THE ORDER OF THE CENTRALIZER OF A REGULAR ELEMENT

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Abstract. In [6] the truncation operation was defined which, given a finite subgroup H of a group K, associates to a class function ψ on K a class function $T_{K/H}(\psi)$ on $N_K(H)$. Let G be an F-split connected reductive algebraic group defined over a finite field of good characteristic and suppose G has connected centre. Let $\phi(w)$ be the number of rational points on a w-twisted rational maximal torus of G. The purpose of this paper is to prove a formula expressing the order of the centralizer in G^F of a regular element of $\mathfrak{g}^F$ in terms of the truncation of ϕ to a certain parabolic subgroup of the Weyl group of G.

Introduction and Notation

We will assume the reader is familiar with the basic theory of linear algebraic groups. Most of what is needed here can be found in Borel [1] or Borel & Tits [2]. Let G be a connected reductive algebraic group defined over a finite field whose characteristic is a good prime for G with corresponding Frobenius endomorphism $F : G \to G$. We will here suppose that G is F-split and that the centre $Z(G)$ of G is connected. We get an induced action of Frobenius on $\mathfrak{g}$, the Lie algebra of G, which we shall also denote by F. We will used the term *rational* for *F-stable*. Fix a maximally split rational maximal torus T and a rational Borel subgroup B containing T. We will denote by Π the set of simple roots defined with respect to the pair (T, B). For $J \subseteq \Pi$ let L_J and B_J be as defined in Carter [4, §2.6]. Let $W = N_G(T)/T$ be the Weyl group of G with respect to T. If $w \in W$, a rational maximal torus T' will be said to be *w-twisted* if $T' = {}^gT$ for some $g \in G$ with $g^{-1}F(g) = \dot{w}$, a representative of w in $N = N_G(T)$. A *regular* element X of $\mathfrak{g}$ is one for which $\dim C_G(X) = \operatorname{rk} G$.

The Main Theorem of this paper occurs in the author's PhD thesis [7, §5.4]. Whilst studying the Fourier transforms of generalized Gel'fand-Graev functions it was observed that several results involving regular elements could be obtained in a canonical way using the notion of truncation from the corresponding results for regular *semisimple* elements. The Main Theorem was obtained by applying this process to the formula which expresses the order of the connected centralizer of a regular semisimple element as the number of rational points of a w-twisted rational maximal torus.

The author would like to ackowledge the support and encouragement of Bob Howlett and Gus Lehrer.

The Main Theorem

First let us recall the definition of the truncation operation.

Definition ([6]). Let H be a finite subgroup of a group K and suppose that ϕ is a class function on K. Define the *truncation* $T_{K/H}(\phi)$ of ϕ to H to be the class function on $N_K(H)$ given by

$$T_{K/H}(\phi)(x) = |H|^{-1} \sum_{y \in H} \phi(xy).$$

It is a simple matter to verify that $T_{K/H}(\phi)$ is indeed a class function on $N_K(H)$. It was shown in [6] that if M is a K-module with character ϕ then M^H, the space of elements of M fixed by H, is an $N_K(H)$-module with character $T_{K/H}(\phi)$.

It was also shown in [6] that, under the given conditions on the characteristic, if $X \in \mathfrak{g}^F$ with semisimple part X_s then $C_G(X_s)^\circ$ is of the form ${}^xL_J$ for some $J \subseteq \Pi$ and $x \in G$ with $x^{-1}F(x) = \dot{w}_1$ and $w_1 \in N_W(J) = \{w \in W \mid w(J) = J\}$. Moreover the conjugacy class C of w_1 in $N_W(J)$ is uniquely determined. In [6] the pair (J, C) is called the *genus* of X.

Main Theorem. *Let G be an F-split connected reductive algebraic group defined over a finite field whose characteristic is a good prime for G and suppose G has connected centre. Let X be a regular element of genus (J, C) in $\mathfrak{g}^F$ and ϕ be the class function on W given by $w \mapsto |T_w^F|$ where T_w is a w-twisted F-stable maximal torus of G. Then $|C_{G^F}(X)| = T_{W/W_J}(\phi)(C)$.*

Note. We have $J = \emptyset$ if and only if X is a regular semisimple element and in this case $T_{W/W_J}(\phi) = \phi$. We see that using the truncation operation provides a canonical means of generalizing results concerning regular semisimple elements to regular elements in general: simply replace all occurences of $\phi(w)$ by $T_{W/W_J}(\phi)(C)$.

First we prove a twisted version of Carter [4, Theorem 3.7.6]. We assume $w \in W$ acts on T from the right via $t \cdot w = t^w$.

Lemma 1. *Let G be a connected reductive group with connected centre. Let T be an maximally split F-stable maximal torus of G, $J \subseteq \Pi$ and $w_1 \in N_W(W_J)$. Then*

$$\frac{1}{|W_J|} \sum_{w \in W_J} |T^{w^{-1}w_1^{-1} \circ F}| = q^{|J|} |Z(L_J)^{w_1^{-1} \circ F}|.$$

Proof. Let $g \in G$ be such that $g^{-1}F(g) = \dot{w}_1$. Then ${}^gT$ is an F-stable maximal torus contained in the F-stable Borel subgroup ${}^gB_J$ of ${}^gL_J$ and is hence a maximally split maximal torus in ${}^gL_J$. Let $T' = {}^{ghg}T = {}^{gh}T$ be an F-stable maximal torus of ${}^gL_J$ where $h \in L_J$. Suppose that T' is twisted from $T_1 = {}^gT$ by ${}^g(w_1 w w_1^{-1}) \in N(T_1)/T_1$ so that $({}^gh)^{-1}F({}^gh) = {}^g(\dot{w}_1 \dot{w} \dot{w}_1^{-1})$. Then $h^{-1}\dot{w}_1 F(h) = \dot{w}_1 \dot{w}$ and $|T'^F|$ is the number of $t \in T$ such that $F(t) = t^{w_1 w}$, that is, $|T^{w^{-1}w_1^{-1} \circ F}|$. On the other hand, since T' is twisted from T_1 by ${}^g(w_1 w w_1^{-1})$, we have

$$|T'^F| = |T_1^{g(w_1 w^{-1} w_1^{-1}) \circ F}|.$$

Hence

$$\begin{aligned} \frac{1}{|W_J|} \sum_{w \in W_J} |T^{w^{-1}w_1^{-1} \circ F}| &= \frac{1}{|W_J|} \sum_{w \in W_J} |T_1^{g(w_1 w^{-1} w_1^{-1}) \circ F}| \\ &= \frac{1}{|W_J|} \sum_{w \in W_J} |T_1^{g w^{-1} \circ F}| \end{aligned}$$

upon change of variable. Now, according to the proof of Carter [4, Theorem 3.7.6] applied to the group ${}^g L_J$, this equals $q^{|J|}|Z({}^g L_J)^F| = q^{|J|}|Z(L_J)^{w_1^{-1} \circ F}|$ as required. $\square$

We now prove what is essentially the Lie algebra analogue of Digne & Michel [5, Proposition 14.24].

Lemma 2. *Let G be a connected reductive algebraic group. If $X \in \mathfrak{g}^F$ is regular nilpotent then, regardless of the characteristic,*

$$|C_{G^F}(X)| = q^l |Z(G)^{\circ F}||H^1(F, Z(G))|$$

where l is the semisimple rank of G and $H^1(F, Z(G))$ is the set of F-conjugacy classes in $Z(G)$.

Proof. According to Springer [10, 5.9(c)], all the regular nilpotent elements of $\mathfrak{g}$ are conjugate under $\operatorname{Ad} G$. Hence, according to Digne & Michel [5, 3.21], the $\operatorname{Ad} G^F$-orbits of regular nilpotent elements in $\mathfrak{g}^F$ which are $\operatorname{Ad} G$-conjugate to X are parametrized by $H^1(F, A(X))$ where $A(X) = C_G(X)/C_G(X)^\circ$ and moreover these $\operatorname{Ad} G^F$-orbits have the same cardinality. By Springer & Steinberg [11, 3.7(a)], we have $C_G(X) = Z(G) \cdot C_U(X)$ (one does not need the restriction that p be good for this). By Springer & Steinberg [11, 3.7(b)] and Keny [8, p. 197, line 8], $C_U(X)$ is connected regardless of the characteristic and hence $A(X) \cong Z(G)/Z(G)^\circ$. Thus $H^1(F, A(X)) \cong H^1(F, Z(G)/Z(G)^\circ) \cong H^1(F, Z(G))$ according to Springer & Steinberg [11, 2.6]. The proof of Digne & Michel [5, 14.23] carries across directly to the Lie algebra and we find that the number of regular nilpotent elements in $\mathfrak{g}^F$ is $|G^F|/|Z(G)^{\circ F}|q^l$ (that regular nilpotent elements exist is proved by Springer [10, 5.9(b)] for good characteristic and by Keny [8] for bad characteristic). This completes the proof. $\square$

Proof of Main Theorem. Let $C_G(X_s)^\circ = {}^g L_J$ with $g^{-1}F(g) = \dot{w}_1$. Since X is regular, X_n is regular in $C_{\mathfrak{g}}(X_s)$ and hence, according to Lemma 2,

$$|C_{G^F}(X)| = q^{|J|}|Z({}^g L_J)^{\circ F}||H^1(F, Z({}^g L_J))|.$$

But, according to Carter [4, Prop. 8.1.4], L_J has connected centre and hence

$$|C_{G^F}(X)| = q^{|J|}|Z({}^g L_J)^F|.$$

On the other hand,

$$\begin{aligned} T_{W/W_J}(\phi)(w_1) &= \frac{1}{|W_J|}\sum_{w\in W_J}|T^F_{w_1 w}| \\ &= \frac{1}{|W_J|}\sum_{w\in W_J}|T^{w^{-1}w_1^{-1}\circ F}| \\ &= q^{|J|}|Z({}^g L_J)^F| \end{aligned}$$

by Lemma 1. The result then follows. □

We have the following interesting corollary of the main Theorem.

Corollary. *Let $V = Y(T)\otimes_{\mathbb{Z}}\mathbb{R}$. For $J\subseteq\Pi$, let V_J be the subspace of V spanned by J. If X is a regular element of $\mathfrak{g}^F$ of genus (J, C) with $w_1 \in C$ then*

$$|C_{G^F}(X)| = |Z(G)^F|\sum_{i=0}^{l}(-1)^{l-i}q^i\,\mathrm{Tr}(w_1, \Lambda^{l-i}(V_J^{\perp}))$$

where $\Lambda^i(V_J^{\perp})$ is the ith exterior power of $V_J^{\perp}$.

Before giving the proof of this corollary we need the following lemma.

Lemma 3. *With notation as above, we have $(\Lambda^i V)^{W_J} = \Lambda^i(V_J^{\perp})$ for $0\leqslant i\leqslant l$.*

Proof. We have a W_J-stable decomposition $V = V_J\oplus V_J^{\perp}$ in which W_J acts trivially on $V_J^{\perp}$ and as the reflection representation on V_J. Now

$$\Lambda^i V \cong \bigoplus_{j=0}^{i}\Lambda^j V_J\otimes\Lambda^{i-j}V_J^{\perp}.$$

So let $v\in(\Lambda^i V)^{W_J}$ and write $v = \sum_{j=0}^{i} v_j\wedge v'_{i-j}$ where $v_k\in\Lambda^k V_J$ and $v'_k\in\Lambda^k V_J^{\perp}$. Then for any $w\in W_J$ we have

$$\sum_{j=0}^{i} v_j\wedge v'_{i-j} = \sum_{j=0}^{i} wv_j\wedge v'_{i-j}$$

and hence $v_j\wedge v'_{i-j} = wv_j\wedge v'_{i-j}$ for all $j\in\{0,\dots,i\}$. Hence $v_j\in(\Lambda^j V_J)^{W_J}$ and according to Bourbaki [3, Chapitre V, Exercice §2, 3(a)] this implies that $v_j = 0$ if $j\geqslant 1$. Since $(\Lambda^0 V_J)^{W_J}\cong\mathbb{R}$ the result follows. □

Proof of Corollary. Since G is F-split, the action of Frobenius on $Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is just multiplication by q. Let $S = T \cap G'$. Then $T = SZ$ and $V = Y(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is an l-dimensional W-module affording the reflection representation. According to Carter [4, 3.3.8], we then have

$$\begin{aligned} |T_w^F| &= |Z(G)^F| \det_V(q - w) \\ &= |Z(G)^F| \sum_{i=0}^{l} (-1)^{l-i} q^i \rho_{l-i}(w) \end{aligned}$$

where ρ_i is the character of the ith exterior power of V (cf. Lehrer [9, Corollary 2.3]). Hence

$$|C_{G^F}(X)| = |Z(G)^F| \sum_{i=0}^{l} (-1)^{l-i} q^i T_{W/W_J}(\rho_{l-i})(w_1).$$

Since ρ_{l-i} is the character of the W-module $\Lambda^{l-i} V$, $T_{W/W_J}(\rho_{l-i})$ is the character of the $N_W(W_J)$-module $(\Lambda^{l-i} V)^{W_J}$. The result then follows from Lemma 3. □

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