_{p/q}\eta}{[2]_{p^2q}\xi\theta} & -f_{16}(-p^{-1},q) & \dfrac{[3]_{p^2}[2]_{p/q}\eta}{[2]_{p^2/q}\theta} & \dfrac{3[2]_{p/q}\eta}{[2]_{p^2q}} \\
\dfrac{[3]_{q^2}[2]_{p/q}}{[3]_q[2]_{p^2q}\xi} & \dfrac{[3]_{q^2}[2]_{p/q}\theta}{[3]_q[2]_{p^2/q}\eta} & -f_{16}(p,-q^{-1}) & -\dfrac{3[2]_{p/q}\theta}{[2]_{p^2q}} \\
\dfrac{[3]_{p^2}[3]_{q^2}[2]_{pq}}{3[3]_q[2]_{p^2q}\xi\theta} & \dfrac{[3]_{q^2}[2]_{pq}}{[3]_q[2]_{p^2/q}\eta} & -\dfrac{[3]_{p^2}[2]_{pq}}{[2]_{p^2/q}\theta} & f_{16}(-p^{-1},-q^{-1})
\end{pmatrix},$$

and

$$f_{16}(x,y) = \frac{-2x/y + xy + 1/xy - 1/xy^3 - y/x - 1/x^3y^3 + y/x^3}{[2]_{x^2y}}.$$

The representation φ^{17}.

$$\begin{aligned}
\varphi^{17}(T_1) &= \operatorname{diag}(p, p, p, -p^{-1}),\\
\varphi^{17}(T_2) &= \operatorname{diag}(p, p, M_2(p,1)),\\
\varphi^{17}(T_3) &= \operatorname{diag}(q, M_{(2,1)}, q),\\
\varphi^{17}(T_4) &= \operatorname{diag}(M_{17}, q, q),
\end{aligned}$$

where

$$M_{17} = \frac{1}{[2]_{p^2q}}\begin{pmatrix} 1 + p^{-2}q^{-1}[0]_q & -[3]_p\alpha \\ (1 - [2]_{p^2q^2})\alpha^{-1} & -1 + p^2q[0]_q \end{pmatrix}.$$

The representation φ^{21}.

$$\begin{aligned}
\varphi^{21}(T_1) &= \operatorname{diag}(p, -p^{-1}, p, p, -p^{-1}, p, -p^{-1}, -p^{-1}),\\
\varphi^{21}(T_2) &= \operatorname{diag}(M_2(p,1), p, M_2(p,1), M_2(p,1), -p^{-1}),\\
\varphi^{21}(T_3) &= \operatorname{diag}(q, q, M_{(2,1)}, q, q, M_{(1^2,1)}),\\
\varphi^{21}(T_4)^{[3]} &= \varphi^{21}(T_4)^{[8]} = q,\\
\varphi^{21}(T_4)^{[1,4,6]} &= \varphi^{21}(T_4)^{[2,5,7]} = M_{21},
\end{aligned}$$

where

$$M_{21} = \frac{1}{[2]_p[2]_{pq}[2]_{p/q}} \begin{pmatrix} (q[2]_{p^2} + q^{-2}[0]_q)[2]_p & -[2]_{q^3}[2]_p\xi\eta^{-1} & -[2]_{q^3}[2]_p\xi \\ -[3]_p[2]_{pq}\eta\xi^{-1} & (p^{-1}q[2]_p[0]_q + 1)[2]_{pq} & -[3]_p[2]_{pq}\eta \\ -[3]_p[2]_{p/q}\xi^{-1} & -[3]_p[2]_{p/q}\eta^{-1} & (pq[2]_p[0]_q + 1)[2]_{p/q} \end{pmatrix}.$$

The representation φ^{23}.

$$\begin{aligned}
\varphi^{23}(T_1) &= \operatorname{diag}(p, p, p, -p^{-1}, p, p, -p^{-1}, p),\\
\varphi^{23}(T_2) &= \operatorname{diag}(p, p, M_2(p,1), p, M_2(p,1), p),\\
\varphi^{23}(T_3) &= \operatorname{diag}(q, M_{(2,1)}, q, M_{(1,2)}, -q^{-1}, -q^{-1}),\\
\varphi^{23}(T_4)^{[3,6]} &= \varphi^{23}(T_4)^{[4,7]} = M_2(q, \eta/([2]_q - 1)\theta),\\
\varphi^{23}(T_4)^{[1,2,5,8]} &= M_{23},
\end{aligned}$$

where

$$M_{23} = \frac{1}{[2]_q} \begin{pmatrix} f_{23}(p,q) & \dfrac{[2]_{pq^2}\xi}{[2]_{p^2q}[2]_{pq}\eta} & \dfrac{[2]_{p/q}[2]_{pq^2}\xi}{[2]_{p^2q}[2]_{pq}^2\theta} & \dfrac{[2]_{pq^2}\xi}{[2]_{p^2q}[2]_{pq}} \\ \dfrac{[3]_{p^2}[2]_p\eta}{[2]_{p^2q}[2]_{p/q}\xi} & g_{23}(p,q) & \dfrac{([2]_{q^2}-1)[2]_p\eta}{[2]_{pq}[2]_{p^2q}\theta} & \dfrac{-[3]_{p^2}[2]_p\eta}{[2]_{p/q}[2]_{p^2q}} \\ \dfrac{[3]_q[3]_{p^2}[2]_p\theta}{[2]_{p/q}[2]_{p^2/q}\xi} & \dfrac{([2]_{q^2}-1)[3]_q[2]_p\theta}{[2]_{p/q}[2]_{p^2/q}\eta} & -g_{23}(p,-1/q) & \dfrac{[3]_{p^2}[2]_p\theta}{[2]_{p/q}[2]_{p^2/q}} \\ \dfrac{[3]_q[2]_{p/q^2}}{[2]_{p/q}[2]_{p^2/q}\xi} & \dfrac{-[3]_q[2]_{p/q^2}}{[2]_{p/q}[2]_{p^2/q}\eta} & \dfrac{[2]_{p/q^2}}{[2]_{pq}[2]_{p^2/q}\theta} & -f_{23}(p,-1/q) \end{pmatrix},$$

where

$$f_{23}(x,y) = \frac{y^4 - x^4y^2 - x^2y^2 - 1}{x^3y^4[2]_{xy}[2]_{x^2y}} \quad \text{and}$$

$$g_{23}(x,y) = \frac{x^4y^6 - x^4y^2 + x^2y^6 - x^2y^4 + x^2y^2 + y^6 + y^2 - 1}{xy^4[2]_{x/y}[2]_{x^2y}}.$$

The representation φ^{25}.

$$\begin{aligned}
\varphi^{25}(T_1) &= \mathrm{diag}(p, -p^{-1}, p, p, -p^{-1}, p, -p^{-1}, -p^{-1}, \\
&\qquad p, p, -p^{-1}, p, -p^{-1}, -p^{-1}, p, -p^{-1}), \\
\varphi^{25}(T_2) &= \mathrm{diag}(M_2(p,1), p, M_2(p,1), M_2(p,1), -p^{-1}, \\
&\qquad p, M_2(p,1), M_2(p,1), -p^{-1}, M_2(p,1)), \\
\varphi^{25}(T_3) &= \mathrm{diag}(q, q, M_{(2,1)}, q, q, M_{(1^2,1)}, M_{(1,2)}, -q^{-1}, -q^{-1}, M_{(1,1^2)}, -q^{-1}, -q^{-1}), \\
\varphi^{25}(T_4)^{[3,9]} &= M_2(q, \alpha/([2]_q - 1)\eta), \\
\varphi^{25}(T_4)^{[8,14]} &= M_2(q, \beta/([2]_q - 1)\theta), \\
\varphi^{25}(T_4)^{[1,4,6,10,12,15]} &= \varphi^{25}(T_4)^{[2,5,7,11,13,16]} = M_{25},
\end{aligned}$$

where

$$
M_{25} = \frac{1}{[2]_q} \left(\begin{matrix}
f_{25}(p,q) & \dfrac{-[2]_{p^2/q}\xi}{[2]_{pq}[2]_{p/q}\alpha} & \dfrac{-[2]_{p^2q}\xi}{[2]_{pq}[2]_{p/q}\beta} \\
\dfrac{-2[3]_p[2]_{p^2/q^2}\alpha}{[2]_p[2]_{p/q}[2]_{p^2q}[2]_{p^2/q}\xi} & g_{25}(p,q) & \dfrac{[3]_p[2]_{p^2/q^2}\alpha}{[2]_p[2]_{p/q}[2]_{p^2/q}\beta} \\
\dfrac{-2[3]_p[2]_{p^2q^2}\beta}{[2]_p[2]_{pq}[2]_{p^2q}[2]_{p^2/q}\xi} & \dfrac{[3]_p[2]_{p^2q^2}\beta}{[2]_p[2]_{pq}[2]_{p^2q}\alpha} & g_{25}(-1/p,q) \\
\dfrac{-2[3]_p[2]_{p^2/q^2}[3]_q\eta}{[2]_p[2]_{pq}[2]_{p^2q}[2]_{p^2/q}\xi} & \dfrac{-([3]_p-[3]_q+2)[3]_q\eta}{[2]_p[2]_{pq}[2]_{p^2q}\alpha} & \dfrac{[3]_p[2]_{p^2/q^2}[3]_q\eta}{[2]_p[2]_{pq}[2]_{p^2/q}\beta} \\
\dfrac{-2[3]_p[2]_{p^2q^2}[3]_q\theta}{[2]_p[2]_{p/q}[2]_{p^2q}[2]_{p^2/q}\xi} & \dfrac{[3]_p[2]_{p^2q^2}[3]_q\theta}{[2]_p[2]_{p/q}[2]_{p^2q}\alpha} & \dfrac{-([3]_p-[3]_q+2)[3]_q\theta}{[2]_p[2]_{p/q}[2]_{p^2/q}\beta} \\
\dfrac{[2]_{p^2q^2}[2]_{p^2/q^2}}{[2]_{pq}[2]_{p/q}[2]_{p^2q}[2]_{p^2/q}\xi} & \dfrac{[2]_{p^2q^2}}{[2]_{pq}[2]_{p/q}[2]_{p^2q}\alpha} & \dfrac{[2]_{p^2/q^2}}{[2]_{pq}[2]_{p/q}[2]_{p^2/q}\beta}
\end{matrix} \right.
$$

$$
\left. \begin{matrix}
\dfrac{-[2]_{p^2q}\xi}{[2]_{pq}[2]_{p/q}\eta} & \dfrac{-[2]_{p^2/q}\xi}{[2]_{pq}[2]_{p/q}\theta} & \dfrac{[3]_q[2]_{p^2q}[2]_{p^2/q}\xi}{[2]_{pq}[2]_{p/q}} \\
\dfrac{-([3]_p-[3]_q+2)\alpha}{[2]_p[2]_{p/q}[2]_{p^2/q}\eta} & \dfrac{[3]_p[2]_{p^2/q^2}\alpha}{[2]_p[2]_{p/q}[2]_{p^2q}\theta} & \dfrac{2[3]_p[3]_q\alpha}{[2]_p[2]_{p/q}} \\
\dfrac{[3]_p[2]_{p^2q^2}\beta}{[2]_p[2]_{pq}[2]_{p^2/q}\eta} & \dfrac{-([3]_p-[3]_q+2)\beta}{[2]_p[2]_{pq}[2]_{p^2q}\theta} & \dfrac{2[3]_p[3]_q\beta}{[2]_p[2]_{pq}} \\
-g_{25}(p,-1/q) & \dfrac{-[3]_p[2]_{p^2/q^2}\eta}{[2]_p[2]_{pq}[2]_{p^2q}\theta} & \dfrac{-2[3]_p[3]_q\eta}{[2]_p[2]_{pq}} \\
\dfrac{-[3]_p[2]_{p^2q^2}\theta}{[2]_p[2]_{p/q}[2]_{p^2/q}\eta} & -g_{25}(-1/p,-1/q) & \dfrac{-2[3]_p[3]_q\theta}{[2]_p[2]_{p/q}} \\
\dfrac{-[2]_{p^2q^2}}{[2]_{pq}[2]_{p/q}[2]_{p^2/q}[3]_q\eta} & \dfrac{-[2]_{p^2/q^2}}{[2]_{pq}[2]_{p/q}[2]_{p^2q}[3]_q\theta} & -f_{25}(p,-1/q)
\end{matrix} \right),
$$

and where

$$
f_{25}(x,y) = -\frac{x^4y^2 + x^2 - x^2y^4 + y^2}{x^2y^4[2]_{xy}[2]_{x/y}}, \quad \text{and}
$$

$$
g_{25}(x,y) = -\frac{x^6y^4 - x^4y^6 - x^4y^2 + x^4 + x^4y^4 + x^2y^2 + x^2 - x^2y^6 + y^2 - y^6}{x^4y^4[2]_x[2]_{x/y}[2]_{x^2y}}.
$$

References

[1] N. Bourbaki. *Groupes et Algèbres de Lie, Chap. IV, V, VI.* Hermann, Paris, 1968.

[2] R. Carter. *Finite groups of Lie type: Conjugacy classes and complex characters.* Wiley, New York, 1985.

[3] Bruce W. Char, Keith O. Geddes, Gaston H. Gonnet, Benton L. Leong, Michael B. Monagan, and Stephen M. Watt. *Maple V Language Reference Manual.* Springer-Verlag, New York, Berlin, Heidelberg, 1991.

[4] C.W. Curtis and I. Reiner. *Methods of representation theory, Vol. I. With applications to finite groups and orders.* Wiley, New York, 1981.

[5] M. Geck. On the character values of Iwahori-Hecke algebras of exceptional type. *Proc. London Math. Soc. (3)*, 68:51–76, 1994.

[6] P.N. Hoefsmit. *Representations of Hecke algebras of finite groups with BN-pairs of classical type.* Ph.D. Thesis, University of British Columbia, 1974.

[7] R. Kilmoyer. *Some irreducible representations of a finite group with BN-pair.* Ph.D. Thesis, Massachusetts Institute of Technology, 1969.

[8] A. Ram. Seminormal representations of Weyl groups and Iwahori-Hecke algebras. *Proc. London Math. Soc. (3)*, 75:999–999, 1997.

[9] H. Wenzl. Hecke algebras of type a_n and subfactors. *Invent. Math.*, 92:349–383, 1988.

[10] A. Young. Quantitative substitutional analysis (sixth paper). *Proc. London Math. Soc. (2)*, 34:196–230, 1932.

Arun Ram, Department of Mathematics, Princeton University, Princeton, NJ 08544-1000

E-mail address: `rama@math.princeton.edu`

D. E. Taylor, School of Mathematics and Statistics, University of Sydney, NSW 2006 Australia

E-mail address: `don@maths.su.oz.au`