RATIONAL REGULAR NILPOTENT ELEMENTS OF A REDUCTIVE LIE ALGEBRA

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Abstract. The aim of this paper is give an exposition of some of the main properties of regular nilpotent elements of the Lie algebra of a reductive algebraic group (with not necessarily connected centre) defined over a finite field. The approach presented here will be similar to that of Steinberg [10] and Digne & Michel [4] for regular unipotent elements.

1. Introduction and Notation

We will assume the reader is familiar with the basic theory of linear algebraic groups. Most of what is needed here can be found in Borel [1] or Borel & Tits [2]. Throughout this paper G will denote a connected reductive algebraic group defined over the finite field $\mathbb{F}_q$ with corresponding Frobenius endomorphism $F : G \to G$. We will suppose here that G is F-split. We will in general denote Lie algebras by lower case gothic letters (so, for example, $\mathfrak{g} = \mathrm{Lie}(G)$). We get an induced action of Frobenius on $\mathfrak{g}$, which we shall also denote by F. Given an F-stable Borel subgroup B and an F-stable Levi decomposition $B = TU$ we will let Φ be the root system of G with respect to T and Π the system of simple roots determined by the pair $T \subseteq B$. Let the root subgroup (resp. root subspace) corresponding to root α be denoted by X_α (resp. $\mathfrak{g}_\alpha$). Let $U' = \prod_{\alpha\in\Phi + -\Pi} X_\alpha$ and Z be the centre of G. We will repeatedly use the fact that $\cap_{\alpha\in\Pi}\ker\alpha = Z$ (cf. Digne & Michel [4, 0.35]). Let $W = N_G(T)/T$ be the Weyl group of G with respect to T. We will denote by s_β the reflection in root β and by $\dot{w}$ a representative in $N_G(T)$ of $w \in W$.

2. Some Preparatory Lemmas

Lemma 1. *Let B, T, U and U' be as above. The derived group of U is U' and*

(i) *If $u \in U$, $Y \in \mathfrak{u}$ then $\mathrm{Ad}\, u(Y) - Y \in \mathfrak{u}'$.*
(ii) *If $g \in B$, $Y \in \mathfrak{u}$ then $\mathrm{Ad}\, g(Y) \in \mathfrak{u}$.*
(iii) *If $g \in B$, $Y' \in \mathfrak{u}'$ then $\mathrm{Ad}\, g(Y') \in \mathfrak{u}'$.*
(iv) *If $g \in B$, $Y \in \mathfrak{b}$ then $\mathrm{Ad}\, g(Y) - Y \in \mathfrak{u}$.*

Proof. The first statement is Digne & Michel [4, 14.17]. For (i), the map $v \mapsto uvu^{-1}v^{-1}$ sends U to U'. According to Humphreys [5, 10.1(c)], its differential is $Y \mapsto \mathrm{Ad}\, u(Y) -$

Y and sends $\mathfrak{u}$ to $\mathfrak{u}'$. In a similar fashion, if $g \in B$, differentiation of the map $U \to U$ given by $x \mapsto gxg^{-1}$ shows that $\operatorname{Ad} g(Y) \in \mathfrak{u}$ proving (ii). If $t \in T$ then differentiating the map $U' \to U'$ given by $x \mapsto txt^{-1}$ shows that $\operatorname{Ad} t(Y') \in \mathfrak{u}'$. Now let $u \in U$. Since $Y' \in \mathfrak{u}$, if follows from (i) that $\operatorname{Ad} u(Y') - Y' \in \mathfrak{u}'$ and hence $\operatorname{Ad} u(Y') \in \mathfrak{u}'$. Statement (iii) then follows since $B = TU$. The last statement follows by differentiating the map $b \mapsto gbg^{-1}b^{-1}$ from B to U. □

Lemma 2. *If B is a Borel subgroup of G then $B = \{g \in G \mid \operatorname{Ad} g(\mathfrak{b}) = \mathfrak{b}\}$.*

Proof. Clearly $B \subseteq \{g \in G \mid \operatorname{Ad} g(\mathfrak{b}) = \mathfrak{b}\}$. Conversely, suppose that $\operatorname{Ad} g(\mathfrak{b}) = \mathfrak{b}$. Let T be any maximal torus in B and let Π be the simple system of roots determined by the pair $T \subseteq B$. For $\alpha \in \Pi$ let Y_α be a non-zero element of $\mathfrak{g}_\alpha$. Then $\operatorname{Ad} g(\sum_{\alpha\in\Pi} Y_\alpha) \in \mathfrak{b}$ and hence $g \in B$ by Borel [1, 14.23]. □

The proofs of the following two results are based on Digne & Michel [4, 14.2, 14.4].

Proposition 3 (Springer [8, Remark following 5.6]). *If G is a linear algebraic group then $\dim C_G(Y) \geqslant \operatorname{rk}(G)$ for any element $Y \in \mathfrak{g}$.*

Proof. By Borel [1, 14.25], Y is contained in $\mathfrak{b} = \operatorname{Lie}(B)$ for some Borel subgroup B of G. Let U be its unipotent radical. By Lemma 1, $\operatorname{Ad} B(Y) \subseteq Y + \mathfrak{u}$ and hence $\dim \operatorname{Ad} B(Y) \leqslant \dim U = \dim B - \operatorname{rk}(G)$. But $\dim \operatorname{Ad} B(Y) = \dim B - \dim C_B(Y)$ and hence $\dim C_B(Y) \geqslant \operatorname{rk}(G)$. A fortiori, we must have $\dim C_G(Y) \geqslant \operatorname{rk}(G)$. □

Definition. An element $Y \in \mathfrak{g}$ is *regular* in $\mathfrak{g}$ if $\dim C_G(Y) = \operatorname{rk}(G)$.

Note. In the case of the group, one can show the existence of regular elements by finding $t \in T$ such that $C_G(t) = T$ (cf. Digne & Michel [4, 14.2(ii)]). The corresponding result for the Lie algebra is not true however. For example, take $\mathfrak{sl}_2$ over a field of characteristic 2. Every semisimple element is then central. We will prove the existence of regular elements of the Lie algebra in Theorem 5.

Proposition 4 (cf. Steinberg [10, 3.7]). *Let $Y = Y_s + Y_n$ be the Jordan decomposition of an element of $\mathfrak{g}$. Then Y is regular in $\mathfrak{g}$ if and only if Y_n is regular in $C_{\mathfrak{g}}(Y_s)$.*

Proof. By Humphreys [5, 4.2 Proposition (b)] we have $C_G(Y) = C_G(Y_s) \cap C_G(Y_n)$, and hence $\dim C_G(Y) = \dim C_{C_G(Y_s)}(Y_n)$. Now let T be a maximal torus of G whose Lie algebra contains Y_s. Then $T \subseteq C_G(Y_s)$ and hence $\operatorname{rk}(G) = \operatorname{rk}(C_G(Y_s))$. Since $C_{\mathfrak{g}}(Y_s) = \operatorname{Lie} C_G(Y_s)$ (cf. Borel [1, 9.1]), the result follows according to the definition of regular applied to the group $C_G(Y_s)$. □

3. The main theorem on regular nilpotent elements

Theorem 5 (cf. Steinberg [10, 3.2, 3.3], Digne & Michel [4, 14.14]). *Let G be a connected reductive group. Then regular nilpotent elements exist and any two regular nilpotent elements are conjugate under $\mathrm{Ad}\, G$. Let Y be a nilpotent element of $\mathfrak{g}$. The following properties are equivalent:*

(i) *Y is regular.*

(ii) *There are only a finite number of Borel subalgebras containing Y.*

(iii) *There is only one Borel subalgebra containing Y.*

(iv) *There exists a Borel subalgebra $\mathfrak{b} = \mathrm{Lie}\, B$ containing Y and a maximal torus T of B such that in the decomposition $Y = \sum_{\alpha\in\Phi^+} Y_\alpha$ where α runs over the positive roots relative to the pair $T \subset B$ and $Y_\alpha \in \mathfrak{g}_\alpha$, none of the Y_α for α simple is equal to zero.*

(v) *Property* (iv) *is true for any Borel subalgebra $\mathfrak{b} = \mathrm{Lie}\, B$ containing Y and any maximal torus of B.*

Proof. We will first show that the statements (i)-(v) are equivalent. In the course of this the first assertion will be proved.

(iv) $\Longrightarrow$ (iii). Let B' be a second Borel subgroup such that $Y \in \mathfrak{b}' = \mathrm{Lie}(B')$. By the conjugacy theorem for Borel subgroups (cf. Borel [1, 11.1]) and the Bruhat decomposition, $B' = u\dot{w}B\dot{w}^{-1}u^{-1}$ for some $w \in W$, $u \in U_{w^{-1}}$. Then $\mathfrak{b}$ contains $\mathrm{Ad}(\dot{w}^{-1}u^{-1})(Y)$. According to Lemma 1, $\mathrm{Ad}\, u^{-1}(Y) - Y \in \mathfrak{u}'$ and hence $\mathrm{Ad}\, u^{-1}(Y) = \sum_{\alpha\in\Phi^+} Y_\alpha$ where $Y_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \neq 0$ for $\alpha \in \Pi$. Since $\mathrm{Ad}\, \dot{w}^{-1}(\mathrm{Ad}\, u^{-1}(Y)) \in \mathfrak{b}$ we must have $w\alpha > 0$ for every $\alpha \in \Pi$ and hence $w = 1$. Thus $B' = B$ and (iii) holds.

(ii) $\Longrightarrow$ (v). Suppose (v) is false; that is, for some Borel subgroup B whose Lie algebra contains Y and some maximal torus T of B, $Y = \sum_{\alpha\in\Phi^+} Y_\alpha$ with $Y_\alpha \in \mathfrak{g}_\alpha$ and $Y_\beta = 0$ for some $\beta \in \Pi$. Then for any $u \in X_\beta$, $\mathrm{Ad}\, u^{-1}(Y) - Y \in \mathfrak{u}'$ by Lemma 1 and hence $\mathrm{Ad}\, u^{-1}(Y) = \sum_{\alpha\in\Phi^+} Y'_\alpha$ with $Y'_\alpha \in \mathfrak{g}_\alpha$ and $Y'_\beta = 0$. Hence $\mathrm{Ad}\, \dot{s}_\beta^{-1}u^{-1}(Y) \in \mathfrak{b}$ since the only positive root made negative by s_β is β, and hence $Y \in \mathrm{Ad}(u\dot{s}_\beta)(\mathfrak{b})$ for any $u \in X_\beta$. These Borel subalgebras are distinct. For if $\mathrm{Ad}\, \dot{s}_\beta(\mathfrak{b}) = \mathrm{Ad}(u\dot{s}_\beta)(\mathfrak{b})$ for some $u \in X_\beta$ then by applying Lemma 2 in the case of the Borel subalgebra $\mathrm{Ad}\, \dot{s}_\beta(\mathfrak{b}) = \mathrm{Lie}({}^{\dot{s}_\beta}B)$ we see that $u \in {}^{\dot{s}_\beta}B \cap X_\beta = {}^{\dot{s}_\beta}(B \cap {}^{\dot{s}_\beta}X_\beta) = {}^{\dot{s}_\beta}(B \cap X_{-\beta})$ and hence $u = 1$. Thus the Borel subalgebras $\mathrm{Ad}(u\dot{s}_\beta)(\mathfrak{b})$ for $u \in X_\beta$ constitute an infinite number of Borel subalgebras containing Y, contradicting (ii).

(iii) $\Longrightarrow$ (ii). This implication follows trivially.

(v) $\Longrightarrow$ (iv). According to Borel [1, 14.25], each nilpotent element of $\mathfrak{g}$ lies in some Borel subalgebra. The implication immediately follows.

We have thus proved the equivalence of (ii), (iii), (iv) and (v).

(i) $\Longrightarrow$ (v). Let Y be a regular nilpotent element. Let $Y \in \mathfrak{b}$ for some Borel subalgebra $\mathfrak{b} = \mathrm{Lie}(B)$. Let T be a maximal torus of B and let $Y = \sum_{\alpha\in\Phi+} Y_\alpha$ where the sum is over positive roots relative to $T \subset B$ and $Y_\alpha \in \mathfrak{g}_\alpha$. Suppose that $Y_\beta = 0$ for some $\beta \in \Pi$. Then $Y \in \mathfrak{p} = \mathrm{Lie}(P)$ where P is the parabolic subgroup $B \cup B\dot{s}_\beta B$. We first show that if $g \in P$ then $\mathrm{Ad}\, g(Y) = \sum_{\alpha\in\Phi+} Y'_\alpha$ with $Y'_\alpha \in \mathfrak{g}_\alpha$ and $Y'_\beta = 0$. First suppose that $g \in B$. Write $g = tu$ with $t \in T$, $u \in U$. Then $\mathrm{Ad}\, u(Y) - Y \in \mathfrak{u}'$ and hence $\mathrm{Ad}\, g(Y) = \mathrm{Ad}\, t\, \mathrm{Ad}\, u(Y) = \sum_{\alpha\in\Phi+} Y'_\alpha$ for some $Y'_\alpha \in \mathfrak{g}_\alpha$ with $Y'_\beta = 0$. Since $Y_\beta = 0$ and the only positive root made negative by s_β is β, if $g = \dot{s}_\beta$ then $\mathrm{Ad}\, \dot{s}_b(Y) = \sum_{\alpha\in\Phi+} Y'_\alpha$ for some $Y'_\alpha \in \mathfrak{g}_\alpha$ with $Y'_\beta = 0$. Hence the $\mathrm{Ad}\, P$-orbit of Y contains only elements of U of the form $\sum_{\alpha\in\Phi+} Y'_\alpha$ with $Y'_\alpha \in \mathfrak{g}_\alpha$ and $Y'_\beta = 0$ and thus $\dim \mathrm{Ad}\, P(Y) \leqslant \dim(U) - 1$. As $\dim P = \dim U + \dim T + 1$,

$$
\begin{aligned}
\dim C_P(Y) &= \dim P - \dim \mathrm{Ad}\, P(Y) \\
&\geqslant \dim P - (\dim U - 1) \\
&= \dim U + \dim T + 1 - \dim U + 1 \\
&= \dim T + 2 \\
&= \mathrm{rk}(G) + 2,
\end{aligned}
$$

a contradiction, as then $\mathrm{rk}(G) = \dim C_G(Y) \geqslant \mathrm{rk}(G) + 2$.

(iv) $\Longrightarrow$ (i) and proof of the initial assertion. We now show that if Y is a regular nilpotent element of $\mathfrak{g}$ and Y' satisfies (iv) then Y and Y' are conjugate under $\mathrm{Ad}\, G$. Since regular nilpotent elements exist (cf. Springer [8, 5.9(b)] and Keny [7]), this will show that any element Y' of $\mathfrak{g}$ satisfying (iv) is regular and hence satisfies condition (i). Note that if Y and Y' are any two regular nilpotent elements of $\mathfrak{g}$ then this will also show that they are conjugate (since we have shown that Y' satisfies condition (iv)), thus proving the initial assertion. Replacing Y by an $\mathrm{Ad}\, G$ conjugate we may assume $Y \in \mathfrak{u}$. We have shown that Y satisfies (v) and hence, according to Humphreys [6, 16.2(C)] and Digne & Michel [4, 0.31(i)], we may replace Y' by an $\mathrm{Ad}\, T$ conjugate in such a way that if $Y = \sum_{\alpha\in\Phi+} Y_\alpha$ and $Y' = \sum_{\alpha\in\Phi+} Y'_\alpha$ with $Y_\alpha, Y'_\alpha \in \mathfrak{g}_\alpha$ then $Y_\alpha = Y'_\alpha$ for $\alpha \in \Pi$. Thus $Y' - Y \in \mathfrak{u}'$. As the orbits of a unipotent affine group acting on a variety are closed (cf. Borel [1, 4.10]), the set $\{\mathrm{Ad}\, u(Y) \mid u \in U\}$ is a closed subvariety of $\mathfrak{u}$ and hence $\{\mathrm{Ad}\, u(Y) - Y \mid u \in U\}$ is also a closed subvariety of $\mathfrak{u}$. Since $\mathrm{Ad}\, u(Y) - Y \in \mathfrak{u}'$ for all $u \in U$ and $\mathfrak{u}'$ is a closed subvariety of $\mathfrak{u}$, this implies that $\{\mathrm{Ad}\, u(Y) - Y \mid u \in U\}$ is a closed subvariety of $\mathfrak{u}'$. Its codimension in $\mathfrak{u}$ is $\dim C_U(Y)$. We will show that this equals the codimension of $\mathfrak{u}'$ so that $\mathfrak{u}' = \{\mathrm{Ad}\, u(Y) - Y \mid u \in U\}$. For the codimension of $\mathfrak{u}'$ is the semisimple rank of G, equal to $\mathrm{rk}(G) - \dim Z$. But, by the following lemma, $C_G(Y) = Z \cdot C_U(Y)$ and hence

$\mathrm{rk}(G) - \dim Z = \dim C_G(Y) - \dim Z = \dim C_U(Y)$. Hence there exists a $u \in U$ such that $\mathrm{Ad}\,u(Y) - Y = Y' - Y$. Hence $\mathrm{Ad}\,u(Y) = Y'$ and Y is $\mathrm{Ad}\,G$-conjugate to Y'.

□

Lemma 6 (cf. Springer [8, 5.9(b)], Springer & Steinberg [9, III, 3.7(a)]). *Let Y be a regular nilpotent element of $\mathfrak{g}$ and let U be the unipotent radical of the unique Borel subgroup B whose Lie algebra contains Y. Then $C_G(Y) = Z \cdot C_U(Y)$ as a direct product.*

Proof. If $\mathrm{Ad}\,g(Y) = Y$ then, since (i) implies (iii) in Theorem 5, we have $\mathrm{Ad}\,g(\mathfrak{b}) = \mathfrak{b}$ and hence $g \in B$ by Lemma 2. Thus $g \in C_B(Y)$. Since $C_B(Y)$ is a linear algebraic group (being a closed subgroup of B), the Jordan decomposition of g in G is of the form $g = tv$ where $t, v \in C_B(Y)$, t semisimple and v unipotent. Being unipotent, v must lie in U and hence in $C_U(Y)$. By property (v), if T is a maximal torus of B containing t and Π is the set of simple roots with respect to $T \subset B$ then $\alpha(t) = 1$ for all $\alpha \in \Pi$ and hence $t \in Z$. The result follows. □

4. Rational conjugacy of regular nilpotent elements

Proposition 7 (cf. Springer & Steinberg [9, III, 1.19(b)]). *The regular nilpotent $\mathrm{Ad}\,G^F$-orbits of $\mathfrak{g}^F$ are parametrized by $H^1(F, Z/Z^\circ)$, the set of F-conjugacy classes in Z/Z°.*

Proof. By Theorem 5, all the regular nilpotent elements of $\mathfrak{g}$ are conjugate under $\mathrm{Ad}\,G$. Hence, according to Digne & Michel [4, 3.21], the $\mathrm{Ad}\,G^F$-orbits of regular nilpotent elements in $\mathfrak{g}^F$ are parametrized by $H^1(F, A(Y))$ where Y is any rational regular nilpotent element and $A(Y) = C_G(Y)/C_G(Y)^\circ$. But according to Lemma 6, $C_G(Y) = Z \cdot C_U(Y)$. By Springer & Steinberg [9, III, 3.7(b)] and Keny [7, p. 197, line 8], $C_U(Y)$ is connected regardless of the characteristic and hence $C_G(Y)/C_G(Y)^\circ \cong Z/Z^\circ$. Thus $H^1(F, A(Y)) \cong H^1(F, Z/Z^\circ)$ giving the result. □

Lemma 8. *Let A be a finite abelian group. If F is an automorphism of A then $|H^1(F, A)| = |A^F|$.*

Proof. Since A is abelian, if $\mathcal{L} : A \to A$ is the map $x \mapsto x^{-1}F(x)$ then two elements of A are F-conjugate if and only if they represent the same coset in $A/\mathcal{L}(A)$. Also, as the kernel of $\mathcal{L}$ is A^F, $\mathcal{L}(A) \cong A/A^F$. Since A is finite we find that $|H^1(F, A)| = |A/\mathcal{L}(A)| = |A|/|A/A^F| = |A^F|$. □

Corollary 9. *We have $|H^1(F, Z)| = |Z^F|/|Z^{\circ F}|$*

Proof. We have $|H^1(F,Z)| = |H^1(F,Z/Z^\circ)|$ by Springer & Steinberg [9, I, 2.6]. Then by the above lemma, $|H^1(F,Z/Z^\circ)| = |(Z/Z^\circ)^F|$. This last quantity equals the desired result by Digne & Michel [4, 3.13] since Z° is connected. □

It is possible to be more precise with the parametrization of the rational regular nilpotent orbits. For this we need to extend the definition of the map $\mathcal{L}$ as follows.

Definition (cf. Digne, Lehrer & Michel [3, 1.1]). If H is any F-stable algebraic subgroup of G, the Lang map $\mathcal{L}_H : H \to H$ is defined by $\mathcal{L}_H(h) = h^{-1}F(h)$. If $H = G$ then we will write $\mathcal{L}$ for $\mathcal{L}_G$.

Lemma 10 (Digne, Lehrer & Michel [3, 1.3]). *Let H be an F-stable connected closed subgroup of G such that $H \supseteq Z$. Then $H^1(F,Z)$ and $\mathcal{L}_H^{-1}(Z)$ are groups, ZH^F is a normal subgroup of $\mathcal{L}_H^{-1}(Z)$ and $H^1(F,Z) \cong \mathcal{L}_H^{-1}(Z)/ZH^F$.*

Proof. It is instructive to present a proof which does not explicitly use the methods of Galois cohomology.

First we need to show that $H^1(F,Z)$ and $\mathcal{L}_H^{-1}(Z)$ are groups. If $z_1, z_2, a, b \in Z$ then $z_1^{-1}aF(z_1)z_2^{-1}bF(z_2) = z^{-1}(ab)F(z)$ where $z = z_1z_2$ and hence F-conjugacy classes of Z multiply, showing that $H^1(F,Z)$ is a group. If $a, b \in \mathcal{L}_H^{-1}(Z)$ then $\mathcal{L}_H(ab^{-1}) = ba^{-1}F(a)F(b^{-1}) = a^{-1}F(a)bF(b^{-1}) \in Z$ and hence $\mathcal{L}_H^{-1}(Z)$ is a group. We must now show that $ZH^F \trianglelefteq \mathcal{L}_H^{-1}(Z)$. If $g \in \mathcal{L}_H^{-1}(Z)$, $z \in Z$ and $h \in H^F$ then $gzhg^{-1} = zghg^{-1}$ and

$$F(ghg^{-1}) = F(g)hF(g)^{-1} = gg^{-1}F(g)h(g^{-1}F(g))^{-1}g^{-1} = ghg^{-1}$$

since $g^{-1}F(g) \in Z$. Hence $ghg^{-1} \in H^F$ and $ZH^F \trianglelefteq \mathcal{L}_H^{-1}(Z)$.

Define $\alpha : \mathcal{L}_H^{-1}(Z) \to H^1(F,Z)$ by sending $h \in \mathcal{L}_H^{-1}(Z)$ to the F-conjugacy class of $\mathcal{L}_H(h)$ in Z. We first show that α is surjective. Let $z \in Z$. Since H is connected and $z \in H$, by the Lang-Steinberg Theorem (cf. Springer-Steinberg [9, I, 2.2]) there exists $h \in H$ such that $\mathcal{L}_H(h) = z$. Then $\alpha(h)$ is the class represented by z. Now $h \in \mathcal{L}_H^{-1}(Z)$ lies in $\ker\alpha$ if and only if $h^{-1}F(h) = a^{-1}F(a)$ for some $a \in Z$, which occurs if and only if $ha^{-1} \in H^F$ for some $a \in Z$, that is, if and only if $h \in ZH^F$. Thus $\ker\alpha = ZH^F$ and α induces an isomorphism $\mathcal{L}_H^{-1}(Z)/ZH^F \cong H^1(F,Z)$. □

Lemma 11 (cf. Steinberg [11, 10.10]). *Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be two F-stable Borel subalgebras of G. Then $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are conjugate under $\operatorname{Ad} G^F$.*

Proof. Let $\mathfrak{b}_i = \operatorname{Lie}(B_i)$. Then there exists $g \in G$ such that $B_2 = {}^gB_1$ and hence $\mathfrak{b}_2 = \operatorname{Ad} g(\mathfrak{b}_1)$. Since $\mathfrak{b}_2$ is F-stable, $\operatorname{Ad}(g^{-1}F(g))(\mathfrak{b}_1) = \mathfrak{b}_1$ and hence $g^{-1}F(g) \in B_1$ by Lemma 2. By the Lang-Steinberg Theorem (cf. Springer-Steinberg [9, I, 2.2]) there exists $h \in B_1$ such that $g^{-1}F(g) = h^{-1}F(h)$. Then $gh^{-1} \in G^F$ and $\mathfrak{b}_2 = \operatorname{Ad}(gh^{-1})(\mathfrak{b}_1)$. □

Theorem 12 (cf. Digne, Lehrer & Michel [3, 3.2]). *Let G be a connected reductive group. Fix an F-stable Borel subgroup B of G with unipotent radical U and an F-stable maximal torus T of B. Let $\mathcal{L}_T : T \to T$ be the map $t \mapsto t^{-1}F(t)$ and let $H^1(F,Z)$ be the set of F-conjugacy classes in $Z = Z$. For each $z \in H^1(F,Z)$, let $t_z \in T$ be such that $t_z^{-1}F(t_z)$ lies in z. Define*

$$\mathfrak{u}_z = \{\operatorname{Ad} g(\operatorname{Ad} t_z(Y_0) + Y') \mid g \in G^F, Y' \in \mathfrak{u}'^F\}$$

where Y_0 is a fixed rational regular nilpotent element. Then the following hold:

(i) *The sets $\mathfrak{u}_z$ for $z \in H^1(F,Z)$ are disjoint.*
(ii) *The set of regular nilpotent elements of $\mathfrak{g}^F$ is the union of the $\mathfrak{u}_z$.*
(iii) *$\mathfrak{u}_z = \operatorname{Ad} t_z(\mathfrak{u}_1)$ and hence each $\mathfrak{u}_z$ has the same cardinality.*
(iv) *The $\mathfrak{u}_z$ are precisely the regular nilpotent orbits in $\mathfrak{g}^F$.*

Proof. Note that t_z is a representative of z in $\mathcal{L}_T^{-1}(Z)$ according to the isomorphism $H^1(F,Z) \cong \mathcal{L}_T^{-1}(Z)/ZT^F$ of Lemma 10. Also note that the sets $\mathfrak{u}_z$ are well-defined. For suppose that t_z and $t_z' \in \mathcal{L}_T^{-1}(Z)$ both represent $z \in H^1(F,Z)$. Then $t_z' = t_z ah$ for some $a \in Z$, $h \in T^F$ and

$$\begin{aligned}\operatorname{Ad} t_z'(Y_0) + Y' &= \operatorname{Ad} ah(\operatorname{Ad} t_z(Y_0) + \operatorname{Ad}(ah)^{-1}(Y')) \\ &= \operatorname{Ad} h(\operatorname{Ad} t_z(Y_0) + \operatorname{Ad} h^{-1}(Y'))\end{aligned}$$

since $\operatorname{Ad} a(Y) = Y$ for $a \in Z$, $Y \in \mathfrak{g}$. Since $\operatorname{Ad} h^{-1}(Y') \in \mathfrak{u}'^F$ and $h \in G^F$ it follows that the definition of $\mathfrak{u}_z$ does not depend on the choice of representative for z in $\mathcal{L}_T^{-1}(Z)$.

(i) Suppose that $\mathfrak{u}_z \cap \mathfrak{u}_{z'} \neq \emptyset$. Then

$$\operatorname{Ad} t_z(Y_0) + Y' = \operatorname{Ad} g(\operatorname{Ad} t_{z'}(Y_0) + Y'')$$

for some $g \in G^F$, $Y', Y'' \in \mathfrak{u}'^F$. Now $\operatorname{Ad} t_z(Y_0) + Y' \in \mathfrak{b}$ and $\operatorname{Ad} t_{z'}(Y_0) + Y'' \in \mathfrak{b}$. By Theorem 5, $\operatorname{Ad} t_{z'}(Y_0) + Y''$ is regular and hence $\mathfrak{b} = \operatorname{Ad} g(\mathfrak{b})$ and consequently, by Lemma 2, $g \in B^F$. Write $g = su$ with $s \in T^F$, $u \in U^F$. Since $\operatorname{Ad} g(Y'') \in \mathfrak{u}'^F$ (by Lemma 1), we see that

$$\operatorname{Ad}(s^{-1}t_z)(Y_0) \equiv \operatorname{Ad}(ut_{z'})(Y_0) \mod \mathfrak{u}'^F.$$

But $ut_{z'} = t_{z'}t_{z'}^{-1}ut_{z'}$ and

$$\begin{aligned}\mathrm{Ad}(ut_{z'})(Y_0) &= \mathrm{Ad}\, t_{z'}(\mathrm{Ad}(t_{z'}^{-1}ut_{z'})(Y_0))\\ &= \mathrm{Ad}\, t_{z'}(Y_0 + \mathrm{Ad}(t_{z'}^{-1}ut_{z'})(Y_0) - Y_0)\\ &\equiv \mathrm{Ad}\, t_{z'}(Y_0) \mod \mathfrak{u}'^F\end{aligned}$$

since $t_{z'}^{-1}ut_{z'} \in U$ and hence, by Lemma 1, $\mathrm{Ad}(t_{z'}^{-1}ut_{z'})(Y_0) - Y_0 \in \mathfrak{u}'$. Thus

$$\mathrm{Ad}(s^{-1}t_z)(Y_0) \equiv \mathrm{Ad}\, t_{z'}(Y_0) \mod \mathfrak{u}'^F$$

which implies that

$$\mathrm{Ad}(t_{z'}^{-1}s^{-1}t_z)(Y_0) \equiv Y_0 \mod \mathfrak{u}'^F.$$

Since Y_0 is regular and $t_{z'}^{-1}s^{-1}t_z \in T$ this implies that $\alpha(t_{z'}^{-1}s^{-1}t_z) = 1$ for all $\alpha \in \Pi$ and hence that $t_{z'}^{-1}s^{-1}t_z \in Z$. Since T is abelian, we see that $t_{z'}ZT^F = t_zZT^F$ and hence $z = z'$.

(ii) By Theorem 5, every regular nilpotent element Y of $\mathfrak{g}^F$ lies in some Borel subalgebra $\mathfrak{b}_1$ of G. Since Y is rational and regular $\mathfrak{b}_1 = F(\mathfrak{b}_1)$. Thus, by Lemma 11, there exists $g \in G^F$ such that $\mathrm{Ad}\, g(\mathfrak{b}_1) = \mathfrak{b}$ and hence $\mathrm{Ad}\, g(Y) \in \mathfrak{b}^F$. Since Y is nilpotent, this implies that $\mathrm{Ad}\, g(Y) \in \mathfrak{u}^F$ (cf. Borel [1, 10.6(4)]) and we may assume that $Y \in \mathfrak{u}^F$. Let T be any rational maximal torus contained in B and let Φ and Π be defined as usual. Now write $Y = \sum_{\alpha\in\Phi^+} Y_\alpha$ where $Y_\alpha \in \mathfrak{g}_\alpha$. Then Y is regular if and only if $Y_\alpha \neq 0$ for all $\alpha \in \Pi$. Thus, by Humphreys [6, 16.2(C)], there exists a $t \in T$ such that $\mathrm{Ad}\, t(Y_0) \equiv Y \mod \mathfrak{u}'$. Since Y is rational we see that $\mathrm{Ad}\, t(Y_0) \equiv \mathrm{Ad}\, F(t)(Y_0) \mod \mathfrak{u}'$ and hence that $\mathrm{Ad}(t^{-1}F(t))(Y_0) \equiv Y_0 \mod \mathfrak{u}'$. But this implies that $\alpha(t^{-1}F(t)) = 1$ for all $\alpha \in \Pi$ and hence that $t^{-1}F(t) \in Z$ and $t \in \mathcal{L}_T^{-1}(Z)$. Thus $Y \in \mathfrak{u}_z$ for some $z \in H^1(F, Z)$.

(iii) Since $t_z^{-1}F(t_z) \in Z$, $\mathrm{Ad}(t_z^{-1}F(t_z))(Y') = Y'$ for $Y' \in \mathfrak{u}'$ and thus $Y' \in \mathfrak{u}'^F$ if and only if $\mathrm{Ad}\, t_z(Y') \in \mathfrak{u}'^F$. Hence

$$\begin{aligned}\mathfrak{u}_z &= \{\mathrm{Ad}\, g(\mathrm{Ad}\, t_z(Y_0) + Y') \mid g \in G^F, Y' \in \mathfrak{u}'^F\}\\ &= \{\mathrm{Ad}\, g(\mathrm{Ad}\, t_z(Y_0 + Y')) \mid g \in G^F, Y' \in \mathfrak{u}'^F\}\\ &= \{\mathrm{Ad}\, t_z\, \mathrm{Ad}(t_z^{-1}gt_z)(Y_0 + Y') \mid g \in G^F, Y' \in \mathfrak{u}'^F\}.\end{aligned}$$

Since $t_z \in \mathcal{L}_T^{-1}(Z)$, $t_z^{-1}F(t_z) \in Z$ and hence $t_z^{-1}gt_z \in G^F$ if and only if $g \in G^F$. Thus

$$\begin{aligned}\mathfrak{u}_z &= \{\mathrm{Ad}\, t_z(\mathrm{Ad}\, g(Y_0 + Y')) \mid g \in G^F, Y' \in \mathfrak{u}'^F\}\\ &= \mathrm{Ad}\, t_z(\mathfrak{u}_1).\end{aligned}$$

(iv) From Proposition 7 we know that there are $|H^1(F,Z)|$ regular nilpotent $\operatorname{Ad} G^F$-orbits on $\mathfrak{g}^F$. On the other hand, by definition, each of the $\mathfrak{u}_z$ is a union of $\operatorname{Ad} G^F$-orbits. Since there are also $|H^1(F,Z)|$ distinct sets $\mathfrak{u}_z$, these must be precisely the regular nilpotent $\operatorname{Ad} G^F$-orbits.

□

Note. This result is valid for all characteristics. The corresponding result for regular unipotent elements (Digne, Lehrer & Michel [3, 3.2]), however, is only true for good characteristic.

Lemma 13 (cf. Digne & Michel [4, 14.23]). *The number of regular nilpotent elements of $\mathfrak{g}^F$ is*

$$\frac{|G^F|}{|Z^{\circ F}|q^l}$$

where l is the semi-simple rank of G.

Proof. According to Theorem 5, if Y is regular nilpotent then it is contained in a unique Borel subalgebra and hence the number of rational regular nilpotent elements is equal to the product of the number of rational Borel subalgebras (which, by Lemma 2 and the conjugacy of Borel subgroups, is $|G^F/B^F|$ for any rational Borel subgroup B) and the number of such elements in a given rational Borel subalgebra. By Borel [1, 10.6(4)] and Theorem 5, if $B = TU$ is any F-stable Levi decomposition with Φ and Π defined as usual then, writing

$$\mathfrak{b} = \operatorname{Lie} T \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha,$$

we see that Y is regular nilpotent in $\mathfrak{b}$ if and only if $Y \in \mathfrak{u}$ and the $\mathfrak{g}_\alpha$ components are nonzero for α a simple root. Hence the number of regular nilpotents in $\mathfrak{b}^F$ is $|\mathfrak{u}'^F|$ times the number of elements of $\mathfrak{u}^F/\mathfrak{u}'^F$ for which every component is nontrivial in the decomposition $\mathfrak{u}/\mathfrak{u}' \cong \bigoplus_{\alpha\in\Pi} \mathfrak{g}_\alpha$. We shall call such elements regular also. By Humphreys [6, 16.2(C)] T acts transitively on the set of regular elements of $\mathfrak{u}/\mathfrak{u}'$ and by Digne & Michel [4, 0.35] the stabilizer of any given regular element is Z. Hence, according to Digne & Michel [4, 3.21], the number of T^F-orbits of rational regular elements of $\mathfrak{u}^F/\mathfrak{u}'^F$ is $|H^1(F,Z)|$. In each orbit however there are $|T^F/Z^F|$ elements. Hence the number of regular nilpotent elements in $\mathfrak{g}^F$ is

$$\frac{|G^F|}{|B^F|}\cdot|\mathfrak{u}'^F|\cdot|H^1(F,Z)|\cdot\frac{|T^F|}{|Z^F|}.$$

Now $|B^F| = |T^F||U^F|$, $|\mathfrak{u}^F/\mathfrak{u}'^F| = q^l$ and according to Corollary 9,

$$|H^1(F, Z)| = \frac{|Z^F|}{|Z^{\circ F}|}.$$

Substituting these values in gives the desired number. □

Corollary 14. *If $Y \in \mathfrak{g}^F$ is regular nilpotent then*

$$|C_{G^F}(Y)| = q^l |Z^{\circ F}||H^1(F, Z)| = q^l |Z^F|.$$

Proof. According to Theorem 12, the regular nilpotent $\operatorname{Ad} G^F$-orbits are the sets $\mathfrak{u}_z$ where $z \in H^1(F, Z)$ and they have the same cardinality. Hence each regular nilpotent $\operatorname{Ad} G^F$-orbit has size $|G^F|/(q^l|Z^{\circ F}||H^1(F, Z)|)$. This gives the first equality. The second equality follows from Corollary 9. □

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