The irreducible Specht modules in characteristic 2

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In the representation theory of finite groups it is useful to know which ordinary irreducible representations remain irreducible modulo a prime p. For the symmetric groups $\mathfrak{S}_n$ this amounts to determining which Specht modules are irreducible over a field of characteristic p. Throughout this note we work in characteristic 2, and in this case we classify the irreducible Specht modules, thereby verifying the conjecture in [3, p. 97].

Recall that a partition is 2–regular if all of its non–zero parts are distinct; otherwise the partition is 2-singular. The irreducible Specht modules S^λ with λ a 2–regular partition were classified in [2]. Let λ' denote the partition conjugate to λ. If S^λ is irreducible then $S^{\lambda'}$ is irreducible, since $S^{\lambda'}$ is isomorphic to the dual of S^λ tensored with the sign representation. It turns out that if neither λ nor λ' is 2–regular then S^λ is irreducible only if $\lambda = (2, 2)$.

In order to state our theorem, for an integer k, we let $l(k)$ be the least non–negative integer such that $k < 2^{l(k)}$.

Main Theorem *Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. Then the Specht module S^λ is irreducible in characteristic 2 if and only if*
(i) $\lambda_i - \lambda_{i+1} \equiv -1 \bmod 2^{l(\lambda_{i+1}-\lambda_{i+2})}$ *for all $i \geq 1$; or,*
(ii) $\lambda'_i - \lambda'_{i+1} \equiv -1 \bmod 2^{l(\lambda'_{i+1}-\lambda'_{i+2})}$ *for all $i \geq 1$, where $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is the partition conjugate to λ; or,*
(iii) $\lambda = (2, 2)$.

In part (i) of the Theorem, λ is necessarily 2–regular and the irreducible Specht modules S^λ for λ a 2–regular partition were classified in [2, Corollary 2.14]. Part (ii) therefore classifies the irreducible Specht modules S^λ when λ' is 2–regular. For the proof of the Theorem we must show that $(2, 2)$ is the only partition λ such that λ and λ' are 2–singular and S^λ is irreducible.

If λ is 2-regular then S^λ modulo its radical is an irreducible module D^λ, and every irreducible module for the symmetric group arises in this way. If both λ and λ' are 2–singular and $\lambda \neq (2, 2)$ we show that S^λ is reducible by exhibiting two distinct composition factors. One of these composition factors is D^{λ^R} where λ^R is the 2–regularization of λ, which we define next.

Let $\trianglerighteq$ be the usual dominance order on partitions; see [3, Definition 3.2].

Given partitions λ and μ write $\lambda \sim \mu$ if, for all k, we have

$$\#\{\,(i,j) \mid 1 \leq j \leq \lambda_i \text{ and } k = i + j\,\} = \#\{\,(i,j) \mid 1 \leq j \leq \mu_i \text{ and } k = i + j\,\}.$$

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It turns out that, given λ, there is a unique partition λ^R such that $\lambda^R \trianglerighteq \mu$ for all partitions μ with $\mu \sim \lambda$.

Proposition 1 [1, Theorem A] *The partition λ^R is 2–regular. Moreover, D^{λ^R} is a composition factor of S^λ.*

In order to produce another composition factor of S^λ, we introduce some definitions.

Definition *Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition and suppose that r is minimal such that $\lambda_{r+1} = 0$. We define integers $N_i(\lambda)$ for $i \geq 1$ as follows.*
(i) *For $i > r$ let $N_i(\lambda) = 0$.*
(ii) *Let $N_{r-1}(\lambda)$ be the largest integer of the form $2^m - 1$ such that $N_{r-1}(\lambda) \leq \lambda_{r-1}$.*
(iii) *Next, define $N_i(\lambda)$ for $i = r-2, r-3, \dots, 1$ to be the largest integer of the form $2^m - 1$ such that $N_i(\lambda) \leq \lambda_i - \lambda_{i+1} + N_{i+1}(\lambda)$.*
(iv) *Finally, let* $N_r(\lambda) = \begin{cases} \min\{\lambda_r, N_{r-1}(\lambda)\}, & \text{if } r > 1, \\ \lambda_1, & \text{otherwise.} \end{cases}$

Lemma 1 *Suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition with $\lambda_r \neq 0$.*
(i) *If $i \geq 1$ then $N_i(\lambda) \leq \lambda_i - \lambda_{i+1} + N_{i+1}(\lambda)$.*
(ii) *If $1 \leq i \leq r-1$ then $N_i(\lambda)$ is the largest integer of the form $2^m - 1$ such that $N_i(\lambda) \leq \lambda_i - \lambda_{i+1} + N_{i+1}(\lambda)$.*
(iii) *If $1 \leq i \leq r-1$ then $\lambda_i - \lambda_{i+1} + N_{i+1} \leq 2N_i(\lambda)$.*
(iv) *We have $\lambda_{r-1} \leq 2N_{r-1}(\lambda)$.*

Proof. (i) This follows easily from the definition of $N_i(\lambda)$.

(ii) If $1 \leq i < r-1$ then the result is just the definition of $N_i(\lambda)$. Next, note that $N_{r-1}(\lambda) \leq \lambda_{r-1} - \lambda_r + N_r(\lambda) \leq \lambda_{r-1}$ since $N_r(\lambda) = \min\{\lambda_r, N_{r-1}(\lambda)\}$. However, $N_{r-1}(\lambda)$ is the largest integer of the form $2^m - 1$ such that $N_{r-1}(\lambda) \leq \lambda_{r-1}$. Therefore, part (ii) holds when $i = r-1$.

(iii) This follows from the maximality of $N_i(\lambda)$ in part (ii).

(iv) This follows from the definition of $N_{r-1}(\lambda)$. □

Observe, also, that $\lambda_i \geq N_i(\lambda) \geq 1$ for $1 \leq i \leq r$ and

$$\lambda_1 - N_1(\lambda) \geq \lambda_2 - N_2(\lambda) \geq \cdots .$$

Therefore, $(\lambda_1 - N_1(\lambda), \lambda_2 - N_2(\lambda), \cdots)$ is a partition.

Definition *Let $\alpha(\lambda) = (\lambda_1 - N_1(\lambda), \lambda_2 - N_2(\lambda), \cdots)$.*
For $i \geq 1$ and $j \geq 1$ let $N_{i,j}(\lambda) = N_i\big(\alpha^{i-1}(\lambda)\big)$.

Note that $N_i(\lambda) = N_{i,1}(\lambda)$.

Example In the figures below, the number of j's in row i is equal to $N_{i,j}(\lambda)$ for $\lambda = (11, 9, 5, 1)$ and for $\lambda = (12, 10, 6, 2)$ respectively.

3	2	2	2	1	1	1	1	1	1	1
3	2	1	1	1	1	1	1	1		
3	2	1	1	1						
1										

3	3	2	2	2	1	1	1	1	1	1	1
2	2	2	1	1	1	1	1	1	1		
2	2	2	1	1	1						
1	1										

Lemma 2 *Suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition with $\lambda_r \neq 0$.*
(i) *If $j \geq 1$ then $N_{1,j}(\lambda) \geq N_{2,j}(\lambda) \geq N_{3,j}(\lambda) \geq \cdots$.*
(ii) *If $i \geq 1$ then $N_{i,1}(\lambda) \geq N_{i,2}(\lambda) \geq N_{i,3}(\lambda) \geq \cdots$.*

Proof. (i) By Lemma 1(iii), if $1 \leq i \leq r-2$ then $N_{i+1,1}(\lambda) \leq 2N_{i,1}(\lambda)$, and hence $N_{i+1,1}(\lambda) \leq N_{i,1}(\lambda)$ since both $N_{i+1,1}(\lambda)$ and $N_{i,1}(\lambda)$ are one less than a power of 2. Also, by definition, $0 = N_{r+1,1}(\lambda) < N_{r,1}(\lambda) \leq N_{r-1,1}(\lambda)$; so we have now proved part (i) for $j = 1$. The general case follows by replacing λ by $\alpha^{j-1}(\lambda)$.

(ii) As in part (i), it suffices to show that $N_{i,1}(\lambda) \geq N_{i,2}(\lambda)$ for all $i \geq 1$.

If $i > r$ then $N_{i,2}(\lambda) = N_{i,1}(\lambda) = 0$.

Next, consider the case $i = r$. If $N_{r,1}(\lambda) = \lambda_r$ then $N_{r,2}(\lambda) = 0 < N_{r,1}(\lambda)$. If $N_{r,1}(\lambda) = N_{r-1,1}(\lambda)$ then $\lambda_r \leq \lambda_{r-1} \leq 2N_{r-1,1}(\lambda)$, by Lemma 1(iv); therefore, $N_{r,2}(\lambda) \leq \lambda_r - N_{r,1}(\lambda) \leq N_{r-1,1}(\lambda) = N_{r,1}(\lambda)$.

Now assume that $1 \leq i < r$. Then $N_{i+1,1}(\lambda) \geq N_{i+1,2}(\lambda)$ by induction. Therefore, $N_{i,1}(\lambda) \geq N_{i+1,2}(\lambda)$ since $N_{i,1}(\lambda) \geq N_{i+1,1}(\lambda)$ by part (i), and

$$\begin{aligned}\lambda_i - \lambda_{i+1} + N_{i+1,1}(\lambda) &\geq \big(\lambda_i - N_{i,1}(\lambda)\big) - \big(\lambda_{i+1} - N_{i+1,1}(\lambda)\big) + N_{i+1,2}(\lambda)\\ &\geq N_{i,2}(\lambda), \text{ by Lemma 1(i).}\end{aligned}$$

If $N_{i,2}(\lambda)$ has the form $2^m - 1$ for some m, then $N_{i,2}(\lambda) \leq N_{i,1}(\lambda)$ by Lemma 1(ii). If $N_{i,2}(\lambda)$ does not have the form $2^m - 1$ then $N_{i+1,1}(\lambda) = \lambda_{i+1}$ and so $\lambda_i \leq 2N_{i,1}(\lambda)$ by Lemma 1(iii); therefore, $N_{i,2}(\lambda) \leq \lambda_i - N_{i,1}(\lambda) \leq N_{i,1}(\lambda)$ as required. □

Lemma 3 *Suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition with $\lambda_r \neq 0$ and that $1 \leq i \leq r-1$. Then*

$$\lambda_i \leq 2N_{i,1}(\lambda) + \sum_{j=i+1}^{r-1} N_{j,1}(\lambda),$$

with equality only if $N_{j,2}(\lambda) = N_{j,1}(\lambda)$, for $i \leq j \leq r-1$, and $N_{r,2}(\lambda) = \lambda_r - N_{r,1}(\lambda)$.

Proof. By parts (iv) and (iii) of Lemma 1 we have

$$\begin{aligned}\lambda_{r-1} &\leq 2N_{r-1,1}(\lambda)\\ \lambda_{r-2} - \lambda_{r-1} &\leq 2N_{r-2,1}(\lambda) - N_{r-1,1}(\lambda)\\ \lambda_{r-3} - \lambda_{r-2} &\leq 2N_{r-3,1}(\lambda) - N_{r-2,1}(\lambda)\\ &\vdots\\ \lambda_i - \lambda_{i+1} &\leq 2N_{i,1}(\lambda) - N_{i+1,1}(\lambda).\end{aligned}$$

Summing these inequalities we obtain

$$\lambda_i \leq 2N_{i,1}(\lambda) + N_{i+1,1}(\lambda) + \cdots + N_{r-1,1}(\lambda)$$

with equality only if we have equalities throughout.

Assume that we have equalities throughout. Recall that $N_{j,2}(\lambda) = N_{j,1}(\alpha(\lambda))$ and write $\alpha(\lambda) = (\alpha(\lambda)_1, \alpha(\lambda)_2, \ldots)$.

First, note that $\alpha(\lambda)_{r-1} = \lambda_{r-1} - N_{r-1,1}(\lambda) = N_{r-1,1}(\lambda) > 0$.

Suppose that $i \le j < r-1$. Then, using the equalities above,

$$\alpha(\lambda)_j = \lambda_j - N_{j,1}(\lambda) = \lambda_{j+1} - N_{j+1,1}(\lambda) + N_{j,1}(\lambda) = \alpha(\lambda)_{j+1} + N_{j,1}(\lambda).$$

Since $\alpha(\lambda)_{r-1} \neq 0$, Lemma 1(ii) implies that $N_{j,2}(\lambda)$ is the largest integer of the form $2^m - 1$ such that

$$N_{j,2}(\lambda) \le \alpha(\lambda)_j - \alpha(\lambda)_{j+1} + N_{j+1,2}(\lambda) = N_{j,1}(\lambda) + N_{j+1,2}(\lambda).$$

Therefore, $N_{j,2}(\lambda) \ge N_{j,1}(\lambda)$, and so $N_{j,2}(\lambda) = N_{j,1}(\lambda)$, by Lemma 2(ii).

Next, consider $N_{r-1,2}(\lambda)$. We have $\alpha(\lambda)_{r-1} = N_{r-1,1}(\lambda)$. Therefore, if $\alpha(\lambda)_r = 0$ then

$$N_{r-1,2}(\lambda) = \min\{N_{r-1,1}(\lambda), N_{r-2,2}(\lambda)\} = N_{r-1,1}(\lambda),$$

since we have already shown that $N_{r-2,2}(\lambda) = N_{r-2,1}(\lambda)$, and $N_{r-2,1}(\lambda) \ge N_{r-1,1}(\lambda)$ by Lemma 2(i). Assume, now, that $\alpha(\lambda)_r \neq 0$. Then $N_{r-1,2}(\lambda)$ is the largest integer of the form $2^m - 1$ such that $N_{r-1,2}(\lambda) \le \alpha(\lambda)_{r-1} = N_{r-1,1}(\lambda)$. Therefore, $N_{r-1,2}(\lambda) = N_{r-1,1}(\lambda)$.

Finally, note that $N_{r,1}(\lambda) = \min\{\lambda_r, N_{r-1,1}(\lambda)\}$; therefore, $\lambda_r - N_{r,1}(\lambda) \le \lambda_{r-1} - N_{r-1,1}(\lambda) = N_{r-1,1}(\lambda)$. Thus,

$$N_{r,2}(\lambda) = \min\{\lambda_r - N_{r,1}(\lambda), N_{r-1,2}(\lambda)\} = \lambda_r - N_{r,1}(\lambda)$$

since $N_{r-1,2}(\lambda) = N_{r-1,1}(\lambda)$. □

Definition *Suppose that λ is a partition of n. Let $\lambda^\flat = (\lambda^\flat_1, \lambda^\flat_2, \ldots)$ where $\lambda^\flat_j = \sum_{i \ge 1} N_{i,j}(\lambda)$.*

By Lemma 2(ii), $\lambda^\flat$ is a partition of n. It is easy to see that $\lambda^\flat \trianglerighteq \lambda'$. The point of the construction of $\lambda^\flat$ is the following.

Proposition 2 [3, Theorem 24.6] *Suppose that λ is a partition of n and let F be a field of characteristic 2. Then $\mathrm{Hom}_{F\mathfrak{S}_n}(M^{\lambda^\flat}, S^\lambda) \neq 0$, where $M^{\lambda^\flat}$ is the permutation representation of $\mathfrak{S}_n$ on the Young subgroup for the partition $\lambda^\flat$.*

Proof. Let T be a $\lambda^\flat$–tableau of type λ such that for all i and j the number of i's in row j of T is $N_{i,j}(\lambda)$. The definition of $N_{i,j}(\lambda)$ implies that for all i, j we have

$$N_{i,j}(\lambda) \equiv -1 \bmod 2^{l(N_{i+1,j}(\lambda))}.$$

(Recall that $l(k)$ is the least integer such that $k < 2^{l(k)}$.) By [3, Theorem 24.6] there exists a non–zero $\theta_T \in \mathrm{Hom}_{F\mathfrak{S}_n}(M^{\lambda^\flat}, S^\lambda)$. (In [3], the hypothesis that T is reverse semistandard is used only to show that $\ker\theta_T \subseteq S^{\lambda^\flat \perp}$; this is not needed here.) □

Corollary *Suppose that $\lambda^R \not\trianglerighteq \lambda^\flat$. Then S^λ is reducible.*

Proof. By [3, Theorem 12.1], if D^μ is a composition factor of $M^{\lambda^\flat}$ then $\mu \trianglerighteq \lambda^\flat$. Therefore, by Proposition 2, S^λ has a composition factor D^μ with $\mu \trianglerighteq \lambda^\flat$; in particular, $\mu \neq \lambda^R$. Hence, S^λ is reducible by Proposition 1. □

The next three Lemmas apply this Corollary in cases which interest us.

Lemma 4 *Suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition with $\lambda_r \neq 0$. Assume that $\lambda'_1 \geq \lambda_1$ and that there exists an $i \geq 2$ such that $\lambda'_i = \lambda'_{i+1} > 0$. Then S^λ is reducible.*

Proof. Take i to be minimal such that $i \geq 2$ and $\lambda'_i = \lambda'_{i+1} > 0$. Let $k = \lambda'_i$.

We claim that $\lambda^R_1 \leq r + k$. To see this, first note that

$$\lambda^R_1 = \max\{\, \lambda'_j + j - 1 \mid \lambda'_j \neq 0 \,\}.$$

Suppose that $\lambda'_j \neq 0$; then $1 \leq j \leq \lambda_1$. If $j \geq i$ then $\lambda'_j \leq \lambda'_i$ so $\lambda'_j + j - 1 \leq \lambda'_i + \lambda_1 - 1 \leq k + r - 1$. On the other hand, if $j < i$ then, because i is minimal, we have $\lambda'_1 \geq \lambda'_2 > \lambda'_3 > \cdots > \lambda'_j$; so $\lambda'_j + j - 1 \leq \lambda'_1 + 1 = r + 1 \leq r + k$. This completes the proof of our claim that $\lambda^R_1 \leq r + k$.

Next, consider $\lambda^\flat$. Since $i \geq 2$ and $\lambda'_i = \lambda'_{i+1} > 0$ we have $\lambda_k \geq 3$. In particular, if $k = r$ then $\lambda_r \geq 3$ and $N_{k,1}(\lambda) \geq 3$. We claim that $N_{k,1}(\lambda) \geq 3$ even when $k < r$. To prove this assume that $k \leq r - 1$; then, by Lemma 1(iii), we have

$$\lambda_k - \lambda_{k+1} + N_{k+1,1}(\lambda) \leq 2N_{k,1}(\lambda).$$

Now, $\lambda_k - \lambda_{k+1} \geq 2$ and $N_{k+1,1}(\lambda) \geq 1$ and $N_{k,1}(\lambda)$ is one less than some power of 2. Therefore, $N_{k,1}(\lambda) \geq 3$ as we claimed above.

Thus, for $1 \leq j \leq k$ we have $N_{j,1}(\lambda) \geq N_{k,1}(\lambda) \geq 3$ by Lemma 2(i). Also, $N_{j,1}(\lambda) \geq 1$ for $k < j \leq r$. Therefore,

$$\lambda^\flat_1 = \sum_{j=1}^{r} N_{j,1}(\lambda) \geq 3k + r - k > r + k.$$

Hence, $\lambda^\flat_1 > \lambda^R_1$, so $\lambda^R \not\trianglerighteq \lambda^\flat$. Thus, S^λ is reducible by the Corollary to Proposition 2. □

Lemma 5 *Suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition with $\lambda_r \neq 0$ and $r \geq 2$. Assume that $\lambda_{r-1} > \lambda_r$ and $\lambda'_1 = \lambda'_2$ and $\lambda'_i > \lambda'_{i+1}$ for all i with $2 \leq i \leq \lambda_1$. Then S^λ is reducible.*

Proof. Since $\lambda_{r-1} > \lambda_r \geq 2$, we have $N_{r-1,1}(\lambda) \geq 3$; so $N_{j,1}(\lambda) \geq 3$ for $1 \leq j \leq r-1$ by Lemma 2(i). Therefore, $\lambda^\flat_1 = N_{r,1}(\lambda) + \sum_{j=1}^{r-1} N_{j,1}(\lambda) \geq 2 + 3(r-1)$. On the other hand, $\lambda^R_1 = \lambda'_1 + 1 = r + 1$ because no two columns after the first two columns have the same length. Once more, $\lambda^R \not\trianglerighteq \lambda^\flat$ and S^λ is reducible. □

Lemma 6 *Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a partition with $\lambda_r \neq 0$ and $r \geq 3$. Assume that $\lambda_1 = \lambda_2$ and $\lambda_1^R = \lambda_1 + 1$ and $\lambda_2 > \lambda_2^R$. Then S^λ is reducible.*

Proof. Since $\lambda_1 = \lambda_2$, we have $N_{1,1}(\lambda) = N_{2,1}(\lambda)$. By Lemma 3,

$$\lambda_1 = \lambda_2 \leq 2N_{2,1}(\lambda) + N_{3,1}(\lambda) + \cdots + N_{r-1,1}(\lambda) = \lambda_1^\flat - N_{r,1}(\lambda),$$

since $r \geq 3$. Thus, $\lambda_1^\flat \geq \lambda_1 + N_{r,1}(\lambda) \geq \lambda_1^R$.

If $\lambda_1^\flat > \lambda_1^R$ then $\lambda^R \not\trianglerighteq \lambda^\flat$ and S^λ is reducible.

Assume, now, that $\lambda_1^\flat = \lambda_1^R$. Since $\alpha(\lambda)_1 = \alpha(\lambda)_2$, we have that

$$\lambda_2^\flat = 2N_{2,2}(\lambda) + N_{3,2}(\lambda) + \cdots + N_{r,2}(\lambda).$$

Since $\lambda_1^\flat = \lambda_1^R$, we have $\lambda_1^\flat = \lambda_1 + N_{r,1}(\lambda)$ and

$$\lambda_2 = 2N_{2,1}(\lambda) + N_{3,1}(\lambda) + \cdots + N_{r-1,1}(\lambda);$$

so, by Lemma 3, $N_{j,2}(\lambda) = N_{j,1}(\lambda)$ for $2 \leq j \leq r-1$ and $N_{r,2}(\lambda) = \lambda_r - N_{r,1}(\lambda)$. Therefore,

$$\begin{aligned} \lambda_2^\flat &= 2N_{2,1}(\lambda) + N_{3,1}(\lambda) + \cdots + N_{r-1,1}(\lambda) + \lambda_r - N_{r,1}(\lambda) \\ &= \lambda_2 + \lambda_r - N_{r,1}(\lambda). \end{aligned}$$

Hence,

$$\lambda_1^\flat + \lambda_2^\flat = \lambda_1^\flat + \lambda_2 + \lambda_r - N_{r,1}(\lambda) = \lambda_1 + \lambda_2 + \lambda_r > \lambda_1^R + \lambda_2^R,$$

since $\lambda_1^R = \lambda_1 + 1$ and $\lambda_2 > \lambda_2^R$.

Thus, $\lambda^R \not\trianglerighteq \lambda^\flat$, and S^λ is reducible. □

Proof of the Main Theorem. By the comments which follow the statement of the Theorem, it remains to consider the Specht modules S^λ where λ and λ' are 2–singular partitions.

Assume that λ and λ' are 2–singular. We may also assume that $\lambda_1 \geq \lambda_1'$ because S^λ is reducible if and only if $S^{\lambda'}$ is reducible.

Suppose first that λ has only two non–zero parts; then $\lambda = (k, k)$ for some $k \geq 2$. By [3, Theorem 24.15] the Specht module S^λ is irreducible if and only if $k = 2$.

We may therefore assume that λ has at least three non–zero parts. If $\lambda_i = \lambda_{i+1} > 0$ for some $i \geq 2$ then $S^{\lambda'}$ is reducible by Lemma 4. Since λ is 2–singular, we can now assume that $\lambda_1 = \lambda_2$ and $\lambda_i > \lambda_{i+1}$ whenever $i \geq 2$ and $\lambda_i \neq 0$. If $\lambda_2 = \lambda_3 + 1$ then $S^{\lambda'}$ is reducible by Lemma 5. If $\lambda_2 \neq \lambda_3 + 1$ then $\lambda_1 = \lambda_2 > \lambda_3 + 1 > \lambda_4 + 1 > \cdots$; in this case, $\lambda_1^R = \lambda_1 + 1$ and $\lambda_2^R = \lambda_2 - 1$, so S^λ is reducible by Lemma 6.

This completes the proof of the Theorem. □

In conclusion, we remark that in characteristic p, with p odd, the irreducibility of Specht modules is much more complicated. We are unable even to conjecture which p–singular Specht modules are irreducible in characteristic p.

References

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