REMARKS ON FINITELY PRESENTED BANDS

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Abstract. For each variety of bands $\mathcal{V}$, we give a formula for $\phi_{\mathcal{V}}(m, k)$, which is the largest integer such that for every band B in $\mathcal{V}$ generated by m generators and k relations, there is a subset of the generators of size $\phi_{\mathcal{V}}(m, k)$ which generates a (relatively) free sub-band of B as a basis. We also determine the semilattice structure of a finitely presented band and the complete structure of a finitely presented normal band.

1. Introduction and Preliminaries. Let $\mathcal{V}$ be a variety of algebras. For integers $m > k$, we define $\phi_{\mathcal{V}}(m, k)$ to be the largest integer such that for every algebra B in $\mathcal{V}$ presented by m generators $a_1, \ldots, a_m$ and k relations, it is possible to choose $\phi_{\mathcal{V}}(m, k)$ elements from among $a_1, \ldots, a_m$ which generate a (relatively) free subalgebra of B as a basis.

The question of finding $\phi_{\mathcal{V}}$ for the variety of groups was posed by Lyndon ([**7**, Question 5.29]). Romanovskii [**8**] showed that $\phi_{\mathcal{V}}(m, k) = m - k$ in this case, but the proof is not constructive. For the variety of semigroups Shneerson [**10**] gave a constructive proof, based on linear algebra, that $\phi_{\mathcal{V}}(m, k) = m - k$. Standard linear algebra methods, applied to abelian groups (see [**5**, Chapter 10]), shows that $\phi_{\mathcal{V}}(m, k) = m - k$ for the variety of abelian groups also. This implies, by an easy exercise, that $\phi_{\mathcal{V}}(m, k) = m - k$ for the variety of commutative semigroups also.

Let $\mathcal{V}$ be a variety of algebras. Let A be an alphabet and R a set of relations. Then $\langle A \mid R \rangle_{\mathcal{V}}$ will denote the algebra in $\mathcal{V}$ generated by A, subject to the relations of R. The subscript $\mathcal{V}$ is sometimes dropped when it is clear from context. We record the following properties of $\phi_{\mathcal{V}}$.

LEMMA 1.1. *For any variety of algebras* $\mathcal{V}$,

(i) $\phi_{\mathcal{V}}(m_1 + m_2, k_1 + k_2) \le \phi_{\mathcal{V}}(m_1, k_1) + \phi_{\mathcal{V}}(m_2, k_2)$, *and*

(ii) $\phi_{\mathcal{V}}(m, k) \le \phi_{\mathcal{V}}(m - 1, k - 1)$.

Proof. If $S_1 = \langle A_1 \mid R_1 \rangle_{\mathcal{V}}$ and $S_2 = \langle A_2 \mid R_2 \rangle_{\mathcal{V}}$ are two algebras defined with disjoint alphabets, then $A_0 \subseteq A_1 \cup A_2$ generates a relatively free subalgebra as a basis in $\langle A_1 \cup A_2 \mid R_1 \cup R_2 \rangle_{\mathcal{V}}$ only if $A_0 \cap A_i$ does so in S_i for $i = 1, 2$. Hence the first inequality.

Suppose $S = \langle A \mid R \rangle_{\mathcal{V}}$. Choose $a \in A$ and a letter b not in A. We prove (ii) by considering $\langle A, b \mid R, a = b \rangle_{\mathcal{V}}$, using similar reasoning to above. $\square$

In this paper, we give a formula for $\phi_{\mathcal{V}}(m, k)$ for each variety of bands $\mathcal{V}$. Section 2 proves a combinatorial lemma which is the main ingredient of the proof of the formula. Section 3 gives an upper bound on $\phi_{\mathcal{V}}(m, k)$ in the case of semilattices. The formula is proved in Section 4 by first elucidating the semilattice structure of a finitely presented band. Finally, the results in Section 4 are improved in Section 5 to give complete information on the structure of a finitely presented normal band.

We shall use the following notation: the free semigroup over a set A is denoted by F_A. If $w \in F_A^1$ we define its *content* $C(w)$ to be the set of elements of A appearing in w. If $w \in F_A$, denote its initial and terminal letters by $i(w)$ and $t(w)$ respectively. Let P_A be the semilattice of finite subsets of A under union.

2. A Combinatorial Lemma.

LEMMA 2.1. *Let A be a set of size m. Let $A_1, A_2, \ldots, A_k$ be subsets of A such that $|A_i| \geq l$ for each i. Then there exists a subset A' of A such that*

(i) $|A'| \leq \min\{\lfloor \frac{m+(l-1)k}{2l-1} \rfloor, k\}$,

(ii) $A' \cap A_i \neq \emptyset$ *for each* i.

Proof. Let $n = lk - m$. We shall deal with the case when $n \leq 0$ first, and then prove the lemma for $n > 0$ by induction.

If $n \leq 0$, then $m \geq lk$ and $\lfloor \frac{m+(l-1)k}{2l-1} \rfloor \geq k$. For each i choose $a_i \in A_i$. The lemma is proved by letting $A' = \{a_1, a_2, \ldots, a_k\}$.

Now suppose $n > 0$. Since $|A_i| \geq l$ for each i and $lk > m$, some $a_0 \in A$ is in (at least) two of the A_i's. In other words,

$$k' = |\{i \mid a_0 \notin A_i\}| \leq k - 2.$$

Since $lk' - (m-1) \leq lk - m - (2l-1) < n$, by induction there is a subset B of $A \setminus \{a_0\}$ such that

$$
\begin{aligned}
|B| &\leq \min\{\lfloor \frac{(m-1)+(l-1)k'}{2l-1} \rfloor, k'\} \\
&\leq \min\{\lfloor \frac{(m+(l-1)k-(2l-1)}{2l-1} \rfloor, k-2\} \\
&\leq \min\{\lfloor \frac{m+(l-1)k}{2l-1} \rfloor, k\} - 1,
\end{aligned}
$$

and $B \cap A_i \neq \emptyset$ if $a_0 \notin A_i$. The lemma is proved by letting $A' = \{a_0\} \cup B$.
□

Obviously, the bound on $|A'|$ can be attained when $m \geq lk$, and when $m = l$, the bound can be improved to $|A'| = 1$. I do not know the necessary and sufficient conditions for the bound in Lemma 2.1 to be sharp.

Using Lemma 2.1 in the case $l = 2$ allows us to prove the following combinatorial lemma, on which the proof of the formula in Section 4 depends.

LEMMA 2.2. *Let A be a set of size m. Suppose $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$ are subsets of A such that for each i, either*

(a) $|A_i| \geq 2$ and $|B_i| \geq 2$, or

(b) $|A_i| = 1$ and $A_i \subseteq B_i$.

Then there exists a subset A' of A such that

(i) $|A'| \leq k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}$,

(ii) $A' \cap A_i \neq \emptyset$ and $A' \cap B_i \neq \emptyset$ for each i.

Proof. By reordering we can assume $|A_i| = 1$ if and only if $i \in \{1, 2, \ldots, k_0\}$ for some $k_0 \leq k$. Let $C = A_1 \cup A_2 \cup \cdots \cup A_{k_0}$. Then by condition (b),

$$k' = |\{i \mid C \cap A_i = \emptyset\}| + |\{i \mid C \cap B_i = \emptyset\}| \leq 2(k - k_0) \leq 2(k - |C|).$$

Applying Lemma 2.1 with $l = 2$, we obtain a subset D of $A \setminus C$ so that

$$\begin{aligned}|D| &\le \min\{\lfloor \frac{(m-|C|)+k'}{3} \rfloor, k'\} \\ &\le \min\{\lfloor \frac{(m-|C|)+2(k-|C|)}{3} \rfloor, 2(k-|C|)\} \\ &= (k-|C|) + \min\{\lfloor \frac{(m-|C|)-(k-|C|)}{3} \rfloor, k-|C|\} \\ &\le k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\} - |C|,\end{aligned}$$

$D \cap A_i \neq \emptyset$ if $C \cap A_i = \emptyset$, and $D \cap B_i \neq \emptyset$ if $C \cap B_i = \emptyset$. The lemma is proved by letting $A' = C \cup D$. $\square$

Note that the proofs of the lemmas provide an algorithm for finding the required A'. The algorithm guarantees an upper bound on the size of A', but may not give the smallest possible A'.

3. Semilattices. In this section let $\mathcal{V}$ denote the variety of semilattices. Let

$$S = \langle A \mid u_i = v_i, i \in I \rangle_{\mathcal{V}}$$

be a finitely presented semilattice. Let α be the congruence on P_A generated by the relation

$$\rho = \{(C(u_i), C(v_i)) \mid i \in I\}.$$

Then $S = P_A/\alpha$.

LEMMA 3.1. *Each α-class $B\alpha$ has a unique maximal element, denoted by $m(B\alpha)$.*

Proof. We give an algorithm for finding $m(B\alpha)$, given a subset B of A. First we let $M = B$. For each $i \in I$, if $C(u_i) \subseteq M$ or $C(v_i) \subseteq M$ (but not both), then replace M by $M \cup C(u_i) \cup C(v_i)$. (This corresponds to the elementary ρ-transition $M \to M \cup C(u_i)$ or $M \to M \cup C(v_i)$.) Repeat this procedure until for each i, either $C(u_i) \cup C(v_i) \subseteq M$ or $C(u_i) \not\subseteq M$ and $C(v_i) \not\subseteq M$.

By the construction, it is clear that M is not α-equivalent to another subset N if $N \not\subseteq M$. Therefore M is maximal and unique. $\square$

The word problem can be easily solved by the procedure given in the proof: $B\alpha C$ if and only if $m(B\alpha) = m(C\alpha)$.

LEMMA 3.2. *Let $\mathcal{V}$ denote the variety of semilattices. Then for $m > k$,*

$$\phi_{\mathcal{V}}(m,k) \leq m - (k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}).$$

Proof. We first prove a special case, then use it to prove the lemma in general.

Let $E = \langle a,b,c,d \mid ab = cd \rangle_{\mathcal{V}}$. If $B \subset \{a,b,c,d\}$ has 3 elements, then either $\{a,b\} \subset B$ or $\{c,d\} \subset B$. Suppose without loss of generality that $\{a,b\} \subset B$. Since $m(\{a,b\}\alpha) = \{a,b,c,d\} = m(B\alpha)$, B does not freely generate a sub-semilattice of E. Therefore $\phi_{\mathcal{V}}(4,1) \leq 2$.

Case 1 Suppose $\lfloor \frac{m-k}{3} \rfloor \geq k$, that is, $m = 4k + l$ for some $l \geq 0$. Then by Lemma 1.1 (i),

$$\phi_{\mathcal{V}}(m,k) \leq \phi_{\mathcal{V}}(l,0) + \sum_{i=1}^{k} \phi_{\mathcal{V}}(4,1) \leq l + 2k = m - 2k.$$

Case 2 If $\lfloor \frac{m-k}{3} \rfloor \leq k$, then, writing $m' = m - (k - \lfloor \frac{m-k}{3} \rfloor)$ and $k' = k - (k - \lfloor \frac{m-k}{3} \rfloor)$,

$$\phi_{\mathcal{V}}(m,k) \leq \phi_{\mathcal{V}}(m',k') \leq m' - 2k' = m - (k + \lfloor \frac{m-k}{3} \rfloor),$$

where the first inequality follows from Lemma 1.1 (ii), and the second inequality follows from case 1, because $\lfloor \frac{m'-k'}{3} \rfloor = \lfloor \frac{m-k}{3} \rfloor = k'$. $\square$

4. Varieties of Bands. In this section we determine a formula of $\phi_{\mathcal{V}}(m,k)$ for each variety of bands $\mathcal{V}$. Along the way we determine the semilattice structure of bands in $\mathcal{V}$ defined by a presentation. We assume familiarity with the basic results from the theory of semigroups and bands, which can be found in [**6**].

Theorem 4.1. *In any band B, $\mathcal{J}$ equals the minimum semilattice congruence on B, and B is a semilattice of rectangular bands.*

Corollary 4.2. *Let T be a sub-band of a band B. Then for all x, y in T, $x\mathcal{J}y$ in T if and only if $x\mathcal{J}y$ in B.*

Proof. If $x\mathcal{J}y$ in a band, then x and y lie in the same rectangular band. Therefore $xyx = x$ and $yxy = y$. $\square$

Let A be a fixed alphabet. For a variety of bands $\mathcal{V}$, let $R_{\mathcal{V}}$ be the fully invariant congruence on F_A generated by the defining identities for $\mathcal{V}$. Thus $F_A/R_{\mathcal{V}}$ is the free band in $\mathcal{V}$ over A.

The lattice **LB** of varieties of bands has been determined by Birjukov [**1**], Fennemore [**2**] and Gerhard [**3**] (see also [**4**]). Only the following fact will be needed in this paper ([**9**], Remark 3.11):

> Let $\mathbf{LB_0}$ be the lattice obtained from **LB** by taking away the four varieties of trivial, right zero, left zero, and rectangular bands. If $\mathcal{V} \in \mathbf{LB_0}$, then $C(w) = C(z)$ if $(w, z) \in R_{\mathcal{V}}$.

We first treat the case of the subvarieties of rectangular bands.

Theorem 4.3. *$\phi_{\mathcal{V}}(m, k) = m - k$ for the variety of left [right] zero semigroups. $\phi_{\mathcal{V}}(m, k) = \max\{m - 2k, 1\}$ for the variety of rectangular bands.*

Proof. Let $\mathcal{LZ}$ and $\mathcal{RZ}$ denote the varieties of left and right zero semigroups respectively. If $\mathcal{V} = \mathcal{LZ}$, then

$$\langle A \mid u_j = v_j, j \in J\rangle_{\mathcal{V}} \simeq \langle A \mid i(u_j) = i(v_j), j \in J\rangle_{\mathcal{V}},$$

so the result is clear. The case of right zero semigroups is dual.

If $\mathcal{V}$ is the variety of rectangular bands, then it is easy to see that

$$\begin{aligned}\langle A \mid u_j = v_j, j \in J\rangle_{\mathcal{V}} &\simeq \langle A \mid i(u_j)t(u_j) = i(v_j)t(v_j), j \in J\rangle_{\mathcal{V}} \\ &\simeq \langle A \mid i(u_j) = i(v_j), j \in J\rangle_{\mathcal{LZ}} \times \langle A \mid t(u_j) = t(v_j), j \in J\rangle_{\mathcal{RZ}},\end{aligned}$$

and the result follows. $\square$

For the rest of the section let $\mathcal{V}$ be a variety in $\mathbf{LB_0}$.

Let B be a band in the variety $\mathcal{V}$ defined by the presentation

$$B = \langle A \mid u_i = v_i, i \in I \rangle_{\mathcal{V}}$$

where I is some index set and $u_i, v_i \in F_A$ for each $i \in I$. Let β be the congruence on F_A generated by the relation

$$R = \{(u_i, v_i) \mid i \in I\} \cup R_{\mathcal{V}}.$$

Then $B = F_A/\beta$. Let α be the congruence on P_A generated by the relation

$$\rho = \{(C(u_i), C(v_i)) \mid i \in I\}.$$

Now any two distinct β-equivalent elements of F_A are connected by a finite sequence of elementary R-transitions. If the transition is of the form

$$puq \to pvq \qquad (p, q \in F_A^1, (u, v) \in R_{\mathcal{V}}),$$

then $C(puq) = C(pvq)$. On the other hand, if the transition is of the form

$$pu_iq \to pv_iq \quad \text{or} \quad pv_iq \to pu_iq \qquad (p, q \in F_A^1)$$

for some $i \in I$, then there is a corresponding elementary ρ-transition

$$C(p)C(u_i)C(q) \to C(p)C(v_i)C(q) \quad \text{or} \quad C(p)C(v_i)C(q) \to C(p)C(u_i)C(q).$$

Therefore

$$w\beta = z\beta \quad \Rightarrow \quad C(w)\alpha = C(z)\alpha.$$

Suppose now that $w, z \in F_A$ are such that $w\beta \mathcal{J} z\beta$ in B. Then there exist $x, y \in F_A^1$ such that $(x\beta)(w\beta)(y\beta) = z\beta$. Therefore

$$C(x)C(w)C(y)\alpha = C(z)\alpha$$

by the above, and $C(w)C(z)\alpha = C(z)\alpha$. Hence $C(z)\alpha \leq C(w)\alpha$ in the semilattice P_A/α. Similarly $C(w)\alpha \leq C(z)\alpha$, and we have shown that

$$w\beta \mathcal{J} z\beta \quad \Rightarrow \quad C(w)\alpha = C(z)\alpha.$$

The converse is also true, and we prove it in two steps. Firstly, since B is a band, $a^2 = a$ and $ab\mathcal{J}ba$ for all a, b in B, so that, clearly, if $w \in F_A$ and $C(w) = \{a_1, a_2, \ldots, a_n\}$, then $w\beta \ \mathcal{J} \ (a_1 a_2 \ldots a_n)\beta$. Secondly note that if $\{a_1, a_2, \ldots, a_n\}, \{b_1, b_2, \ldots, b_m\} \in P_A$ are connected by an elementary ρ-transition, say

$$\begin{aligned}\{a_1, a_2, \ldots, a_n\} = {} & \{c_1, \ldots, c_k\} \cup C(u_i) \\ \to {} & \{c_1, \ldots, c_k\} \cup C(v_i) = \{b_1, b_2, \ldots, b_m\},\end{aligned}$$

then

$$\begin{aligned} a_1 a_2 \ldots a_n\beta \ \ & \mathcal{J} \ \ c_1 \ldots c_k u_i \beta \qquad \text{(by the first step)} \\ & = \ \ c_1 \ldots c_k v_i \beta \\ & \mathcal{J} \ \ b_1 b_2 \ldots b_m \beta \qquad \text{(by the first step)}. \end{aligned}$$

We summarize the result in the following

THEOREM 4.4. *Let $\mathcal{V} \in \mathbf{LB_0}$. Let $w, z \in F_A$. Then $w\beta\mathcal{J}z\beta$ if and only if $C(w)\alpha = C(z)\alpha$. Thus the semilattice of the $\mathcal{J}$-classes of B is isomorphic to the semilattice defined by the presentation $SL(B) = \langle A \mid u_i = v_i, i \in I\rangle$.*

In particular, if B is the free band in $\mathcal{V}$, then $w\beta\mathcal{J}z\beta$ if and only if $C(w) = C(z)$. Before proving the formula for $\phi_{\mathcal{V}}$ we give two corollaries.

COROLLARY 4.5. *Suppose that a subset A_0 of A generates a (relatively) free sub-band of B. Then A_0 generates a free sub-semilattice of $SL(B)$.*

Proof. Let A_1 and A_2 be nonempty subsets of A_0 such that $A_1\alpha = A_2\alpha$. We need to show that $A_1 = A_2$. Choose $w, z \in F_A$ such that $C(w) = A_1$ and $C(z) = A_2$. Because $A_1\alpha = A_2\alpha$, $w\beta\mathcal{J}z\beta$ in B. By Corollary 4.2, $w\beta\mathcal{J}z\beta$ in the sub-band generated by A_0. Therefore $C(w) = C(z)$ by hypothesis. □

COROLLARY 4.6. *Let $w \in F_A$. Then there exists $z \in F_A$ such that $w\beta = z\beta$ and $C(z) = m(C(w)\alpha)$.*

Proof. Choose $y \in F_A$ such that $C(y) = m(C(w)\alpha)$. Then by Theorem 4.4, $w\beta\mathcal{J}y\beta$. Hence we can find $u, v \in F_A^1$ such that $w\beta = (u\beta)(y\beta)(v\beta)$. The corollary is proved with $z = uyv$, since

$$m(C(w)\alpha) = C(y) \subseteq C(uyv) \subseteq m(C(uyv)\alpha) = m(C(w)\alpha). = 0$$

□

THEOREM 4.7. *Suppose $\mathcal{V} \in \mathbf{LB_0}$. Then for $m > k$,*

$$\phi_{\mathcal{V}}(m,k) = m - (k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}).$$

Proof. Let A be an alphabet of size m and let

$$S = \langle A \mid u_1 = v_1, u_2 = v_2, \ldots, u_k = v_k \rangle_{\mathcal{V}}.$$

We need to show that there exists a subset A_0 of A of the correct size which generates a (relatively) free sub-band of S as a basis.

Case 1 Suppose that for each i, either

(a) $|C(u_i)| \geq 2$ and $|C(v_i)| \geq 2$, or

(b) $|C(u_i)| = 1$ and $C(u_i) \subseteq C(v_i)$.

By Lemma 2.2, we can obtain a subset A' of A such that $|A'| \leq k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}$, and if $A_0 = A \setminus A'$, then $u_i, v_i \notin F_{A_0}$ for any i. Therefore, as none of the relations can be applied, A_0 generates a (relatively) free sub-band of S as a basis. (Note that the hypothesis that $\mathcal{V} \in \mathbf{LB_0}$ is used here.)

Case 2 Suppose that one of the relations, say $u_k = v_k$, is of the form $a_0 = w$, where $a_0 \in A \setminus C(w)$. Let u_i' $[v_i']$ be the word in $F_{A\setminus\{a_0\}}$ obtained by substituting each occurence of a_0 in u_i $[v_i]$ by w. Then it is easy to check that

$$S \simeq \langle A \mid u_1' = v_1', u_2' = v_2', \ldots, u_{k-1}' = v_{k-1}', a_0 = w \rangle_{\mathcal{V}} \simeq S',$$

where we put

$$S' = \langle A \setminus \{a_0\} \mid u_1' = v_1', u_2' = v_2', \ldots, u_{k-1}' = v_{k-1}' \rangle_{\mathcal{V}},$$

and $A_0 \subseteq A \setminus \{a_0\}$ generates a (relatively) free sub-band of S as a basis if it generates a (relatively) free sub-band of S' as a basis. Since

$$\begin{aligned}(m-1) - ((k-1) + &\min\{\lfloor \frac{(m-1)-(k-1)}{3} \rfloor, k-1\}) \\ &\geq m - (k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}),\end{aligned}$$

the problem can be reduced to the first case.

Hence we have shown that

$$\phi_{\mathcal{V}}(m,k) \geq m - (k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}).$$

Lemma 3.2 and Corollary 4.5 show that the reverse inequality also holds. So the proof is complete. □

Note that the proof of the theorem, combined with the procedure in Section 2, gives a procedure to obtain a subset of A of size $\phi_{\mathcal{V}}(m,k)$ which generates a relatively free sub-band.

5. Structure of Finitely Presented Normal Bands. In this section we apply results in Section 4 to obtain complete information on the structure of a finitely presented normal band.

Let B be a normal band defined by the finite presentation

$$B = \langle A \mid u_i = v_i, i \in I \rangle.$$

Let the congruences β and α be as defined in Section 4.

For each element τ of P_A/α, denote the $\mathcal{J}$-class corresponding to τ by J_τ. By general theory, J_τ is a rectangular band. Let $RB(\tau)$ be the rectangular band defined by

$$RB(\tau) = \langle m(\tau) \mid u_i = v_i, i \in I' \rangle,$$

where $i \in I'$ if and only if $u_i, v_i \in F_{m(\tau)}$.

THEOREM 5.1. *For each α-class τ, $J_\tau \simeq RB(\tau)$.*

Proof. Let $m(\tau) = \{a_1, \ldots, a_n\}$. By Corollary 4.6, we can assume the elements of J_τ are $w\beta$ where $C(w) = m(\tau)$. Using the identity $abcd = acbd$ for normal bands, we can see that J_τ is generated by $b_k = a_k a_1 \cdots a_n a_k \beta$, $k = 1, \ldots, n$.

For each $j \in I'$, let u'_j $[v'_j]$ be the element in J_τ obtained by substituting each occurence of a_k in u_j $[v_j]$ by b_k. Then

$$\begin{aligned} u'_j = i(u_j)a_1 \cdots a_n t(u_j)\beta &= u_j a_1 \cdots a_n u_j \beta \\ &= v_j a_1 \cdots a_n v_j \beta = i(v_j)a_1 \cdots a_n t(v_j)\beta = v'_j. \end{aligned}$$

Hence, by the universal property of $RB(\tau)$, there is a homomorphism φ from $RB(\tau)$ onto J_τ, mapping a_k to b_k for $k = 1, \ldots, n$.

On the other hand, the map $w\beta \mapsto w$ defines a homomorphism from J_τ to $RB(\tau)$ which is clearly inverse to φ. For this map to be well-defined we need to check that if $w, z \in F_{m(\tau)}$ and $w\beta = z\beta$, then $w = z$ in $RB(\tau)$. This is true because the identities for normal bands are satisfied by rectangular bands, and the only relations $u_i = v_i$ applicable here are when $i \in I'$. $\square$

If τ_1, τ_2 are two α-classes, $\tau_1 \geq \tau_2$, then $m(\tau_1) \subseteq m(\tau_2)$, and there is a homomorphism $\varphi_{\tau_1,\tau_2} : RB(\tau_1) \to RB(\tau_2)$ defined by $a \mapsto a$ for each $a \in m(\tau_1)$.

Thus we have expressed B as a strong semilattice of finitely presented rectangular bands. Corollary 4.5 can be improved in the case of normal bands.

PROPOSITION 5.2. *A subset A_0 of A generates a free normal sub-band of B if and only if A_0 generates a free sub-semilattice of $SL(B)$, and A_0 generates a free sub-rectangular band of $RB(A_0\alpha)$.*

Proof. We shall only prove the 'if' part, the other direction being obvious.

If A_0 generates a free sub-semilattice in $SL(B)$, then Theorem 4.4 shows that for $w, z \in F_{A_0}$, $w\beta = z\beta$ only if $C(w) = C(z)$.

It remains to show that if $C(w) = C(z) = A' \subseteq A_0$, then $w\beta = z\beta$ if and only if the words w and z are equal in the free rectangular band over A'. Since $A' \subseteq A_0$, $m(A'\alpha) \subseteq m(A_0\alpha)$. The homomorphism $\varphi_{A'\alpha, A_0\alpha}$ maps a to a for each $a \in A'$. The hypothesis implies that A' generates a free rectangular band in $RB(A_0\alpha)$, and hence in $RB(A'\alpha)$. $\square$

The results in this section are valid for left [right] normal bands by replacing 'rectangular bands' with 'left [right] zero semigroups'.

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