# A weight basis for representations of even orthogonal Lie algebras

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**Abstract**

A weight basis for each finite-dimensional irreducible representation of the orthogonal Lie algebra $\mathfrak{o}(2n)$ is constructed. The basis vectors are parametrized by the $D$-type Gelfand–Tsetlin patterns. The basis is consistent with the chain of subalgebras $\mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n$, where $\mathfrak{g}_k = \mathfrak{o}(2k)$. Explicit formulas for the matrix elements of generators of $\mathfrak{o}(2n)$ in this basis are given. The construction is based on the representation theory of the Yangians and extends our previous results for the symplectic Lie algebras.

# 1 Introduction

In their pioneering works [3] and [4] Gelfand and Tsetlin proposed a combinatorial method to explicitly construct representations of the classical Lie algebras. For each finite-dimensional irreducible representation of the general linear Lie algebra $\mathfrak{gl}(N)$ and the orthogonal Lie algebra $\mathfrak{o}(N)$ they gave a parametrization of basis vectors and provided explicit formulas for the matrix elements of generators of the Lie algebras in the basis. Derivations of the matrix element formulas in the orthogonal case are given in [14, 16]; see also [5]. A number of different approaches to the problem of constructing representation bases for simple Lie algebras has been developed; see [9] for more references. Note also recent results by Donnelly [2] and Littelmann [6]. In [2] explicit combinatorial constructions of the fundamental representations of the $B$ and $C$ series Lie algebras and of their $q$-analogs are given; in [6] monomial bases parametrized by patterns of Gelfand–Tsetlin type are constructed for all simple complex Lie algebras. In [9] an analog of the Gelfand–Tsetlin basis for the symplectic Lie algebras $\mathfrak{sp}(2n)$ is constructed and explicit formulas for the matrix elements of generators of $\mathfrak{sp}(2n)$ in this basis are given. Bases for the finite-dimensional irreducible representations of the classical Lie algebras of $B$, $C$, and $D$ series can be constructed in a uniform manner with the use of the representation theory of the Yangians, as in [9]. Here we extend the results of [9] to the case of the $D$ series and hope to treat the remaining $B$ series case in a forthcoming publication.

Our basis for $\mathfrak{o}(2n)$ is different from that of Gelfand and Tsetlin [4]. Their basis is consistent with the chain of subalgebras

$$\mathfrak{o}(2) \subset \mathfrak{o}(3) \subset \cdots \subset \mathfrak{o}(N).$$

The reductions $\mathfrak{o}(k) \downarrow \mathfrak{o}(k-1)$ are multiplicity-free which makes the basis orthogonal with respect to a natural contravariant bilinear form. However, the basis vectors are not weight vectors with respect to the Cartan subalgebra of $\mathfrak{o}(N)$. To get a weight (although non-orthogonal) basis we consider the following chain instead:

$$\mathfrak{o}(2) \subset \mathfrak{o}(4) \subset \cdots \subset \mathfrak{o}(2n)$$

so that all the subalgebras belong to the $D$ series.

The reduction $\mathfrak{o}(2n) \downarrow \mathfrak{o}(2n-2)$ is not multiplicity free. This means that the subspace $V(\lambda)^+_\mu$ of $\mathfrak{o}(2n-2)$-highest vectors of a weight $\mu$ in an $\mathfrak{o}(2n)$-module $V(\lambda)$ is not necessarily one-dimensional. However, this space turns out to possess a natural structure of an irreducible representation of a large associative algebra $\mathrm{Y}^+(2)$ called the twisted Yangian (introduced by Olshanski in [13]) and can also be equipped with an action of the $\mathfrak{gl}(2)$-Yangian $\mathrm{Y}(2)$. This allows us to construct a Yangian Gelfand–Tsetlin basis in $V(\lambda)^+_\mu$ associated with an inclusion $\mathrm{Y}(1) \subset \mathrm{Y}(2)$; see [7, 11, 12].

Our calculations are based on the relationship between the twisted Yangian $\mathrm{Y}^+(2)$ and the transvector algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$, $\mathfrak{g}_n = \mathfrak{o}(2n)$. The transvector algebras (they are sometimes called the Mickelsson algebras or $S$-algebras) are studied in detail in [17, 18].

Although the constructions of the bases are very similar for the orthogonal and symplectic cases, there is a slight difference in the calculation of the matrix elements of generators of the Lie algebras in the basis. The Lie algebra $\mathfrak{sp}(2n)$ contains the "second diagonal" generators $F_{-k,k} = 2E_{-k,k}$ where the $E_{ij}$ denote the standard generators of $\mathfrak{gl}(2n)$ (see Section 2 below). The action of these elements in the basis is rather simple and can be easily found. However, their counterparts do not exist in the orthogonal case. Instead, there are second degree elements $\Phi_{-k,k}$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_n)$ which belong to the centralizer of $\mathfrak{g}_{k-1}$ in $\mathrm{U}(\mathfrak{g}_k)$ and play the role similar to that of the elements $F_{-k,k}$ in the symplectic case.

## 2 Notations and preliminary results

We shall enumerate the rows and columns of $2n \times 2n$-matrices over $\mathbb{C}$ by the indices $-n, \dots, -1, 1, \dots, n$. We let the $E_{ij}$, $i, j = -n, \dots, n$ denote the standard basis of the Lie algebra $\mathfrak{gl}(2n)$. We shall also assume throughout the paper that the index 0 is skipped in a sum or in a product. Introduce the elements

$$F_{ij} = E_{ij} - E_{-j,-i}. \tag{2.1}$$

We have $F_{-j,-i} = -F_{ij}$. In particular, $F_{-i,i} = 0$ for all $i$. The orthogonal Lie algebra $\mathfrak{g}_n := \mathfrak{o}(2n)$ can be identified with the subalgebra in $\mathfrak{gl}(2n)$ spanned by the elements $F_{ij}$, $i, j = -n, \dots, n$.

The subalgebra $\mathfrak{g}_{n-1}$ is spanned by the elements (2.1) with the indices $i, j$ running over the set $\{-n+1, \dots, n-1\}$. Denote by $\mathfrak{h} = \mathfrak{h}_n$ the diagonal Cartan subalgebra in $\mathfrak{g}_n$. The elements $F_{11}, \dots, F_{nn}$ form a basis of $\mathfrak{h}$.

The finite-dimensional irreducible representations of $\mathfrak{g}_n$ are in a one-to-one correspondence with $n$-tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ where all the entries $\lambda_i$ are simultaneously integers or half-integers (elements of the set $\frac{1}{2} + \mathbb{Z}$) and the following inequalities hold:

$$-|\lambda_1| \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Such an $n$-tuple $\lambda$ is called the highest weight of the corresponding representation which we shall denote by $V(\lambda)$. It contains a unique, up to a multiple, nonzero vector $\xi$ (the highest vector) such that $F_{ii}\,\xi = \lambda_i\,\xi$ for $i = 1, \dots, n$ and $F_{ij}\,\xi = 0$ for $-n \leq i < j \leq n$.

Denote by $V(\lambda)^+$ the subspace of $\mathfrak{g}_{n-1}$-highest vectors in $V(\lambda)$:

$$V(\lambda)^+ = \{\eta \in V(\lambda) \mid F_{ij}\,\eta = 0, \qquad -n < i < j < n\}.$$

Given a $\mathfrak{g}_{n-1}$-highest weight $\mu = (\mu_1, \dots, \mu_{n-1})$ we denote by $V(\lambda)^+_\mu$ the corresponding weight subspace in $V(\lambda)^+$:

$$V(\lambda)^+_\mu = \{\eta \in V(\lambda)^+ \mid F_{ii}\,\eta = \mu_i\,\eta, \qquad i = 1, \dots, n-1\}.$$

Consider the extension of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_n)$

$$\mathrm{U}'(\mathfrak{g}_n) = \mathrm{U}(\mathfrak{g}_n) \otimes_{\mathrm{U}(\mathfrak{h})} \mathrm{R}(\mathfrak{h}),$$

where $\mathrm{R}(\mathfrak{h})$ is the field of fractions of the commutative algebra $\mathrm{U}(\mathfrak{h})$. Let $\mathrm{J}$ denote the left ideal in $\mathrm{U}'(\mathfrak{g}_n)$ generated by the elements $F_{ij}$ with $-n < i < j < n$. Set

$$\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1}) = \{x \in \mathrm{U}'(\mathfrak{g}_n)/\mathrm{J} \mid F_{ij}\,x \equiv 0, \quad -n < i < j < n\}.$$

Then $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ is an algebra with the multiplication inherited from $\mathrm{U}'(\mathfrak{g}_n)$. We call it the *transvector algebra*; see [17, 18] for further details.

Set

$$f_i = F_{ii} - i + 1, \qquad f_{-i} = -f_i$$

for $i = 1, \dots, n$. Let $p$ denote the *extremal projection* for the Lie algebra $\mathfrak{g}_{n-1}$; see [1, 18]. The projection $p$ naturally acts in the space $\mathrm{U}'(\mathfrak{g}_n)/\mathrm{J}$ and its image coincides with $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$. The elements

$$pF_{ia}, \qquad\qquad a = -n, n, \qquad i = -n+1, \dots, n-1$$

are generators of $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ [18]. They can be given by the following explicit formulas (modulo J):

$$pF_{ia} = \sum_{i > i_1 > \cdots > i_s > -n} F_{ii_1} F_{i_1 i_2} \cdots F_{i_{s-1} i_s} F_{i_s a} \frac{1}{(f_i - f_{i_1}) \cdots (f_i - f_{i_s})},$$

where $s = 0, 1, \dots$ (it is assumed that index 0 is excluded in the sum). We shall use the normalized generators of $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ defined by

$$
\begin{aligned}
z_{ia} &= pF_{ia}(f_i - f_{i-1}) \cdots (\widehat{f_i - f_{-i}}) \cdots (f_i - f_{-n+1}), \\
z_{ai} &= pF_{ai}(f_i - f_{i+1}) \cdots (\widehat{f_i - f_{-i}}) \cdots (f_i - f_{n-1}),
\end{aligned}
\tag{2.2}
$$

where the hats indicate the factors to be omitted if they occur. We obviously have $z_{ai} = (-1)^{n-i}\, z_{-i,-a}$. The following equivalent formula holds for $z_{ai}$:

$$
\begin{aligned}
z_{ai} = (f_i - f_{i+1}) \cdots (\widehat{f_i - f_{-i}}) \cdots (f_i - f_{n-1}) \\
\times \sum_{n > i_1 > \cdots > i_s > i} \frac{1}{(f_i - f_{i_1}) \cdots (f_i - f_{i_s})} F_{a i_1} F_{i_1 i_2} \cdots F_{i_{s-1} i_s} F_{i_s i}.
\end{aligned}
\tag{2.3}
$$

The elements $z_{ia}$ and $z_{ai}$ naturally act in the space $V(\lambda)^+$ and are called the *raising* and *lowering operators*. One has for $i = 1, \dots, n-1$:

$$
z_{ia} : V(\lambda)^+_\mu \to V(\lambda)^+_{\mu+\delta_i}, \qquad z_{ai} : V(\lambda)^+_\mu \to V(\lambda)^+_{\mu-\delta_i},
$$

where $\mu \pm \delta_i$ is obtained from $\mu$ by replacing $\mu_i$ with $\mu_i \pm 1$.

Note the following relations between these operators; cf. [18]. For $a, b \in \{-n, n\}$ and $i + j \neq 0$ one has

$$
z_{aj} z_{bi} (f_i - f_j + 1) = z_{bi} z_{aj} (f_i - f_j) + z_{ai} z_{bj}.
$$

In particular, $z_{ai}$ and $z_{aj}$ commute for $i + j \neq 0$. One easily verifies that $z_{ai}$ and $z_{bi}$ also commute for all $a, b$. We shall use the following element which can be checked to belong to the algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$

$$
z_{n,-n} = \sum_{n > i_1 > \cdots > i_s > -n} F_{n i_1} F_{i_1 i_2} \cdots F_{i_s, -n} \frac{(f_n - f_{j_1}) \cdots (f_n - f_{j_k})}{2 f_n},
\tag{2.4}
$$

where $s = 1, 2, \dots$ and $\{j_1, \dots, j_k\}$ is the complement to the subset $\{i_1, \dots, i_s\}$ in $\{-n+1, \dots, n-1\}$. There is an equivalent formula for $z_{n,-n}$ which can either be proved directly (cf. [9, Section 2]), or can be deduced from (4.6) below (use the fact that $Z_{n,-n}(g_n) = Z_{n,-n}(-g_n)$):

$$
z_{n,-n} = \sum_{n > i_1 > \cdots > i_s > -n} F_{n i_1} F_{i_1 i_2} \cdots F_{i_s, -n} \frac{(f_{-n} - f_{j_1} - 1) \cdots (f_{-n} - f_{j_k} - 1)}{2 f_{-n} - 2}.
$$

This formula together with (2.3) is used in the derivation of the following relations from (2.2) (cf. [9, Proposition 2.1]): for $a=-n,n$

$$F_{n-1,a}=\sum_{i=-n+1}^{n-1} z_{n-1,i}\, z_{ia} \prod_{j=-n+1,\, j\neq\pm i}^{n-1} \frac{1}{f_i-f_j}, \tag{2.5}$$

where $z_{n-1,n-1}:=1$ and the equalities are considered in $\mathrm{U}'(\mathfrak{g}_n)$ modulo the ideal J.

Let us now introduce the $\mathfrak{gl}(2)$-*Yangian* $\mathrm{Y}(2)$ and the (*orthogonal*) *twisted Yangian* $\mathrm{Y}^{+}(2)$; see [10] for more details. The Yangian $\mathrm{Y}(2)$ is the complex associative algebra with the generators $t_{ab}^{(1)},t_{ab}^{(2)},\dots$ where $a,b\in\{-n,n\}$, and the defining relations

$$[t_{ab}(u),t_{cd}(v)]=\frac{1}{u-v}\Big(t_{cb}(u)t_{ad}(v)-t_{cb}(v)t_{ad}(u)\Big), \tag{2.6}$$

where

$$t_{ab}(u):=\delta_{ab}+t_{ab}^{(1)}u^{-1}+t_{ab}^{(2)}u^{-2}+\cdots\in \mathrm{Y}(2)[[u^{-1}]].$$

Introduce the series $s_{ab}(u)$, $a,b\in\{-n,n\}$ by

$$s_{ab}(u)=t_{an}(u)t_{-b,-n}(-u)+t_{a,-n}(u)t_{-b,n}(-u).$$

Write $s_{ab}(u)=\delta_{ab}+s_{ab}^{(1)}u^{-1}+s_{ab}^{(2)}u^{-2}+\cdots$. The twisted Yangian $\mathrm{Y}^{+}(2)$ is defined as the subalgebra of $\mathrm{Y}(2)$ generated by the elements $s_{ab}^{(1)},s_{ab}^{(2)},\dots$ where $a,b\in\{-n,n\}$. Note that $\mathrm{Y}^{+}(2)$ can be equivalently defined as an abstract algebra with these generators and certain linear and quadratic defining relations; see [10, Section 3].

The Yangian $\mathrm{Y}(2)$ is a Hopf algebra with the coproduct

$$\Delta(t_{ab}(u))=t_{an}(u)\otimes t_{nb}(u)+t_{a,-n}(u)\otimes t_{-n,b}(u). \tag{2.7}$$

The twisted Yangian $\mathrm{Y}^{+}(2)$ is a left coideal in $\mathrm{Y}(2)$ with

$$\Delta(s_{ab}(u))=\sum_{c,d\in\{-n,n\}} t_{ac}(u)t_{-b,-d}(-u)\otimes s_{cd}(u). \tag{2.8}$$

Given a pair of complex numbers $(\alpha,\beta)$ such that $\alpha-\beta\in\mathbb{Z}_+$ we denote by $L(\alpha,\beta)$ the irreducible representation of the Lie algebra $\mathfrak{gl}(2)$ with the highest weight $(\alpha,\beta)$ with respect to the upper triangular Borel subalgebra. We have $\dim L(\alpha,\beta)=\alpha-\beta+1$. We may regard $L(\alpha,\beta)$ as a $\mathrm{Y}(2)$-module by using the algebra homomorphism $\mathrm{Y}(2)\to\mathrm{U}(\mathfrak{gl}(2))$ given by

$$t_{ab}(u)\mapsto\delta_{ab}+E_{ab}u^{-1},\qquad a,b\in\{-n,n\}. \tag{2.9}$$

The coproduct (2.7) allows one to construct representations of $\mathrm{Y}(2)$ of the form

$$L = L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_k, \beta_k).$$

For any $\gamma \in \mathbb{C}$ denote by $W(\gamma)$ the one-dimensional representation of $\mathrm{Y}^+(2)$ spanned by a vector $w$ such that

$$s_{nn}(u)\, w = \frac{u+\gamma}{u+1/2}\, w, \qquad s_{-n,-n}(u)\, w = \frac{u-\gamma+1}{u+1/2}\, w,$$

and $s_{a,-a}(u)\, w = 0$ for $a = -n, n$. By (2.8) we can regard the tensor product $L \otimes W(\gamma)$ as a representation of $\mathrm{Y}^+(2)$. Representations of this type essentially exhaust all finite-dimensional irreducible representations of $\mathrm{Y}^+(2)$ [8]. The vector space isomorphism

$$L \otimes W(\gamma) \to L, \qquad v \otimes w \mapsto v, \qquad v \in L \tag{2.10}$$

provides $L \otimes W(\gamma)$ with an action of $\mathrm{Y}(2)$.

# 3 Construction of the basis

Introduce the following series with coefficients in the transvector algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$: for $a, b \in \{-n, n\}$

$$\begin{aligned} Z_{ab}(u) = \Bigg( - \Big(\delta_{ab}(u - n + 3/2) + F_{ab}\Big) \prod_{i=-n+1}^{n-1} (u + g_i) \\ + \sum_{i=-n+1}^{n-1} z_{ai} z_{ib}\, (u + g_{-i}) \prod_{j=-n+1,\, j \neq \pm i}^{n-1} \frac{u + g_j}{g_i - g_j} \Bigg) \frac{1}{2u+1}, \end{aligned} \tag{3.1}$$

where $g_i := f_i + 1/2$ for all $i$.

As we shall see below (Corollary 3.3) the space $V(\lambda)^+_\mu$ is nonzero only if there exist $\nu_1, \dots, \nu_{n-1}$ such that the inequalities (3.10) hold. We shall be assuming that this condition is satisfied.

**Proposition 3.1** (i) *The mapping*

$$s_{ab}(u) \mapsto -2u^{-2n+2}\, Z_{ab}(u), \qquad a, b \in \{-n, n\} \tag{3.2}$$

*defines an algebra homomorphism* $\mathrm{Y}^+(2) \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$.

(ii) *The representation of* $\mathrm{Y}^+(2)$ *in the space* $V(\lambda)^+_\mu$ *defined via the homomorphism (3.2) is irreducible.*

*Proof.* We use the same arguments as for the proof of the corresponding statements in the symplectic case; see [9, Section 5]. So we shall only give a few key formulas; the details can be restored by using [9].

Introduce the $2n \times 2n$-matrix $F = (F_{ij})$ whose $ij$th entry is the element $F_{ij} \in \mathfrak{g}_n$ and set

$$F(u) = 1 + \frac{F}{u + 1/2}.$$

Denote by $\widehat{F}(u)$ the corresponding *Sklyanin comatrix*; see [8, Section 2]. The mapping

$$s_{ab}(u) \mapsto c(u)\, \widehat{F}(-u + n - 1)_{ab}, \qquad a, b \in \{-n, n\}, \tag{3.3}$$

where

$$c(u) = \prod_{k=1}^{n-1} (1 - (k - 1/2)^2\, u^{-2}),$$

defines an algebra homomorphism from $\mathrm{Y}^+(2)$ to the centralizer $\mathrm{C}_n$ of $\mathfrak{g}_{n-1}$ in $\mathrm{U}(\mathfrak{g}_n)$ [8, Proposition 2.1]; cf. [13]. Further, a slight generalization of [9, Proposition 3.1] implies the following expression for the $ab$-entries of the matrix $\widehat{F}(u + 1/2)$:

$$\widehat{F}(u + 1/2)_{ab} = \left( \delta_{ab} - \sum_{k=1}^{\infty} F_{ab}^{(k)}\, u^{-k} \right) \cdot \operatorname{sdet} F^{(n-1)}(u - 1/2).$$

Here $\operatorname{sdet} F^{(n-1)}(u)$ is the *Sklyanin determinant* of the matrix obtained from $F(u)$ by deleting the $(\pm n)$th rows and columns [8, Section 2], and

$$F_{ab}^{(k)} = \sum F_{a i_1} F_{i_1 i_2} \cdots F_{i_{k-1} b},$$

summed over the indices $i_m \in \{-n + 1, \ldots, n - 1\}$. Finally, calculating the images of $F_{ab}^{(k)}$ and $\operatorname{sdet} F^{(n-1)}(u)$ with respect to the natural homomorphism $\pi : \mathrm{C}_n \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ (cf. [9, Section 5]), we find that the composition of $\pi$ and (3.3) yields (3.2). □

The next theorem provides an identification of the $\mathrm{Y}^+(2)$-module $V(\lambda)^+_\mu$.

**Theorem 3.2** *We have an isomorphism of* $\mathrm{Y}^+(2)$*-modules*

$$V(\lambda)^+_\mu \simeq L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_{n-1}, \beta_{n-1}) \otimes W(-\alpha_0) \tag{3.4}$$

*where* $\alpha_1 = \min\{-|\lambda_1|, -|\mu_1|\} - 1/2, \quad \alpha_0 = \alpha_1 + |\lambda_1 + \mu_1|,$

$$\begin{aligned} \alpha_i &= \min\{\lambda_i, \mu_i\} - i + 1/2, & i &= 2, \dots, n-1, \\ \beta_i &= \max\{\lambda_{i+1}, \mu_{i+1}\} - i + 1/2, & i &= 1, \dots, n-1 \end{aligned}$$

*with* $\mu_n := -\infty$. *In particular,* $V(\lambda)^+_\mu$ *is equipped with an action of* $\mathrm{Y}(2)$ *defined by (2.10).*

*Proof.* Consider the following vector in $V(\lambda)^+_\mu$

$$\xi_\mu = \prod_{i=1}^{n-1} \left( z_{ni}^{\max\{\lambda_i, \mu_i\} - \mu_i} z_{i,-n}^{\max\{\lambda_i, \mu_i\} - \lambda_i} \right) \xi. \tag{3.5}$$

Repeating the arguments of the proof of Theorem 5.2 in [9] we show that $\xi_\mu$ is the highest vector of the $\mathrm{Y}^+(2)$-module $V(\lambda)^+_\mu$. That is, $\xi_\mu$ is annihilated by $s_{-n,n}(u)$, and $\xi_\mu$ is an eigenvector for $s_{nn}(u)$. Namely, $s_{nn}(u)\xi_\mu = \mu(u)\xi_\mu$, where the highest weight $\mu(u)$ is given by

$$\mu(u) = (1 - \alpha_0 u^{-1}) \cdots (1 - \alpha_{n-1} u^{-1})(1 + \beta_1 u^{-1}) \cdots (1 + \beta_{n-1} u^{-1})(1 + \frac{1}{2} u^{-1})^{-1}. \tag{3.6}$$

This is proved simultaneously with the following relations by induction on the degree of the monomial in (3.5): for $i = 1, \dots, n-1$

$$\begin{aligned} z_{in}\, \xi_\mu = -(m_i + \widetilde{\alpha}_1 + 1) \cdots (\widehat{m_i + \alpha_i} + 1) \cdots (m_i + \alpha_{n-1} + 1) \\ \times (m_i - \beta_0 + 1) \cdots (m_i - \beta_{n-1} + 1)\, \xi_{\mu + \delta_i}, \end{aligned}$$

and

$$z_{-ni}\, \xi_\mu = -(m_i - \widetilde{\alpha}_1) \cdots (m_i - \alpha_{n-1})(m_i + \beta_0) \cdots (\widehat{m_i + \beta_{i-1}}) \cdots (m_i + \beta_{n-1})\, \xi_{\mu - \delta_i},$$

where we have used the notation

$$\begin{aligned} m_i &= \mu_i - i + 1/2, \qquad i = 1, \dots, n-1, \\ \widetilde{\alpha}_1 &= \min\{\lambda_1, \mu_1\} - 1/2, \qquad \beta_0 = \max\{\lambda_1, \mu_1\} + 1/2, \end{aligned}$$

(note that $\{\widetilde{\alpha}_1, -\beta_0\} = \{\alpha_0, \alpha_1\}$). On the other hand, it follows from [8, Corollary 6.6] that the tensor product in (3.4) is an irreducible representation of $\mathrm{Y}^+(2)$. Its highest weight can be easily calculated and is given by the same formula (3.6). $\square$

Set $T_{ab}(u) = u^{n-1}\, t_{ab}(u)$ for $a, b \in \{-n, n\}$. By (2.7) and (2.9), $T_{ab}(u)$, as an operator in $V(\lambda)^+_\mu$, is a polynomial in $u$:

$$T_{ab}(u) = \delta_{ab} u^{n-1} + t^{(1)}_{ab} u^{n-2} + \cdots + t^{(n-1)}_{ab}. \tag{3.7}$$

By (2.6), (2.8) and (3.2) we have an equality of operators in $V(\lambda)^+_\mu$:

$$Z_{n,-n}(u) = \frac{(u-\alpha_0) T_{n,-n}(-u) T_{nn}(u) + (u+\alpha_0) T_{n,-n}(u) T_{nn}(-u)}{(-1)^n\, 2u}. \tag{3.8}$$

Therefore, $Z_{n,-n}(u)$ is a polynomial in $u^2$ of degree $n-2$. On the other hand, we find from (3.1) that $Z_{n,-n}(-g_i) = z_{ni} z_{i,-n}$. Thus, by the Lagrange interpolation formula, $Z_{n,-n}(u)$ can also be given by

$$Z_{n,-n}(u) = \sum_{i=1}^{n-1} z_{ni} z_{i,-n} \prod_{j=1,\, j\neq i}^{n-1} \frac{u^2 - g_j^2}{g_i^2 - g_j^2}. \tag{3.9}$$

*Remark.* To make the above evaluation $Z_{n,-n}(-g_i)$ well-defined we agree to consider the series $Z_{ab}(u)$ with $a, b \in \{-n, n\}$ as elements of the *right* module over the field of rational functions in $\mathfrak{h}$ and $u$ generated by monomials in the $z_{ia}$. □

Theorem 3.2 implies that basis vectors of $V(\lambda)^+_\mu$ can be naturally parametrized by $(n-1)$-tuples $(\nu_1, \dots, \nu_{n-1})$, where all the entries are simultaneously integers or half-integers together with the $\lambda_i$ and the $\mu_i$, and the following inequalities hold:

$$\begin{aligned} -|\lambda_1| \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} \geq \nu_{n-1} \geq \lambda_n, \\ -|\mu_1| \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \mu_3 \geq \cdots \geq \mu_{n-1} \geq \nu_{n-1}. \end{aligned} \tag{3.10}$$

For $i \geq 1$ set

$$\gamma_i = \nu_i - i + 1/2, \qquad l_i = \lambda_i - i + 1/2.$$

Introduce the vectors

$$\xi_{\nu\mu} = \prod_{i=1}^{n-1} Z_{n,-n}(\gamma_i - 1) \cdots Z_{n,-n}(\beta_i + 1) Z_{n,-n}(\beta_i)\, \xi_\mu.$$

Using (3.9) we can write an equivalent expression; cf. [9, Section 6]:

$$\xi_{\nu\mu} = \prod_{i=1}^{n-1} z_{ni}^{\nu_{i-1}-\mu_i} z_{i,-n}^{\nu_{i-1}-\lambda_i} \cdot \prod_{k=l_n+1}^{\gamma_{n-1}-1} Z_{n,-n}(k)\, \xi, \tag{3.11}$$

where $\nu_0 := \max\{\lambda_1, \mu_1\}$. The vectors $\xi_{\nu\mu}$ with $\nu$ satisfying (3.10) form a basis of the space $V(\lambda)^+_\mu$; see [9, Proposition 6.1]. We shall use the following normalized basis vectors

$$\zeta_{\nu\mu} = \prod_{1\le i<j\le n-1} (-\gamma_i - \gamma_j)!\ \xi_{\nu\mu}.$$

The generators of the Yangian Y(2) act in the basis $\{\zeta_{\nu\mu}\}$ by the rule: for $i = 1, \dots, n-1$

$$\begin{aligned} T_{nn}(u)\,\zeta_{\nu\mu} &= (u+\gamma_1)\cdots(u+\gamma_{n-1})\,\zeta_{\nu\mu}, \\ T_{n,-n}(-\gamma_i)\,\zeta_{\nu\mu} &= \frac{1}{\gamma_i-\alpha_0}\,\zeta_{\nu+\delta_i,\mu}, \\ T_{-n,n}(-\gamma_i)\,\zeta_{\nu\mu} &= \prod_{k=0}^{n-1}(\alpha_k-\gamma_i+1)\prod_{k=1}^{n-1}(\beta_k-\gamma_i)\,\zeta_{\nu-\delta_i,\mu}; \end{aligned} \tag{3.12}$$

cf. [9, Proposition 4.2]. The action of $T_{-n,-n}(u)$ can be found by using the *quantum determinant*

$$d(u) = T_{-n,-n}(u+1)T_{nn}(u) - T_{n,-n}(u+1)T_{-n,n}(u) \tag{3.13}$$

$$= T_{-n,-n}(u)T_{nn}(u+1) - T_{-n,n}(u)T_{n,-n}(u+1); \tag{3.14}$$

see, e.g. [10, Section 2]. The coefficients of the quantum determinant belong to the center of Y(2) and so, $d(u)$ acts in $V(\lambda)^+_\mu$ as a scalar which can be found by the application of (3.13) to the highest weight vector $\xi_\mu$. So, we have

$$d(u)\,\zeta_{\nu\mu} = (u+\alpha_1+1)\cdots(u+\alpha_{n-1}+1)(u+\beta_1)\cdots(u+\beta_{n-1})\,\zeta_{\nu\mu}.$$

Now, using (3.12) and (3.14) we obtain

$$\begin{aligned} T_{-n,-n}(u)\,\zeta_{\nu\mu} = \prod_{i=1}^{n-1}\frac{(u+\alpha_i+1)(u+\beta_i)}{u+\gamma_i+1}\,\zeta_{\nu\mu} \\ + \prod_{i=1}^{n-1}\frac{1}{u+\gamma_i+1}\,T_{-n,n}(u)T_{n,-n}(u+1)\,\zeta_{\nu\mu}. \end{aligned} \tag{3.15}$$

The operators $T_{-n,n}(u)$ and $T_{n,-n}(u)$ are polynomials in $u$ of degree $\le n-2$; see (3.7). Therefore, their action can be found from (3.12) by using the Lagrange interpolation formula.

The following branching rule for the reduction $\mathfrak{g}_n \downarrow \mathfrak{g}_{n-1}$ is implied by Theorem 3.2; cf. [9, Corollary 5.3].

**Corollary 3.3** *The restriction of $V(\lambda)$ to the subalgebra $\mathfrak{g}_{n-1}$ is isomorphic to the direct sum $\bigoplus c(\mu)V'(\mu)$ of finite-dimensional irreducible representations $V'(\mu)$ of $\mathfrak{g}_{n-1}$ where the multiplicity $c(\mu)$ equals the number of $(n-1)$-tuples $\nu$ satisfying the inequalities (3.10).*

*Proof.* We have $c(\mu) = \dim V(\lambda)^+_\mu$. By Theorem 3.2,

$$\dim V(\lambda)^+_\mu = \prod_{i=1}^{n-1}(\alpha_i - \beta_i + 1),$$

if there exists $\nu$ satisfying (3.10). Otherwise, the space $V(\lambda)^+_\mu$ is trivial. This is proved by comparison of the dimensions of $V(\lambda)$ and $\bigoplus c(\mu)V'(\mu)$ with the use of [15, Chapter VII, Section 9]. $\square$

Applying the above construction of the vectors $\zeta_{\nu\mu}$ to the subalgebras of the chain

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n, \qquad \mathfrak{g}_k = \mathfrak{o}(2k)$$

we obtain a basis of $V(\lambda)$ parametrized by the *D-type Gelfand–Tsetlin patterns* (cf. [6]) which we denote by $\Lambda$:

$$\begin{array}{ccccccccc}
\lambda_{n1} & & \lambda_{n2} & & & \cdots & & & \lambda_{nn} \\
& \lambda'_{n-1,1} & & & \cdots & & & \lambda'_{n-1,n-1} & \\
\lambda_{n-1,1} & & & \cdots & & \lambda_{n-1,n-1} & & & \\
& & \cdots & & \cdots & & & & \\
\lambda_{21} & & \lambda_{22} & & & & & & \\
& \lambda'_{11} & & & & & & & \\
\lambda_{11} & & & & & & & &
\end{array}$$

Here the upper row coincides with $\lambda$, all the entries are simultaneously integers or half-integers and the following inequalities hold

$$-|\lambda_{k1}| \geq \lambda'_{k-1,1} \geq \lambda_{k2} \geq \lambda'_{k-1,2} \geq \cdots \geq \lambda_{k,k-1} \geq \lambda'_{k-1,k-1} \geq \lambda_{kk},$$
$$-|\lambda_{k-1,1}| \geq \lambda'_{k-1,1} \geq \lambda_{k-1,2} \geq \lambda'_{k-1,2} \geq \cdots \geq \lambda_{k-1,k-1} \geq \lambda'_{k-1,k-1}$$

for $k = 2, \ldots, n$. Set

$$l_{ki} = \lambda_{ki} - i + 1, \qquad l'_{ki} = \lambda'_{ki} - i + 1, \qquad 1 \leq i \leq k \leq n \tag{3.16}$$

and introduce the vectors

$$\xi_\Lambda = \prod_{k=2,\dots,n}^{\rightarrow} \left( \prod_{i=1}^{k-1} z_{ki}^{\lambda'_{k-1,i-1}-\lambda_{k-1,i}} z_{i,-k}^{\lambda'_{k-1,i-1}-\lambda_{ki}} \prod_{q=l_{kk}+1}^{l'_{k-1,k-1}-1} Z_{k,-k}(q-\frac{1}{2}) \right) \xi$$

with $\lambda'_{k-1,0} := \max\{\lambda_{k1}, \lambda_{k-1,1}\}$. Finally, set

$$\zeta_\Lambda = N_\Lambda \, \xi_\Lambda, \qquad N_\Lambda = \prod_{k=2}^{n-1} \prod_{1\le i<j\le k} (-l'_{ki} - l'_{kj} + 1)!$$

The following proposition is implied by Corollary 3.3.

**Proposition 3.4** *The vectors $\zeta_\Lambda$ parametrized by the Gelfand–Tsetlin patterns $\Lambda$ form a basis of the representation $V(\lambda)$.* □

# 4 Matrix element formulas

Introduce the following elements of $\mathrm{U}(\mathfrak{g}_n)$:

$$\Phi_{-k,k} = \sum_{i=1}^{k-1} F_{-k,i} F_{ik}, \qquad k = 2, \dots, n.$$

We shall find the action of $\Phi_{-k,k}$ in the basis $\{\zeta_\Lambda\}$, which will be used later on. Since $\Phi_{-k,k}$ commutes with the subalgebra $\mathfrak{g}_{k-1}$ it suffices to consider the case $k = n$. The image of $\Phi_{-n,n}$ under the natural homomorphism $\pi : \mathrm{C}_n \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ coincides with the coefficient at $u^{2n-4}$ of the polynomial $Z_{-n,n}(u)$; see the proof of Proposition 3.1. The following analog of (3.8) is obtained from (2.6), (2.8) and (3.2):

$$Z_{-n,n}(u) = \frac{(u-\alpha_0) T_{-n,-n}(-u) T_{-n,n}(u) + (u+\alpha_0) T_{-n,-n}(u) T_{-n,n}(-u)}{(-1)^n \, 2u}.$$

Therefore, we have an equality of operators in $V(\lambda)^+_\mu$:

$$\Phi_{-n,n} = -t^{(2)}_{-n,n} + t^{(1)}_{-n,n} t^{(1)}_{-n,-n} + (1+\alpha_0)\, t^{(1)}_{-n,n}. \tag{4.1}$$

The image of $s^{(1)}_{nn}$ under the homomorphism (3.3) is $F_{nn}$. On the other hand, by (2.8) we have

$$s^{(1)}_{nn} = t^{(1)}_{nn} - t^{(1)}_{-n,-n} - \alpha_0 - 1/2,$$

as operators in $V(\lambda)^+_\mu$. Therefore, (4.1) can be written as

$$\Phi_{-n,n} = -t^{(2)}_{-n,n} + t^{(1)}_{-n,n} t^{(1)}_{nn} - (F_{nn} + 3/2)\, t^{(1)}_{-n,n}.$$

Finally, relations (3.12) imply that

$$\Phi_{-n,n}\zeta_{\nu\mu} = \sum_{i=1}^{n-1} \theta_i\,(F_{nn} - \gamma_i + 3/2)\,\zeta_{\nu-\delta_i,\mu}, \tag{4.2}$$

where

$$\theta_i = -\prod_{k=0}^{n-1}(\alpha_k - \gamma_i + 1)\prod_{k=1}^{n-1}(\beta_k - \gamma_i)\prod_{j=1,\, j\neq i}^{n-1}(\gamma_j - \gamma_i)^{-1}.$$

The action of $F_{nn}$ in $V(\lambda)^+_\mu$ is immediately found so that

$$F_{nn}\,\zeta_{\nu\mu} = \left(2\sum_{i=0}^{n-1}\nu_i - \sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n-1}\mu_i\right)\zeta_{\nu\mu}.$$

*Remark.* One can introduce the elements $\Phi_{k,-k}$ by

$$\Phi_{k,-k} = \sum_{i=1}^{k-1} F_{ki}F_{i,-k}.$$

The action of $\Phi_{n,-n}$ on $V(\lambda)^+_\mu$ is found in the same way as that of $\Phi_{-n,n}$:

$$\Phi_{n,-n}\zeta_{\nu\mu} = \sum_{i=1}^{n-1}\prod_{j=1,\, j\neq i}^{n-1}\frac{1}{\gamma_j - \gamma_i}\,\zeta_{\nu+\delta_i,\mu},$$

although this will not be used. □

The operator $F_{n-1,-n}$ preserves the subspace of $\mathfrak{g}_{n-2}$-highest vectors in $V(\lambda)$. Therefore it suffices to calculate its action on the basis vectors of the form

$$\xi_{\nu\mu\nu'} = X_{\mu\nu'}\,\xi_{\nu\mu}, \tag{4.3}$$

where $X_{\mu\nu'}$ denotes the operator

$$X_{\mu\nu'} = \prod_{i=1}^{n-2} z_{n-1,i}^{\nu'_{i-1}-\mu'_i}\, z_{i,-n+1}^{\nu'_{i-1}-\mu_i} \cdot \prod_{a=m_{n-1}+1}^{\gamma'_{n-1}-1} Z_{n-1,-n+1}(a),$$

$\nu'$ and $\mu'$ are $(n-2)$-tuples of integers or half-integers such that the inequalities (3.10) are satisfied with $\lambda$, $\nu$, $\mu$ respectively replaced by $\mu$, $\nu'$, $\mu'$; we set $\gamma'_i = \nu'_i - i + 1/2$ and $\nu'_0 = \max\{\mu_1, \mu'_1\}$. The operator $F_{n-1,-n}$ is permutable with the elements $z_{n-1,i}$, $z_{i,-n+1}$ and $Z_{n-1,-n+1}(u)$ which follows from their explicit formulas. Hence, we can write

$$F_{n-1,-n}\, \xi_{\nu\mu\nu'} = X_{\mu\nu'}\, F_{n-1,-n}\, \xi_{\nu\mu}. \tag{4.4}$$

By (2.5) (with $a = -n$) we need to express

$$X_{\mu\nu'}\, z_{n-1,i} z_{i,-n}\, \xi_{\nu\mu}, \qquad i = -n+1, \dots, n-1 \tag{4.5}$$

as a linear combination of the vectors $\xi_{\nu\mu\nu'}$. If $i \neq \pm 1$ then the calculation is exactly the same as in [9, Section 6] where one uses the relations

$$Z_{n,-n}(-g_n) = z_{n,-n}, \qquad Z_{n,-n}(-g_i) = z_{ni} z_{i,-n}, \tag{4.6}$$

which follow from (2.4) and (3.9). Now consider (4.5) with $i = -1$. We have

$$X_{\mu\nu'}\, z_{n-1,-1} z_{-1,-n}\, \xi_{\nu\mu} = -X_{\mu\nu'}\, z_{1,-n+1} z_{n1}\, \xi_{\nu\mu}.$$

If $\lambda_1 \geq \mu_1$ then $z_{n1}\, \xi_{\nu\mu} = \xi_{\nu,\mu-\delta_1}$ while for $\lambda_1 < \mu_1$ we derive from (4.6) that

$$z_{n1}\, \xi_{\nu\mu} = \sum_{i=1}^{n-1} \prod_{a=1,\, a\neq i}^{n-1} \frac{m_1^2 - \gamma_a^2}{\gamma_i^2 - \gamma_a^2}\, \xi_{\nu+\delta_i,\mu-\delta_1}.$$

Similarly, if $\mu'_1 \geq \mu_1$ then $X_{\mu\nu'}\, z_{1,-n+1} = X_{\mu-\delta_1,\nu'}$ while for $\mu'_1 < \mu_1$ one has

$$X_{\mu\nu'}\, z_{1,-n+1} = \sum_{r=1}^{n-2} \prod_{a=1,\, a\neq r}^{n-2} \frac{m_1^2 - {\gamma'_a}^2}{{\gamma'_r}^2 - {\gamma'_a}^2}\, X_{\mu-\delta_1,\nu'+\delta_r}.$$

Finally, take $i = 1$ in (4.5). If $\lambda_1 \leq \mu_1$ then $z_{1,-n}\, \xi_{\nu\mu} = \xi_{\nu,\mu+\delta_1}$, and if $\lambda_1 > \mu_1$ then

$$z_{1,-n}\, \xi_{\nu\mu} = \sum_{i=1}^{n-1} \prod_{a=1,\, a\neq i}^{n-1} \frac{(m_1+1)^2 - \gamma_a^2}{\gamma_i^2 - \gamma_a^2}\, \xi_{\nu+\delta_i,\mu+\delta_1}.$$

Similarly, if $\mu'_1 \leq \mu_1$ then $X_{\mu\nu'}\, z_{n-1,1} = X_{\mu+\delta_1,\nu'}$ and if $\mu'_1 > \mu_1$ then

$$X_{\mu\nu'}\, z_{n-1,1} = \sum_{r=1}^{n-2} \prod_{a=1,\, a\neq r}^{n-2} \frac{(m_1+1)^2 - {\gamma'_a}^2}{{\gamma'_r}^2 - {\gamma'_a}^2}\, X_{\mu+\delta_1,\nu'+\delta_r}.$$

The action of the elements $F_{n-1,n}$ on the vectors (4.3) can be expressed in two different ways. First we sketch a calculation similar to the one used above which leads to (rather complicated) explicit formulas for the matrix elements. Then we give slightly less explicit but more convenient formulas where $F_{n-1,n}$ is represented by a commutator-like expression of simpler operators.

We have the following analog of (4.4):

$$F_{n-1,n}\,\xi_{\nu\mu\nu'} = X_{\mu\nu'}\,F_{n-1,n}\,\xi_{\nu\mu}. \tag{4.7}$$

Now use (2.5) with $a = n$. Here we need to calculate $z_{in}\,\xi_{\nu\mu}$ instead of $z_{i,-n}\,\xi_{\nu\mu}$ in the previous case. Suppose that $i > 1$. We have

$$z_{in}\,\xi_{\nu\mu} = z_{in}z_{ni}\,\xi_{\nu,\mu+\delta_i} = z_{-n,-i}z_{-i,-n}\,\xi_{\nu,\mu+\delta_i} = Z_{-n,-n}(-g_{-i})\,\xi_{\nu,\mu+\delta_i};$$

see (3.1). However,

$$-g_{-i}\,\xi_{\nu,\mu+\delta_i} = (m_i+1)\,\xi_{\nu,\mu+\delta_i}.$$

To calculate $Z_{-n,-n}(m_i+1)\,\xi_{\nu,\mu+\delta_i}$ we use the following equality of operators in $V(\lambda)^+_\mu$:

$$Z_{-n,-n}(u) = \frac{(u-\alpha_0)T_{-n,n}(u)T_{n,-n}(-u) + (u+\alpha_0+1)T_{-n,-n}(u)T_{nn}(-u)}{(-1)^n\,(2u+1)},$$

see (2.6), (2.8) and (3.2); and then apply formulas (3.12) and (3.15).

To calculate $z_{-i,n}\,\xi_{\nu\mu}$ we first permute $z_{-i,n}$ with the generators $z_{nj}$ and $z_{j,-n}$ with $j = 1,\dots,i-1$ in (3.11). Further, we use the relation

$$z_{-i,n}z_{i,-n} = (-1)^{n-i}z_{-n,i}z_{i,-n} = (-1)^{n-i}Z_{-n,-n}(-g_i)$$

and complete the calculation in a similar manner. To find $z_{\pm 1,n}\,\xi_{\nu\mu}$ we need to consider a few different cases which depend on the relationship between the parameters $\lambda_1$, $\mu_1$ and $\mu'_1$ and then proceed exactly as above in the calculation of the action of $F_{n-1,-n}$.

We now give an alternative way of computing the action of $F_{n-1,n}$. The basic idea is to replace the operator $z_{in}$ in the above calculation of $z_{in}\,\xi_{\nu\mu}$ by the following expression: for $i = -n+1,\dots,n-1$

$$z_{in} = [z_{i,-n}, \Phi_{-n,n}]\,\frac{1}{f_i + F_{nn}} \tag{4.8}$$

and then use the formulas for the action of $z_{i,-n}$ and $\Phi_{-n,n}$; see (4.2). More precisely, we regard (4.8) as a relation in the transvector algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ which can be proved as follows. First, we calculate the commutator $[F_{i,-n}, \Phi_{-n,n}]$ in $\mathrm{U}(\mathfrak{g}_n)$ then consider it modulo the ideal J and apply the extremal projection $p$ (see Section 2).

We have $\Phi_{-n,n} F_{nn} = (F_{nn}+2)\,\Phi_{-n,n}$ and so, (2.5), (4.7) and (4.8) imply that

$$F_{n-1,n}\,\xi_{\nu\mu\nu'} = X_{\mu\nu'}\,\left(\Phi_{n-1,-n}(2)\,\Phi_{-n,n} - \Phi_{-n,n}\Phi_{n-1,-n}(0)\right)\xi_{\nu\mu}, \tag{4.9}$$

where

$$\Phi_{n-1,-n}(u) = \sum_{i=-n+1}^{n-1} z_{n-1,i}\, z_{i,-n} \prod_{a=-n+1,\, a\neq \pm i}^{n-1} \frac{1}{f_i - f_a} \cdot \frac{1}{u + f_i + F_{nn}}.$$

The action of $\Phi_{n-1,-n}(u)$ is found exactly as that of $F_{n-1,-n}$. Note that the operators $X_{\mu\nu'}$ and $\Phi_{-n,n}$ commute.

*Remark.* The operator $\Phi_{n-1,-n}(u)$ is a rational function in $u$ which can have singularities at the values $u=0$ and $u=2$ in (4.9). However, the operator

$$\Phi_{n-1,-n}(u+2)\,\Phi_{-n,n} - \Phi_{-n,n}\Phi_{n-1,-n}(u)$$

is regular at $u=0$ and coincides with $F_{n-1,n}$. Note the similarity with the symplectic case [9], where we have $2F_{n-1,n} = [F_{n-1,-n}, F_{-n,n}]$. □

The elements $F_{k-1,-k}$, $F_{k-1,k}$ with $k = 2,\dots,n$ and $F_{21}$, $F_{-2,1}$ generate $\mathfrak{g}_n$ as a Lie algebra. Summarizing the above calculations we obtain the following formulas for the matrix elements of the generators. Given a pattern $\Lambda$ we use the notation (3.16) and set for $1 \le i < k \le n$:

$$A_{ki} = \prod_{a=1,\, a\neq i}^{k-1} \frac{1}{l_{k-1,i}^2 - l_{k-1,a}^2},$$

$$B_{ki}(x) = \prod_{a=1,\, a\neq i}^{k-1} \frac{(x + l'_{k-1,a})(x - l'_{k-1,a} + 1)}{l'_{k-1,a} - l'_{k-1,i}},$$

and

$$\begin{aligned} C_{ki} = &(\max\{\lambda_{k1}, \lambda_{k-1,1}\} + l'_{k-1,i} - 1)(\min\{\lambda_{k1}, \lambda_{k-1,1}\} - l'_{k-1,i} + 1) \\ &\times \prod_{a=2}^{k} (l_{ka} - l'_{k-1,i} + 1) \prod_{a=2}^{k-1} (l_{k-1,a} - l'_{k-1,i} + 1) \prod_{a=1,\, a\neq i}^{k-1} \frac{1}{l'_{k-1,a} - l'_{k-1,i}}. \end{aligned}$$

We denote by $\Lambda \pm \delta_{ki}$ and $\Lambda \pm \delta'_{ki}$ the arrays obtained from $\Lambda$ by replacing $\lambda_{ki}$ and $\lambda'_{ki}$ by $\lambda_{ki} \pm 1$ and $\lambda'_{ki} \pm 1$ respectively. Consider the basis $\{\zeta_\Lambda\}$ of the representation $V(\lambda)$; see Proposition 3.4. We shall suppose that $\zeta_\Lambda = 0$ if the array $\Lambda$ is not a pattern.

**Theorem 4.1** *The action of the generators of the Lie algebra $\mathfrak{o}(2n)$ in the vasis $\{\zeta_\Lambda\}$ is given by the following formulas.*

$$F_{kk}\,\zeta_\Lambda = \left(2\sum_{i=1}^{k}\lambda'_{k-1,i-1} - \sum_{i=1}^{k}\lambda_{ki} - \sum_{i=1}^{k-1}\lambda_{k-1,i}\right)\zeta_\Lambda,$$

$$F_{k-1,-k}\,\zeta_\Lambda = \sum_{i=1}^{k-1} A_{ki}\left(\zeta_\Lambda^+(k,i) - \zeta_\Lambda^-(k,i)\right).$$

*Here*

$$\zeta_\Lambda^+(k,i) = \sum_{j=1}^{k-1}\sum_{m=1}^{k-2} B_{kj}(l_{k-1,i})B_{k-1,m}(l_{k-1,i})\,\zeta_{\Lambda+\delta'_{k-1,j}+\delta_{k-1,i}+\delta'_{k-2,m}}$$

*for $i=2,\ldots,k-1$; and for $i=1$ if $\lambda_{k-1,1}<\lambda_{k1},\lambda_{k-2,1}$. Otherwise,*

$$\begin{aligned}
\zeta_\Lambda^+(k,1) &= \zeta_{\Lambda+\delta_{k-1,1}} && \textit{if}\quad \lambda_{k-1,1}\ge\lambda_{k1},\lambda_{k-2,1},\\
&= \sum_{j=1}^{k-1} B_{kj}(l_{k-1,1})\,\zeta_{\Lambda+\delta'_{k-1,j}+\delta_{k-1,1}} && \textit{if}\quad \lambda_{k-2,1}\le\lambda_{k-1,1}<\lambda_{k1},\\
&= \sum_{m=1}^{k-2} B_{k-1,m}(l_{k-1,1})\,\zeta_{\Lambda+\delta_{k-1,1}+\delta'_{k-2,m}} && \textit{if}\quad \lambda_{k1}\le\lambda_{k-1,1}<\lambda_{k-2,1}.
\end{aligned}$$

*Furthermore,*

$$\zeta_\Lambda^-(k,i) = \zeta_{\Lambda-\delta_{k-1,i}}$$

*for $i=2,\ldots,k-1$; and for $i=1$ if $\lambda_{k-1,1}\le\lambda_{k1},\lambda_{k-2,1}$. Otherwise,*

$$\begin{aligned}
\zeta_\Lambda^-(k,1) &= \sum_{j=1}^{k-1} B_{kj}(l_{k-1,1}-1)\,\zeta_{\Lambda+\delta'_{k-1,j}-\delta_{k-1,1}} && \textit{if}\quad \lambda_{k1}<\lambda_{k-1,1}\le\lambda_{k-2,1},\\
&= \sum_{m=1}^{k-2} B_{k-1,m}(l_{k-1,1}-1)\,\zeta_{\Lambda-\delta_{k-1,1}+\delta'_{k-2,m}} && \textit{if}\quad \lambda_{k-2,1}<\lambda_{k-1,1}\le\lambda_{k1},\\
&= \sum_{j=1}^{k-1}\sum_{m=1}^{k-2} B_{kj}(l_{k-1,1}-1)B_{k-1,m}(l_{k-1,1}-1)\zeta_{\Lambda+\delta'_{k-1,j}-\delta_{k-1,1}+\delta'_{k-2,m}} \\
& && \textit{if}\quad \lambda_{k-1,1}>\lambda_{k1},\lambda_{k-2,1}.
\end{aligned}$$

*The action of $F_{k-1,k}$ is found from the relation*

$$F_{k-1,k} = \Big[\Phi_{k-1,-k}(u+2)\,\Phi_{-k,k} - \Phi_{-k,k}\Phi_{k-1,-k}(u)\Big]_{u=0},$$

*where*

$$\Phi_{-k,k}\,\zeta_\Lambda = \sum_{i=1}^{k-1} C_{ki}\,(F_{kk} - l'_{k-1,i} + 2)\,\zeta_{\Lambda-\delta'_{k-1,i}}$$

*and*

$$\Phi_{k-1,-k}(u)\,\zeta_\Lambda = \sum_{i=1}^{k-1} A_{ki}\left(\frac{1}{u + l_{k-1,i} + F_{kk} - 1}\,\zeta_\Lambda^+(k,i) - \frac{1}{u - l_{k-1,i} + F_{kk} - 1}\,\zeta_\Lambda^-(k,i)\right).$$

□

*Example.* Let $n = 2$. We have $z_{21} = F_{21}$, $z_{1,-2} = F_{1,-2}$ and $Z_{2,-2}(u) = F_{21}F_{1,-2}$. Therefore, the basis vectors are given by

$$\zeta_\Lambda = F_{21}^{\lambda'_{10}-\lambda_{11}} F_{1,-2}^{\lambda'_{10}-\lambda_{21}} (F_{21}F_{1,-2})^{\lambda'_{11}-\lambda_{22}}\,\xi,$$

where $\lambda'_{10} = \max\{\lambda_{21}, \lambda_{11}\}$. The Lie algebra $\mathfrak{o}(4)$ is isomorphic to the direct sum of two copies of $\mathfrak{sl}(2)$ and the action of their generators in the basis $\{\zeta_\Lambda\}$ is easily found. The resulting formulas also hold for the action of the elements of the subalgebra $\mathfrak{g}_2 \subset \mathfrak{g}_n$ in the basis $\{\zeta_\Lambda\}$ of the $\mathfrak{g}_n$-module $V(\lambda)$. □

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