THE RATIONAL ADJOINT ORBITS OF A LIE ALGEBRA OF TYPE G_2

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ABSTRACT. Let G be the Chevalley group of type G_2 over the algebraic closure of a finite field of characteristic not 2 or 3. Let F be the Frobenius endomorphism on G. In this paper we study and classify the $\operatorname{Ad} G^F$-orbits on $\mathfrak{g}^F$ and also the nilpotent $\operatorname{Ad} G$-orbits on $\mathfrak{g}$.

1. INTRODUCTION

In 1968, Chang [5] classified the rational conjugacy classes of a Chevalley group G of type G_2. Here we study the analogous problem of classifying the rational adjoint orbits of G on its Lie algebra $\mathfrak{g}$. The approach followed here will closely parallel that of Chang. The method is based on the observation that if $X \in \mathfrak{g}^F$ and $X = X_s + X_n$ is its Jordan decomposition then $X_n \in C_{\mathfrak{g}}(X_s) = \operatorname{Lie}(C_G(X_s))$.

First we classify the rational semisimple orbits. We deal with the regular semisimple orbits first (see Theorem 14), then the irregular semisimple orbits (see Theorem 16). If $X \in \mathfrak{g}^F$ has a nonzero irregular semisimple part then $C_G(X_s)$ is a nontrivial proper Levi subgroup of G, hence isomorphic to GL_2. Using information about the nilpotent orbits in $\mathfrak{gl}_2$ we may thus classify the nonsemisimple nonnilpotent orbits (see Theorem 20). Substantially more work has to be done to classify the nilpotent orbits. The results are contained in Theorems 31 and 32.

2. NOTATION

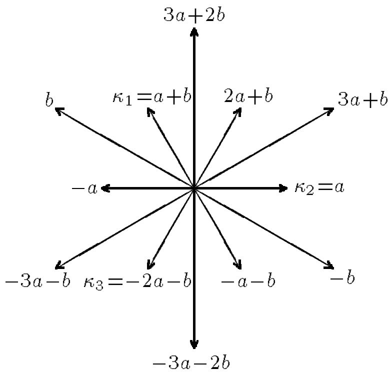
Let q be a power of a prime p and let $\mathbb{F}_q$ denote the finite field of q elements. Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. If $a \in \mathbb{F}_q$, let $\left(\frac{a}{q}\right)$ denote the Legendre symbol which takes the value -1, 1 or 0 according as a is a nonsquare in $\mathbb{F}_q$, a nonzero square in $\mathbb{F}_q$, or 0. According to context, G will denote either an arbitrary connected reductive algebraic group or the connected reductive algebraic group of type G_2 defined over $\mathbb{F}_q$. For the general theory of reductive algebraic groups refer to Borel [1], Springer [14] and Borel & Tits [2]. We will often abbreviate the term "reductive algebraic" to merely "reductive". We will denote the identity of G by e. Gothic lower case characters will be used to denote Lie algebras so, for example, $\mathfrak{g}$ (resp. $\mathfrak{u}$ etc.) denotes the Lie algebra of G (resp. U etc.). If H is a reductive group then H° will denote the connected component containing the identity and, for $X \in \mathfrak{h}$,

$$C_H(X) = \{h \in H \mid \operatorname{Ad} h(X) = X\}$$
$$C_{\mathfrak{h}}(X) = \{Y \in \mathfrak{h} \mid [Y, X] = 0\}.$$

We have $\operatorname{Lie}(C_G(X)) = C_{\mathfrak{g}}(X)$ if $X \in \mathfrak{g}$ is semisimple (see Borel [1, 9.1(2)]).

We will denote the root system of G by Φ and we will let Π be a choice of simple roots.

The connected reductive group of type G_2 over $\mathbb{F}_q$ can also be defined as the Chevalley group of type G_2 over $\overline{\mathbb{F}}_q$ and we shall use the notations of Carter [3]. Specifically, if Φ denotes the root system of type G_2, then in $\mathfrak{g}$ we have the elements e_r and h_r for $r \in \Phi$ and, if $t \in \overline{\mathbb{F}}_q$, we have in G the elements $x_r(t)$ and $h_r(t)$. The elements e_r and h_r satisfy the commutation relations $[h_r, e_s] = A_{rs}e_s = \frac{2(r,s)}{(r,r)}e_s$ and $[e_r, e_s] = N_{rs}e_{r+s}$ if $r \neq -s$. If $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$ is a root then we will define the height of β to be the integer $\sum_{\alpha\in\Pi} n_\alpha$. We will label the root system of type G_2 as in the following diagram.



It will be convenient to define $\kappa_1 = a + b$, $\kappa_2 = a$, $\kappa_3 = -2a - b$ and $\kappa_0 = 0$. For this labelling of the root system the elements e_r will be chosen so that the structure constants N_{rs} are as given on page 211 of Carter [3]. The action of G on the elements e_r, h_r is given by the relations

$$\begin{aligned} \operatorname{Ad} x_r(t)(e_r) &= e_r, \\ \operatorname{Ad} x_r(t)(e_{-r}) &= e_{-r} + th_r - t^2 e_r, \\ \operatorname{Ad} x_r(t)(h_s) &= h_s - A_{sr} t e_r \end{aligned}$$

and if r, s are linearly independent

$$\operatorname{Ad} x_r(t)(e_s) = \sum_{0 \leqslant i \leqslant m} M_{r,s,i} t^i e_{ir+s}$$

where m is the largest integer such that $mr + s$ is a root and

$$M_{r,s,i} = \frac{1}{i!} N_{r,s} N_{r,r+s} \dots N_{r,(i-1)r+s}.$$

We also have the Chevalley commutator relations

$$x_r(t)x_s(u)x_r(t)^{-1} = x_s(u) \prod_{i,j>0} x_{ir+js}(C_{ijrs} t^i u^j)$$

for $r \neq \pm s$, where $C_{ijrs} \in \{\pm 1, \pm 2, \pm 3\}$. The Weyl group will be denoted by W and the reflection corresponding to the root r will be denoted by w_r. If we define $n_r = x_r(1)x_{-r}(-1)x_r(1)$ then n_r is a representative in $N_G(T)$ of w_r and

$$\operatorname{Ad} n_r(e_s) = \eta_{r,s} e_{w_r(s)}$$

where $\eta_{r,s} = \pm 1$. The values of $\eta_{r,s}$ are given in Ree [12, (3.4)]. For convenience, tables of $A_{r,s}$, $N_{r,s}$, $\eta_{r,s}$, $\operatorname{Ad} x_r(t)(e_s)$ and $\operatorname{Ad} x_r(t)(h_s)$ are given in full in the Appendix.

For $r \in \Phi$, let H_r denote the subgroup of T consisting of the elements $h_r(s)$ for $s \in \overline{\mathbb{F}}_q$ and X_r the subgroup of G consisting of the elements $x_r(s)$ for $s \in \overline{\mathbb{F}}_q$. If

$w \in W$ then U_w will denote the subgroup of G generated by those X_r for which r is a positive root but $w(r)$ is a negative root.

There are two finite groups which feature prominently in the present work: the Weyl group $W \cong D_{12}$ (the dihedral group of order 12) and S_3 (the symmetric group on three letters). A set of conjugacy class representatives for W is given by

$$\{1, w_a, w_b, w_6 := w_a w_b, w_6^4, w_6^3\}.$$

Unless specified otherwise, w_6^3 and w_6^4 will have the values as defined above.

Given a maximal torus T, we shall denote by $\dot{w}$ a representative in $N = N_G(T)$ of the element w of $W(G,T) = N_G(T)/T$. We shall use the following choice of representatives. If $r \in \Phi$, let $\omega_r = n_r$ represent w_r. We may represent w_6 by the element $\omega_6 = n_a n_b$ and hence w_6^i is represented by ω_6^i.

Let $F : G \to G$ be the Frobenius morphism on G (see Digne & Michel [6, §3]). The set G^F of fixed points of G under F is a finite group (sometimes called a *finite group of Lie type* or *finite reductive group*). For type G_2, this group is the Chevalley group of type G_2 over $\mathbb{F}_q$. An element g (resp. subgroup H) of G that is fixed by F will be said to be a *rational* element (resp. subgroup) of G and G^F is the group of rational points of G.

If X is an element of $\mathfrak{g}$ (resp. $\mathfrak{g}^F$) then the set $\{\operatorname{Ad} g(X) \mid g \in G\}$ (resp. $\{\operatorname{Ad} g(X) \mid g \in G^F\}$) is called the $\operatorname{Ad} G$- (resp. $\operatorname{Ad} G^F$-)orbit of X. We will call an $\operatorname{Ad} G^F$-orbit in $\mathfrak{g}^F$ a *rational* orbit. If $g, h \in G$ we will use the notation $\operatorname{Int} g(h) = {}^g h$.

Two elements a, b of a group H are said to be *F-conjugate* if there exists $x \in H$ such that $b = x^{-1} a F(x)$. We shall denote by $H^1(F, H)$ the set of F-conjugacy classes of an algebraic group H defined over $\mathbb{F}_q$.

If $X \in \mathfrak{g}$, we will denote the semisimple and nilpotent parts of X by X_s and X_n respectively.

3. Generalities on semisimple orbits

First we state and prove some basic general results concerning semisimple classes. For the moment G will denote an arbitrary connected reductive group. We will fix an F-stable maximal torus T and an F-stable Borel subgroup B containing T.

Lemma 1. *Let $X, Y \in \mathfrak{t}$ with $Y = \operatorname{Ad}(g)(X)$ for some $g \in G$. If $g = u\dot{w}tu'$ with $u \in U$, $w \in W$, $t \in T$ and $u' \in U_w$, then $\operatorname{Ad}\dot{w}(X) = Y$, $\operatorname{Ad} u'(X) = X$ and $\operatorname{Ad} u(Y) = Y$.*

Proof. We have

$$\operatorname{Ad}(\dot{w}t) \operatorname{Ad} u'(X) = \operatorname{Ad} u^{-1}(Y).$$

The map $s \mapsto u' s u'^{-1} s^{-1}$ (resp. $s \mapsto u^{-1} s u s^{-1}$) takes T to U_w (resp. U) and hence upon differentiation we find that $\operatorname{Ad} u'(X) - X \in \mathfrak{u}_w$ (resp. $\operatorname{Ad} u^{-1}(Y) - Y \in \mathfrak{u}$). Hence $\operatorname{Ad}(\dot{w}t)(\operatorname{Ad} u'(X) - X) \in \mathfrak{u}^-$. Now

$$\operatorname{Ad}(\dot{w}t)(X) + \operatorname{Ad}(\dot{w}t)(\operatorname{Ad} u'(X) - X) = Y + \operatorname{Ad} u^{-1}(Y - \operatorname{Ad} u(Y)).$$

Since $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$ we must thus have $\operatorname{Ad}\dot{w}(X) = \operatorname{Ad}(\dot{w}t)(X) = Y$, $\operatorname{Ad} u'(X) = X$ and $\operatorname{Ad} u(Y) = Y$. $\square$

Lemma 2. *If $w \in W$, $\alpha \in \Phi$, then there exists $c \neq 0$ such that, for all t, ${}^{\dot{w}} x_\alpha(t) = x_{w(\alpha)}(ct)$ and $\operatorname{Ad}\dot{w}(e_\alpha) = c e_{w(\alpha)}$.*

Proof. Since ${}^{\dot w}X_\alpha = X_{w(\alpha)}$ and $\operatorname{Aut}(G_a) \equiv GL_1$ (see Borel [1, §10.10]), ${}^{\dot w}x_\alpha(t_\alpha) = x_{w(\alpha)}(ct_\alpha)$ for some $c \neq 0$. Consequently, on passing to the Lie algebra, $\operatorname{Ad}\dot w(e_\alpha) = ce_{w(\alpha)}$. □

Proposition 3 (cf. Steinberg [16, 2.7(a), 2.8]). *Let G be a connected reductive group. Let T be a fixed maximal torus. If $X \in \mathfrak{t}$ then*

(i) $C_G(X)^\circ = \langle T, X_\alpha \mid \alpha(X) = 0\rangle$.
(ii) $C_G(X) = \langle T, X_\alpha, \dot w \mid \alpha(X) = 0, \operatorname{Ad}\dot w(X) = X\rangle$.

Moreover $C_G(X)^\circ$ is reductive.

Proof. The corresponding result for centralizers of elements of T follows in a straightforward manner from the Bruhat decomposition (see, for example, Carter [Theorem 2.5.14]carter:1993). This approach must be modified in the present situation. Although well-known, there does not appear to be a proof of this result in the standard literature.

We will deal with (ii) first. Observe that T, the X_α with $\alpha(X) = 0$ and the $\dot w$ with $\operatorname{Ad}\dot w(X) = X$ all lie in $C_G(X)$.

Conversely, suppose that $g \in C_G(X)$. By the Bruhat decomposition we may write $g = u\dot w t u'$ with $u \in U$, $w \in W$, $t \in T$, $u' \in U_w$. By Lemma 1 we then have $\operatorname{Ad}\dot w(X) = X$, $\operatorname{Ad}u(X) = X$ and $\operatorname{Ad}u'(X) = X$. We want to show that if $\operatorname{Ad}u(X) = X$ then u lies in the subgroup generated by the X_α for which $\alpha(X) = 0$. So suppose that u does not lie in this subgroup. According to Borel [1, §14.4], $U = \prod_{\alpha\in\Phi^+} X_\alpha$ with uniqueness for any ordering of the roots in the product. Choose the ordering so that the roots have weakly decreasing height and write $u = \prod_{\alpha\in\Phi^+} x_\alpha(t_\alpha)$. Then for some α, $t_\alpha \neq 0$ and $\alpha(X) \neq 0$. Let i be the minimal height of such a root. We may then assume that $u = u_{>i}u_i$ where

$$u_{>i} = \prod_{\substack{\alpha\in\Phi^+\\ \operatorname{ht}(\alpha)>i}} x_\alpha(t_\alpha) \text{ and } u_i = \prod_{\substack{\alpha\in\Phi^+\\ \operatorname{ht}(\alpha)=i}} x_\alpha(t_\alpha).$$

We then have

$$\begin{aligned}\operatorname{Ad}u(X) - X = (\operatorname{Ad}u_{>i}(\operatorname{Ad}u_i(X) - X) - (\operatorname{Ad}u_i(X) - X))\\ + (\operatorname{Ad}u_{>i}(X) - X) + (\operatorname{Ad}u_i(X) - X).\end{aligned}$$

By differentiating the Chevalley commutator relations we see that, if

$$\mathfrak{g}_{>i} = \bigoplus_{\substack{\alpha\in\Phi^+\\ \operatorname{ht}(\alpha)>i}} \mathfrak{g}_\alpha,$$

with a similar definition being made for $\mathfrak{g}_{\geqslant i}$, then $\operatorname{Ad}u_i(X) - X \in \mathfrak{g}_{\geqslant i}$,

$$\operatorname{Ad}u_{>i}(\operatorname{Ad}u_i(X) - X) - (\operatorname{Ad}u_i(X) - X) \in \mathfrak{g}_{>2i}$$

and $\operatorname{Ad}u_{>i}(X) - X \in \mathfrak{g}_{>i}$. Since $i > 0$, we have $2i > i$ and

$$\operatorname{Ad}u(X) - X \equiv \operatorname{Ad}u_i(X) - X \mod \mathfrak{g}_{>i}.$$

A similar argument then shows that

$$\begin{aligned}\operatorname{Ad} u(X) - X &\equiv \sum_{\substack{\alpha\in\Phi^+\\ \mathrm{ht}(\alpha)=i}} (\operatorname{Ad} x_\alpha(t_\alpha)(X) - X) \mod \mathfrak{g}_{>i}\\ &\equiv \sum_{\substack{\alpha\in\Phi^+\\ \mathrm{ht}(\alpha)=i}} -\alpha(X)t_\alpha e_\alpha \mod \mathfrak{g}_{>i}.\end{aligned}$$

We have assumed however that, for some α of height i, $t_\alpha \neq 0$ and $\alpha(X) \neq 0$ and thus $\operatorname{Ad} u(X) - X \neq 0$, a contradiction. Thus $\alpha(X) = 0$ whenever $t_\alpha \neq 0$ and u lies in the subgroup generated by the X_α with $\alpha(X) = 0$. A similar result holds for u', whence $g = u\dot{w}tu'$ lies in the subgroup $\langle T, X_\alpha, \dot{w} \mid \alpha(X) = 0, \operatorname{Ad}\dot{w}(X) = X\rangle$.

The group $\langle T, X_\alpha \mid \alpha(X) = 0\rangle$ is connected, being generated by connected groups T, X_α (see Humphreys [7, §7.5, Corollary]). Now let $\operatorname{Ad}\dot{w}(X) = X$ and suppose that $\alpha(X) = 0$. Then, for all $t_\alpha \in \overline{\mathbb{F}}_q$, $\operatorname{Ad} x_\alpha(t_\alpha)(X) = X$ and hence

$$\operatorname{Ad} {}^{\dot{w}}x_\alpha(t_\alpha)(X) = \operatorname{Ad}(\dot{w}x_\alpha(t_\alpha))(X) = \operatorname{Ad}\dot{w}(X) = X.$$

On the other hand, from Lemma 2,

$$\operatorname{Ad} {}^{\dot{w}}x_\alpha(t_\alpha)(X) = X - w(\alpha)(X)ct_\alpha e_\alpha$$

and thus $w(\alpha)(X) = 0$. Hence $\dot{w}$ normalizes $\langle T, X_\alpha \mid \alpha(X) = 0\rangle$. This implies that $\langle T, X_\alpha \mid \alpha(X) = 0\rangle$ is a normal subgroup of $C_G(X)$ of finite index, the index being at most $|W|$. This subgroup is therefore the connected component $C_G(X)^\circ$ (see Borel [1, 1.2(c)]). That $C_G(X)^\circ$ is reductive follows from Digne & Michel [6, Prop. 1.14] since $\{\alpha \in \Phi \mid \alpha(X) = 0\}$ is a closed symmetric subset of Φ. □

We have the following theorem, stated in Springer & Steinberg [15, II, 3.19] without proof.

Theorem 4. *Assume that G is semisimple and simply connected. If the characteristic is not a torsion prime* (*see Springer & Steinberg* [15, I, 4.1]), *then $C_G(X)$ is connected and reductive for every semisimple element X of $\mathfrak{g}$, and conversely.*

Note that for type G_2 the only torsion prime is 2.

Proposition 5 (cf. Springer & Steinberg [15, II, 3.16]). *The map $\mathcal{O} \mapsto \mathcal{O} \cap \mathfrak{t}$ yields a bijection from the F-stable semisimple orbits in $\mathfrak{g}$ to the F-stable W-orbits on $\mathfrak{t}$.*

If G is as in Theorem 4 then

(i) *there is a bijection between F-stable semisimple classes in $\mathfrak{g}$ and semisimple $\operatorname{Ad} G^F$-orbits on $\mathfrak{g}^F$ given by $\mathcal{O} \mapsto \mathcal{O} \cap \mathfrak{g}^F$;*

(ii) *there is a bijection between semisimple $\operatorname{Ad} G^F$-orbits on $\mathfrak{g}^F$ and F-stable W-orbits in $\mathfrak{t}$ given by $\mathcal{O} \mapsto \operatorname{Ad} G(\mathcal{O}) \cap \mathfrak{t}$.*

Proof. The first statement is precisely Springer & Steinberg [15, II, 3.16]. The proof of the second statement follows from Theorem 4 and Digne & Michel [6, 3.21]. □

Definition. We shall say that a (not necessarily rational) element X of $\mathfrak{t}$ *represents* the $\operatorname{Ad} G^F$-orbit $\mathcal{O}$ if $\mathcal{O} = \operatorname{Ad} G(X) \cap \mathfrak{g}^F$.

Lemma 6. *Let $X \in \mathfrak{t}$. Then*

(i) *the W-orbit of X is F-stable if and only if $F(X) = \operatorname{Ad}\dot{w}(X)$ for some $w \in W$;*

(ii) *if* $w \in W$ *satisfies* $F(X) = \operatorname{Ad}\dot{w}(X)$ *and if* $w' \in W$ *is conjugate to* w *then* $F(X') = \operatorname{Ad}\dot{w}'(X')$ *for some* X' *in the* W*-orbit of* X.

Proof. Let us prove (i) first. If the orbit is F-stable, then $F(X)$ must be an element of the orbit and hence $F(X) = \operatorname{Ad}\dot{w}(X)$ for some $w \in W$. Conversely, if $F(X) = \operatorname{Ad}\dot{w}(X)$ and $w' \in W$ then

$$F(\operatorname{Ad}\dot{w}'(X)) = \operatorname{Ad}F(\dot{w}')(F(X)) = \operatorname{Ad}(F(\dot{w}')\dot{w})(X)$$

and hence the W-orbit of X is F-stable.

For (ii), if $w' = w_1 w w_1^{-1}$ then simply take $X' = \operatorname{Ad}\dot{w}_1(X)$. □

4. Semisimple orbits (type G_2)

For the rest of this paper, G will denote the Chevalley group of type G_2 over the field $\overline{\mathbb{F}}_q$ of characteristic $p \neq 2, 3$.

Lemma 7. *If* $\alpha \in \Phi$ *then* h_α *is an integral linear combination of* h_a *and* h_b *with coefficients given as follows:*

$$h_{a+b} = h_a + 3h_b \qquad h_{2a+b} = 2h_a + 3h_b$$
$$h_{3a+b} = h_a + h_b \qquad h_{3a+2b} = h_a + 2h_b$$

the remainder being given by $h_{-\alpha} = -h_\alpha$.

Proof. These relations hold for the corresponding elements of $\mathfrak{g}(\mathbb{C})$ by virtue of the relation $h_\alpha = \frac{2\alpha}{(\alpha,\alpha)}$ (see Carter [3, 3.3, 3.5]). Since the coefficients are integers they carry across to the elements h_α of $\mathfrak{g} = \mathfrak{g}(\overline{\mathbb{F}}_q)$. □

Lemma 8 (Carter [3, Prop. 6.4.2]). *If* $w \in W$, $\alpha \in \Phi$ *then* $\operatorname{Ad}\dot{w}(h_\alpha) = h_{w(\alpha)}$.

Exploiting the symmetry among κ_1, κ_2 and κ_3, we find that every element of $\mathfrak{t}$ may be represented uniquely in the form $\frac{1}{3}(\lambda_1 h_{\kappa_1} + \lambda_2 h_{\kappa_2} + \lambda_3 h_{\kappa_3})$ where $\lambda_1 + \lambda_2 + \lambda_3 = 0$. We will denote such an element of $\mathfrak{t}$ by $(\lambda_1, \lambda_2, \lambda_3)$. Note that the restriction $p \neq 3$ is important here for otherwise h_{κ_1}, h_{κ_2} and h_{κ_3} do not span $\mathfrak{t}$. According to Carter [3, 7.1.1] every group homomorphism from $\mathbb{Z}\Phi$ to $\overline{\mathbb{F}}_q^\times$ gives rise to an element of T. If κ_i gets sent to z_i under such a homomorphism then this element will be denoted by $h(z_1, z_2, z_3)$. The relation $\kappa_1 + \kappa_2 + \kappa_3 = 0$ implies that $z_1 z_2 z_3 = 1$. If $\chi_{\alpha,s}$ is the homomorphism from $\mathbb{Z}\Phi$ to $\overline{\mathbb{F}}_q^\times$ whose value on β is $s^{A_{\alpha\beta}}$ then the corresponding element of T is $h_\alpha(s)$. Consequently, $h_a(s) = h(s^{-1}, s^2, s^{-1})$ and $h_b(s) = h(s, s^{-1}, 1)$. Lemma 8 implies the following result.

Lemma 9. *If* $w \in W$ *and* $(\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{t}$ *then* $\operatorname{Ad}\dot{w}(\lambda_1, \lambda_2, \lambda_3)$ *is given according to the following table.*

w	$\operatorname{Ad}\dot{w}(\lambda_1, \lambda_2, \lambda_3)$	w	$\operatorname{Ad}\dot{w}(\lambda_1, \lambda_2, \lambda_3)$
1	$(\lambda_1, \lambda_2, \lambda_3)$	w_{3a+2b}	$(\lambda_3, \lambda_2, \lambda_1)$
w_a	$(-\lambda_3, -\lambda_2, -\lambda_1)$	w_6	$(-\lambda_2, -\lambda_3, -\lambda_1)$
w_b	$(\lambda_2, \lambda_1, \lambda_3)$	w_6^2	$(\lambda_3, \lambda_1, \lambda_2)$
w_{a+b}	$(-\lambda_1, -\lambda_3, -\lambda_2)$	w_6^3	$(-\lambda_1, -\lambda_2, -\lambda_3)$
w_{2a+b}	$(-\lambda_2, -\lambda_1, -\lambda_3)$	w_6^4	$(\lambda_2, \lambda_3, \lambda_1)$
w_{3a+b}	$(\lambda_1, \lambda_3, \lambda_2)$	w_6^5	$(-\lambda_3, -\lambda_1, -\lambda_2)$

According to Lemma 6, in order to determine the semisimple $\operatorname{Ad} G^F$-orbits on $\mathfrak{g}^F$ it is sufficient to determine representatives $X = (\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{t}$ for W-orbits in $\mathfrak{t}$ such that $F(X) = \operatorname{Ad} \dot{w}(X)$ where $w \in W$ runs over a chosen set of conjugacy class representatives. We will choose the set $\{1, w_a, w_b, w_6^3, w_6^4, w_6\}$ of conjugacy class representatives.

Lemma 10. *If $w \in W$ is a conjugacy class representative of W and $X \in \mathfrak{t}$ is such that $F(X) = \operatorname{Ad} \dot{w}(X)$ then X is of the form given in the following table.*

w	X	
1	$(\lambda_1, \lambda_2, \lambda_3)$	$\lambda_i^q = \lambda_i$
w_a	$(\lambda, \lambda^q - \lambda, -\lambda^q)$	$\lambda^{q^2} = \lambda$
w_b	$(\lambda, \lambda^q, -\lambda - \lambda^q)$	$\lambda^{q^2} = \lambda$
w_6^3	$(\lambda_1, \lambda_2, \lambda_3)$	$\lambda_i^q = -\lambda_i$
w_6^4	$(\lambda, \lambda^q, -\lambda - \lambda^q)$	$\lambda^{q^2} + \lambda^q + \lambda = 0$
w_6	$(\lambda, -\lambda^q, \lambda^q - \lambda)$	$\lambda^{q^2} - \lambda^q + \lambda = 0$

Proof. The result follows by application of Lemma 9. □

Lemma 11. *The values of the roots on $\mathfrak{t}$ are given by*

$$\kappa_i(\lambda_1, \lambda_2, \lambda_3) = \lambda_i$$
$$(\kappa_i - \kappa_j)(\lambda_1, \lambda_2, \lambda_3) = \lambda_i - \lambda_j.$$

Proof. Clearly we only need to prove the first formula. If $\alpha, \beta \in \Phi$ then $\beta(h_\alpha) = A_{\alpha\beta}$. For simplicity consider the case $i = 1$.

$$\begin{aligned}
\kappa_1(\lambda_1, \lambda_2, \lambda_3) &= \frac{1}{3}(\lambda_1 \kappa_1(h_{\kappa_1}) + \lambda_2 \kappa_1(h_{\kappa_2}) + \lambda_3 \kappa_1(h_{\kappa_3})) \\
&= \frac{1}{3}(\lambda_1 A_{\kappa_1\kappa_1} + \lambda_2 A_{\kappa_2\kappa_1} + \lambda_3 A_{\kappa_3\kappa_1}) \\
&= \frac{1}{3}(2\lambda_1 - \lambda_2 - \lambda_3) \\
&= \lambda_1.
\end{aligned}$$

The other two cases are dealt with similarly. □

We are now in a position to determine the regular semisimple $\operatorname{Ad} G^F$-orbits. Let $X \in \mathfrak{g}^F$ be semisimple and let $Y \in G \cdot X \cap \mathfrak{t}$. According to Proposition 3, X is regular if and only if $\alpha(Y) \neq 0$ for all $\alpha \in \Phi$. If $Y = (\lambda_1, \lambda_2, \lambda_3)$ then this condition becomes $\lambda_i \neq 0$ and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Definition. If the regular element $Y \in \mathfrak{t}$ represents the semisimple $\operatorname{Ad} G^F$-orbit of X and $F(Y) = \operatorname{Ad} \dot{w}(Y)$ then X and Y will be said to be *regular of type w*.

Lemma 12. *If $X_i \in \mathfrak{t}$ $(i = 1, 2)$ are regular and $F(X_i) = \operatorname{Ad} \dot{w}_i(X_i)$ for some $w_i \in W$ then X_1 and X_2 are in the same W-orbit if and only if w_1 and w_2 are conjugate.*

Proof. Lemma 6 gives one direction. Now suppose that $X_2 = \operatorname{Ad} \dot{w}(X_1)$. Then $F(\operatorname{Ad} \dot{w}(X_1)) = \operatorname{Ad} \dot{w}_2(\operatorname{Ad} \dot{w}(X_1))$ and hence we have $F(X_1) = \operatorname{Ad}(\dot{w}^{-1}\dot{w}_2\dot{w})(X_1)$. But $F(X_1) = \operatorname{Ad} \dot{w}_1(X_1)$ and hence $\dot{w}_1^{-1}\dot{w}^{-1}\dot{w}_2\dot{w} \in C_G(X_1)$. According to Proposition 3 and Theorem 4, $C_G(X_1) = T$ and hence $w_1 = w^{-1}w_2w$. □

Some elementary calculations in the group D_{12} yield the following lemma.

Lemma 13. *For type $W = D_{12}$, the groups $C_W(w)$, where w is a conjugacy class representative of W, are given by the following table.*

w	$C_W(w)$
1	W
w_a	$\{1, w_a, w_{3a+2b}, w_6^3\}$
w_b	$\{1, w_b, w_{2a+b}, w_6^3\}$
w_6^3	W
w_6^4	$\{1, w_6, w_6^2, w_6^3, w_6^4, w_6^5\}$
w_6	$\{1, w_6, w_6^2, w_6^3, w_6^4, w_6^5\}$

Theorem 14. *Let G be the connected reductive group of type G_2 over the field $\mathbb{F}_q$ of characteristic $p \neq 2, 3$. The regular semisimple $\operatorname{Ad} G^F$-orbits $\mathcal{O}$ on $\mathfrak{g}^F$ are given by the following table. If $Y \in \mathfrak{t}$ is an entry from the first column then the corresponding regular semisimple $\operatorname{Ad} G^F$-orbit is given by $\mathcal{O} = \operatorname{Ad} G(Y) \cap \mathfrak{g}^F$. In the table, X refers to any element of $\mathcal{O}$.*

representative	*type*	$\lvert C_{G^F}(X)\rvert$
$(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i^{q-1} = 1$, $\lambda_i \neq \lambda_j \quad (i \neq j)$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$	1	$(q-1)^2$
$(\lambda, \lambda^q - \lambda, -\lambda^q)$, $\lambda^{q^2-1} = 1$, $\lambda^{q-1} \neq \pm 1$	w_a	$q^2 - 1$
$(\lambda, \lambda^q, -\lambda^q - \lambda)$, $\lambda^{q^2-1} = 1$, $\lambda^{q-1} \neq \pm 1$	w_b	$q^2 - 1$
$(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i^{q-1} = -1$, $\lambda_i \neq \lambda_j \quad (i \neq j)$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$	w_6^3	$(q+1)^2$
$(\lambda, \lambda^q, -\lambda^q - \lambda)$, $\lambda^{q^2-1} + \lambda^{q-1} + 1 = 0$	w_6^4	$q^2 + q + 1$
$(\lambda, -\lambda^q, -\lambda^q - \lambda)$, $\lambda^{q^2-1} - \lambda^{q-1} + 1 = 0$	w_6	$q^2 - q + 1$

In this table representatives from different rows represent different $\operatorname{Ad} G^F$-orbits but different representatives of the same type may represent the same $\operatorname{Ad} G^F$-orbit. For each conjugacy class representative $w \in W$ there exists a function P_w which associates to an element $X \in \mathfrak{t}$ of type w a set $P_w(X)$ such that if $X_1, X_2 \in \mathfrak{t}$ are both of type w then X_1 and X_2 represent the same $\operatorname{Ad} G^F$-orbit if and only if $P_w(X_1) = P_w(X_2)$.

The parametrizing sets $P_w(X)$ are given as follows.

representative	*type*	*parametrizing set*	*no. of* $\operatorname{Ad} G^F$-*orbits of given form*
$(\lambda_1,\lambda_2,\lambda_3)$, $\lambda_i^{q-1}=1$, $\lambda_i\neq\lambda_j \quad (i\neq j)$, $\lambda_1+\lambda_2+\lambda_3=0$	1	$\{\{\lambda_1,\lambda_2,\lambda_3\},\{-\lambda_1,-\lambda_2,-\lambda_3\}\}$	$\frac{(q-1)(q-5)}{12}$
$(\lambda,\lambda^q-\lambda,-\lambda^q)$, $\lambda^{q^2-1}=1$, $\lambda^{q-1}\neq\pm1$	w_a	$\{\pm\lambda,\pm\lambda^q\}$	$\frac{(q-1)^2}{4}$
$(\lambda,\lambda^q,-\lambda^q-\lambda)$, $\lambda^{q^2-1}=1$, $\lambda^{q-1}\neq\pm1$	w_b	$\{\pm\lambda,\pm\lambda^q\}$	$\frac{(q-1)^2}{4}$
$(\lambda_1,\lambda_2,\lambda_3)$, $\lambda_i^{q-1}=-1$, $\lambda_i\neq\lambda_j \quad (i\neq j)$, $\lambda_1+\lambda_2+\lambda_3=0$	w_6^3	$\{\{\lambda_1,\lambda_2,\lambda_3\},\{-\lambda_1,-\lambda_2,-\lambda_3\}\}$	$\frac{(q-1)(q-5)}{12}$
$(\lambda,\lambda^q,-\lambda^q-\lambda)$, $\lambda^{q^2-1}+\lambda^{q-1}+1=0$	w_6^4	$\{\pm\lambda,\pm\lambda^q,\pm(\lambda^q+\lambda)\}$	$\frac{q^2-1}{6}$
$(\lambda,-\lambda^q,-\lambda^q-\lambda)$, $\lambda^{q^2-1}-\lambda^{q-1}+1=0$	w_6	$\{\pm\lambda,\pm\lambda^q,\pm(\lambda^q-\lambda)\}$	$\frac{q^2-1}{6}$

Proof. Proposition 5 and Lemma 6 imply that every regular semisimple $\operatorname{Ad} G^F$-orbit $\mathcal{O}$ is represented by an element of $\mathfrak{t}$ of type w where w is one of the chosen conjugacy class representatives in W. Proposition 5 and Lemma 12 then show that the representatives from different rows represent different $\operatorname{Ad} G^F$-orbits. The different parametrizing sets result from Lemmas 12 and 13.

If X is a rational element of the orbit represented by an element of $\mathfrak{t}$ of type w then $|C_G^F(X)| = |T_w^F|$ where T_w is an F-stable maximal torus twisted by w. By Carter [4, Prop. 3.3.5], $|T_w^F| = f(q)$ where $f(x)$ is the characteristic polynomial of w acting on $Y(T)\otimes_{\mathbb{Z}}\mathbb{R}$, $Y(T)$ being the cocharacter group of T. Now, the nontrivial elements of the set $\{1,\omega_a,\omega_b,\omega_6,\omega_6^4,\omega_6^3\}$ act on T as follows:

$$\begin{aligned}
\omega_a h(z_1,z_2,z_3)\omega_a^{-1} &= h(z_3^{-1},z_2^{-1},z_1^{-1})\\
\omega_b h(z_1,z_2,z_3)\omega_b^{-1} &= h(z_2,z_1,z_3)\\
\omega_6 h(z_1,z_2,z_3)\omega_6^{-1} &= h(z_3^{-1},z_1^{-1},z_2^{-1})\\
\omega_6^4 h(z_1,z_2,z_3)\omega_6^{-4} &= h(z_3,z_1,z_2)\\
\omega_6^3 h(z_1,z_2,z_3)\omega_6^{-3} &= h(z_1^{-1},z_2^{-1},z_3^{-1}).
\end{aligned}$$

The space $Y(T)\otimes\mathbb{R}$ is spanned over $\mathbb{R}$ by the elements

$$\begin{aligned}
h_a &: \lambda\mapsto h(\lambda^{-1},\lambda^2,\lambda^{-1})\\
h_b &: \lambda\mapsto h(\lambda,\lambda^{-1},1).
\end{aligned}$$

As an example of the calculation of $|T_w^F|$ let us look at the case $w = w_a$. We have $\omega_a h_a(\lambda)\omega_a^{-1} = h_a(\lambda^{-1}) = (-h_a)(\lambda)$, where we have used additive notation on $Y(T)\otimes\mathbb{R}$. Similarly, $\omega_a h_b(\lambda)\omega_a^{-1} = h_a(\lambda)h_b(\lambda) = (h_a+h_b)(\lambda)$. Thus the matrix of ω_a with respect to the basis (h_a, h_b) of $Y(T)\otimes\mathbb{R}$ is $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$. The characteristic polynomial of this matrix is $f(x) = x^2 - 1$ and thus $|T_{w_a}^F| = q^2 - 1$. Similarly, we find that $|T_1^F| = (q-1)^2$, $|T_{w_b}^F| = q^2 - 1$, $|T_{w_6}^F| = q^2 - q + 1$, $|T_{w_6^4}^F| = q^2 + q + 1$, $|T_{w_6^3}^F| = (q+1)^2$.

We will now determine the number of regular semisimple $\operatorname{Ad} G^F$-orbits of type w. If $(\lambda_1, \lambda_2, \lambda_3)$ is of type 1 or w_6^3, then $\lambda_3 = -\lambda_1 - \lambda_2$ and $\lambda_3 \neq \lambda_1$ if and only if $\lambda_2 \neq -2\lambda_1$. Also $\lambda_3 \neq \lambda_2$ if and only if $\lambda_2 \neq -\frac{1}{2}\lambda_1$ and $\lambda_3 \neq 0$ if and only if $\lambda_2 \neq -\lambda_1$. Since $|P_w(\lambda_1, \lambda_2, \lambda_3)| = 12$ for $w = 1, w_6^3$ the number of $\operatorname{Ad} G^F$-orbits of type 1 or w_6^3 is given by

$$\frac{1}{12}|\{(\lambda_1, \lambda_2) \mid \lambda_i^{q-1} = 1, \lambda_2 \neq \pm\lambda_1, -\frac{1}{2}\lambda_1, -2\lambda_1\}| = \frac{1}{12}(q-1)(q-5)$$

(note that if the characteristic is not 2 or 3 then ± 1, $-\frac{1}{2}$ and -2 are all distinct).

Similarly, the number of regular semisimple $\operatorname{Ad} G^F$-orbits of type w_a or w_b is given by

$$\frac{1}{4}|\{\lambda \mid \lambda^{q^2-1} = 1, \lambda^{q-1} \neq \pm 1\}| = \frac{(q-1)^2}{4}.$$

For types w_6^4 and w_6 the equations $\lambda^{q^2-1} \pm \lambda^{q-1} + 1 = 0$ are of degree $q^2 - 1$ with no repeated roots. Hence the number of $\operatorname{Ad} G^F$-orbits of type w_6^4 or w_6 is $\frac{q^2-1}{6}$. □

We are now able to determine the irregular semisimple $\operatorname{Ad} G^F$-orbits on $\mathfrak{g}^F$. By Proposition 5 and Proposition 3, we need to find representatives X for F-stable W-orbits on $\mathfrak{t}$ such that some root vanishes on X.

If two linearly independent roots vanish on $X = (\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{t}$ then all roots vanish on X and hence, by Lemma 11, we must have $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and thus $X = 0$.

Lemma 15. *If $w \in W$, $\alpha \in \Phi$ and $X \in \mathfrak{t}$ then $(w\alpha)(\operatorname{Ad}\dot{w}(X)) = \alpha(X)$.*

Proof. On the one hand, by Lemma 2,

$$\begin{aligned} \operatorname{Ad}(\dot{w}x_\alpha(t))(X) &= \operatorname{Ad}\dot{w}(X - \alpha(X)te_\alpha) \\ &= \operatorname{Ad}\dot{w}(X) - \alpha(X)cte_{w(\alpha)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \operatorname{Ad}(\dot{w}x_\alpha(t))(X) &= \operatorname{Ad}{}^{\dot{w}}x_\alpha(t)(\operatorname{Ad}\dot{w}(X)) \\ &= \operatorname{Ad}x_{w(\alpha)}(ct)(\operatorname{Ad}\dot{w}(X)) \\ &= \operatorname{Ad}\dot{w}(X) - w(\alpha)(\operatorname{Ad}\dot{w}(X))cte_{w(\alpha)}. \end{aligned}$$

Since $c \neq 0$, we must have $w(\alpha)(\operatorname{Ad}\dot{w}(X)) = \alpha(X)$. □

Theorem 16. *Let G be the connected reductive group of type G_2 over the field $\mathbb{F}_q$ of characteristic $p \neq 2,3$. The irregular semisimple $\operatorname{Ad} G^F$-orbits $\mathcal{O}$ on $\mathfrak{g}^F$ are given by the following table. If $Y \in \mathfrak{t}$ is an entry from the first column then the corresponding semisimple $\operatorname{Ad} G^F$-orbit is given by $\mathcal{O} = \operatorname{Ad} G(Y) \cap \mathfrak{g}^F$. In the table, X refers to any element of $\mathcal{O}$.*

representative	$\lvert C_{G^F}(X)\rvert$	*no. of reps of given form*
$(\lambda,\lambda,-2\lambda), \lambda^{q-1}=1$	$q(q-1)^2(q+1)$	$\frac{q-1}{2}$
$(\lambda,\lambda,-2\lambda), \lambda^{q-1}=-1$	$q(q-1)(q+1)^2$	$\frac{q-1}{2}$
$(\lambda,0,-\lambda), \lambda^{q-1}=1$	$q(q-1)^2(q+1)$	$\frac{q-1}{2}$
$(\lambda,0,-\lambda), \lambda^{q-1}=-1$	$q(q-1)(q+1)^2$	$\frac{q-1}{2}$
$(0,0,0)$	$q^6(q^2-1)(q^6-1)$	1

In this table representatives from different rows represent different $\operatorname{Ad} G^F$-orbits and two elements from the same row corresponding to $\lambda=\lambda_1$ and $\lambda=\lambda_2$ represent the same $\operatorname{Ad} G^F$-orbit if and only if $\lambda_1 = \pm\lambda_2$.

Proof. Since there are two orbits of W on Φ, namely the sets of long and short roots, Lemma 15 implies that if one root vanishes on X, we may assume that this root is either a or b.

If b is the only root which vanishes on X then $\lambda_i \neq 0$ and $\lambda_1=\lambda_2$ and hence X is of the form $(\lambda,\lambda,-2\lambda)$ with $\lambda\neq 0$. The W-orbit of X is then

$$\{(\lambda,\lambda,-2\lambda),(\lambda,-2\lambda,\lambda),(-2\lambda,\lambda,\lambda),(-\lambda,-\lambda,2\lambda),(-\lambda,2\lambda,-\lambda),(2\lambda,-\lambda,-\lambda)\}.$$

But the restrictions on the characteristic imply that $\lambda\neq\pm 2\lambda$ and hence for the W-orbit of X to be F-stable, there are only two possibilities, namely $F(X)=X$ in which case $\lambda^{q-1}=1$ and $F(X)=-X$ in which case $\lambda^{q-1}=-1$.

If a is the only root which vanishes on X then $\lambda_2=0$ and $\lambda_1,\lambda_3\neq 0$ so that X is of the form $(\lambda,0,-\lambda)$ with $\lambda\neq 0$. A similar argument to the above then shows that $\lambda^{q-1}=\pm 1$.

It remains to determine the centralizers. Let $Y=(\lambda,\lambda,-2\lambda)$ with $\lambda^{q-1}=1$. Then we may take $X=Y$ and by Proposition 3,

$$C_G(X)=\langle T, X_b, X_{-b}\rangle = H_{2a+b}\cdot\langle X_b, X_{-b}\rangle.$$

The product here is not direct since $H_{2a+b}\cap\langle X_b,X_{-b}\rangle=\{h_{2a+b}(\pm 1)\}$. By Digne & Michel [6, Prop. 3.19(ii)], since W is F-stable, we see that $\lvert C_{G^F}(X)\rvert = q(q-1)^2(q+1)$. Similarly, if $X=(\lambda,0,-\lambda)$ with $\lambda^{q-1}=1$ then

$$C_G(X)=\langle T, X_a, X_{-a}\rangle = H_{3a+2b}\cdot\langle X_a, X_{-a}\rangle$$

and by the same reasoning as above, $\lvert C_{G^F}(X)\rvert = q(q-1)^2(q+1)$.

Now suppose that $Y=(\lambda,\lambda,-2\lambda)$ with $\lambda^{q-1}=-1$. Then $F(Y)=\operatorname{Ad}\dot{w}_{2a+b}(Y)$. According to the Lang-Steinberg Theorem (see, for example, Digne & Michel [6, 3.10]) $x^{-1}F(x)=\dot{w}_{2a+b}^{-1}$ for some $x\in G$, in which case we may take $X=\operatorname{Ad}x(\lambda,\lambda,-2\lambda)\in\mathfrak{g}^F$. From the above, we find that $C_G(X)={}^{x}H_{2a+b}\cdot{}^{x}\langle X_b,X_{-b}\rangle$. Now $w_{2a+b}(b)=b$ and hence ${}^{x}(TX_b)$ is an F-stable Borel subgroup of $C_G(X)$ containing the F-stable maximal torus ${}^{x}T$. Also, if $w=1$ or $w=w_b$ then $F({}^{x}\dot{w}_b)={}^{F(x)}\dot{w}_b={}^{x\dot{w}_{2a+b}^{-1}}\dot{w}_b={}^{x}\dot{w}_b{}^{x}t$

for some $t \in T$, since w_{2a+b} and w_b commute. Hence the Weyl group of $C_G(X)$ with respect to ${}^xT$ is F-stable and, by Digne & Michel [6, Prop. 3.19(ii)],

$$|C_{G^F}(X)| = q(q+1)|({}^xT)^F| = q(q+1)|T^F_{w_a}| = q(q-1)(q+1)^2.$$

An analogous calculation applies when $Y = (\lambda, 0, -\lambda)$ with $\lambda^{q-1} = -1$ by replacing b by a and w_{2a+b} by w_{3a+2b}. □

5. Regular nilpotent elements

Lemma 17. *Let A be a finite abelian group. If F is an automorphism of A then $|H^1(F, A)| = |A^F|$.*

Proof. Since A is abelian, if $\mathcal{L} : A \to A$ is the map $x \mapsto x^{-1}F(x)$ then two elements of A are F-conjugate if and only if they represent the same coset in $A/\mathcal{L}(A)$. Also, as the kernel of $\mathcal{L}$ is A^F, $\mathcal{L}(A) \cong A/A^F$. Since A is finite we find that $|H^1(F, A)| = |A/\mathcal{L}(A)| = |A|/|A/A^F| = |A^F|$. □

We now prove what is essentially the Lie algebra analogue of Digne & Michel [6, Proposition 14.24].

Proposition 18. *Let G be a connected reductive algebraic group. Regardless of characteristic, the following results hold:*

(i) A nilpotent element $X \in \mathfrak{g}$ is regular if and only if it is conjugate under $\operatorname{Ad} G$ to an element of the form $\sum_{\alpha\in\Phi^+} X_\alpha$ where $X_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Phi^+$ and $X_\alpha \neq 0$ for $\alpha \in \Pi$.

(ii) If $X \in \mathfrak{g}^F$ is regular nilpotent then

$$|C_{G^F}(X)| = q^l|Z(G)^F|$$

where l is the semisimple rank of G.

Proof. That regular nilpotent elements exist is proved by Springer [13, 5.9(b)] for good characteristic and by Keny [11] for bad characteristic. Moreover by Springer [13, 5.8, 5.9(c)] and Keny [11] any element of the form given in (i) is regular and all regular nilpotent elements of $\mathfrak{g}$ are conjugate. This deals with (i).

Now according to Digne & Michel [6, 3.21], the $\operatorname{Ad} G^F$-orbits of regular nilpotent elements in $\mathfrak{g}^F$ which are $\operatorname{Ad} G$-conjugate to X are parametrized by $H^1(F, A(X))$ where $A(X) = C_G(X)/C_G(X)^\circ$ and moreover these $\operatorname{Ad} G^F$-orbits have the same cardinality. By Springer & Steinberg [15, 3.7(a)], we have $C_G(X) = Z(G) \cdot C_U(X)$ (one does not need the restriction that p be good for this). By Springer & Steinberg [15, 3.7(b)] and Keny [11, p. 197, line 8], $C_U(X)$ is connected regardless of the characteristic and hence $A(X) \cong Z(G)/Z(G)^\circ$. Thus $H^1(F, A(X)) \cong H^1(F, Z(G)/Z(G)^\circ)$. Since $Z(G)/Z(G)^\circ$ is a finite abelian group we may apply Lemma 17 and Digne & Michel [6, 3.13] to deduce that

$$|H^1(F, A(X))| = |Z(G)^F|/|Z(G)^{\circ F}|.$$

The proof of Digne & Michel [6, 14.23] carries across directly to the Lie algebra and we find that the number of regular nilpotent elements in $\mathfrak{g}^F$ is $|G^F|/|Z(G)^{\circ F}|q^l$. Thus $|C_{G^F}(X)| = |Z(G)^{\circ F}|q^l \cdot |H^1(F, A(X))| = q^l|Z(G)^F|$. □

6. Non-semisimple, non-nilpotent orbits

Lemma 19. *If $X' = (\lambda, \lambda, -2\lambda)$ or $X' = (\lambda, 0, -\lambda)$ with $\lambda^{q-1} = \pm 1$ then there is exactly one non-semisimple $\operatorname{Ad} G^F$-orbit $\mathcal{O}$ in $\mathfrak{g}^F$ such that if $X \in \mathcal{O}$ then the semisimple part of X is $\operatorname{Ad} G$-conjugate to X'.*

Proof. For both values of X' we have $C_G(X') \cong \mathrm{GL}_2$. The result follows from the fact that in GL_2 there is only one non-semisimple $\operatorname{Ad}\mathrm{GL}_2$-orbit on $\mathfrak{gl}_2$ with semisimple part X'. □

Theorem 20. *Let G be the connected reductive group of type G_2 over the field $\mathbb{F}_q$ of characteristic $p \neq 2, 3$. The non-nilpotent non-semisimple $\operatorname{Ad} G^F$-orbits $\mathcal{O}$ on $\mathfrak{g}^F$ are given by the following table. If $Y \in \mathfrak{t}$ is an entry from the first column then the corresponding $\operatorname{Ad} G^F$-orbit is given by $\mathcal{O} = G \cdot Y \cap \mathfrak{g}^F$. In the table, X refers to any element of $\mathcal{O}$.*

representative	$\lvert C_{G^F}(X)\rvert$	*no. of reps of given form*
$(\lambda, \lambda, -2\lambda) + e_b, \lambda^{q-1} = 1$	$q(q-1)$	$\frac{q-1}{2}$
$(\lambda, \lambda, -2\lambda) + e_b, \lambda^{q-1} = -1$	$q(q+1)$	$\frac{q-1}{2}$
$(\lambda, 0, -\lambda) + e_a, \lambda^{q-1} = 1$	$q(q-1)$	$\frac{q-1}{2}$
$(\lambda, 0, -\lambda) + e_a, \lambda^{q-1} = -1$	$q(q+1)$	$\frac{q-1}{2}$

Representatives from different rows represent different $\operatorname{Ad} G^F$-orbits and two elements from the same row corresponding to $\lambda = \lambda_1$ and $\lambda = \lambda_2$ represent the same $\operatorname{Ad} G^F$-orbit if and only if $\lambda_1 = \pm\lambda_2$.

Proof. The entries in the first column result from Lemma 19. The relationships amongst the different entries in the first column result from Theorem 16. It remains to calculate $|C_{G^F}(X)|$. Suppose that $Y_s = (\lambda, \lambda, -2\lambda)$ with $\lambda^{q-1} = 1$. According to Proposition 18,

$$\begin{aligned} |C_{G^F}(X)| &= |C_{C_{G^F}(X_s)}(X_n)| \\ &= q|Z(C_G(X_s))^F| \\ &= q|H_{2a+b}^F| \\ &= q(q-1). \end{aligned}$$

Now suppose that $Y_s = (\lambda, \lambda, -2\lambda)$ with $\lambda^{q-1} = -1$. Then,

$$|C_{G^F}(X)| = q|Z(C_{G^F}(X_s))^F| = q|({}^x H_{2a+b})^F|$$

where $x^{-1}F(x) = \dot{w}_{2a+b}^{-1}$. Now

$$\begin{aligned} F({}^x h_{2a+b}(t)) &= {}^{x\dot{w}_{2a+b}^{-1}} h_{2a+b}(t^q) \\ &= {}^x h_{2a+b}(t^{-q}). \end{aligned}$$

Thus $|({}^x H_{2a+b})^F| = |\{t \in \overline{\mathbb{F}}_q \mid t = t^{-q}\}| = q+1$ and $|C_{G^F}(X)| = q(q+1)$. Similar calculations apply when $Y = (\lambda, 0, -\lambda)$. □

7. Nilpotent orbits

7.1. **A grading of $\mathfrak{g}$.** We shall first define a useful grading of $\mathfrak{g}$. Define the function $h : \Phi \to \mathbb{Z}$ by

$$h(ia + jb) = 2j.$$

For $i \in \mathbb{Z}$, let

$$\Phi_i = \{\alpha \in \Phi \mid h(\alpha) = i\},$$
$$\Phi_{\geqslant i} = \{\alpha \in \Phi \mid h(\alpha) \geqslant i\},$$
$$\mathfrak{g}_i = \begin{cases} \bigoplus_{\alpha \in \Phi_i} \mathfrak{g}_\alpha & \text{if } i \neq 0, \\ \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha & \text{if } i = 0, \end{cases}$$
$$\mathfrak{g}_{\geqslant i} = \bigoplus_{j \geqslant i} \mathfrak{g}_j.$$

Let

$$G_0 = \langle T, X_\alpha \mid \alpha \in \Phi_0 \rangle$$

and if $i \geqslant 0$

$$G_{\geqslant i} = \begin{cases} \langle X_\alpha \mid \alpha \in \Phi_{\geqslant i} \rangle & \text{if } i > 0, \\ \langle T, X_\alpha \mid \alpha \in \Phi_{\geqslant 0} \rangle & \text{if } i = 0. \end{cases}$$

Then G_0 and $G_{\geqslant i}$ are closed connected subgroups of G such that $\operatorname{Lie} G_0 = \mathfrak{g}_0$ and $\operatorname{Lie} G_{\geqslant i} = \mathfrak{g}_{\geqslant i}$. Moreover $G_{\geqslant i}$ for $i > 0$ are subgroups of U and are hence unipotent subgroups.

Note. According to Dynkin-Kostant theory in positive characteristic (see, for example Humphreys [8, Chapter 7]) the nilpotent orbits of $\mathfrak{g}$ are parametrized by certain weighted Dynkin diagrams each of which gives rise to a grading of $\mathfrak{g}$. The grading introduced here is precisely that corresponding to the $\operatorname{Ad} G$-orbit of e_1 (see Section 7.7.3 for the definition of e_1).

Lemma 21. *The following relations hold*:

(i) $\operatorname{Ad} G_0(\mathfrak{g}_i) \subseteq \mathfrak{g}_i$.
(ii) $\operatorname{Ad} G_{\geqslant 0}(\mathfrak{g}_j) \subseteq \mathfrak{g}_{\geqslant j}$.
(iii) $\operatorname{Int} G_0(G_{\geqslant i}) = G_{\geqslant i}$ *if* $i \geqslant 0$.

Proof. All of these relations are simple consequences of the Chevalley relations. □

7.2. **The action of G_0 on $\mathfrak{g}_2$.**

Definition. If $r \in \Phi$, let $\phi_r : \mathrm{SL}_2 \to \langle X_r, X_{-r} \rangle$ be the isomorphism that sends $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ to $x_a(t)$, $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ to $x_{-a}(t)$ and $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ to $h_a(t)$.

Note. With this definition of ϕ_r,

$$\omega_r = x_r(1)x_{-r}(-1)x_r(1) = \phi_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proposition 22 (cf. Chang [5, 3.5]). *Let* $t_0e_b + t_1e_{a+b} + t_2e_{2a+b} + t_3e_{3a+b} \in \mathfrak{g}_2$ *be represented by the column vector* $\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix}$. *Then, by Lemma 21,* $\langle X_a, X_{-a}\rangle$ *acts on* $\mathfrak{g}_2$ *and the action of* $\phi_a\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \langle X_a, X_{-a}\rangle$ *on* $\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix}$ *is given by*

$$\phi_a\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \delta^3 & -3\gamma\delta^2 & 3\gamma^2\delta & \gamma^3 \\ -\beta\delta^2 & \delta(\alpha\delta + 2\beta\gamma) & -\gamma(2\alpha\delta+\beta\gamma) & -\alpha\gamma^2 \\ \beta^2\delta & -\beta(2\alpha\delta+\beta\gamma) & \alpha(\alpha\delta+2\beta\gamma) & \alpha^2\gamma \\ \beta^3 & -3\alpha\beta^2 & 3\alpha^2\beta & \alpha^3 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

Proof. The Bruhat decomposition in SL_2 is given explicitly by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & \alpha\gamma^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\gamma^{-1} & 0 \\ 0 & -\gamma \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \gamma^{-1}\delta \\ 0 & 1 \end{pmatrix} & \text{if } \gamma \neq 0, \\ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 & \beta\alpha^{-1} \\ 0 & 1 \end{pmatrix} & \text{if } \gamma = 0, \end{cases}$$

and hence

$$\phi_a\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{cases} x_a(\alpha\gamma^{-1})h_a(-\gamma^{-1})\omega_a x_a(\gamma^{-1}\delta) & \text{if } \gamma \neq 0, \\ h_a(\alpha)x_a(\beta\alpha^{-1}) & \text{if } \gamma = 0. \end{cases} \tag{7.1}$$

Now

$$\begin{aligned}\operatorname{Ad} x_a(z)(t_0e_b + t_1e_{a+b} + t_2e_{2a+b} + t_3e_{3a+b}) &= t_0e_b + (t_1 - t_0z)e_{a+b} \\ &\quad + (t_2 - 2t_1z + t_0z^2)e_{2a+b} + (t_3 + 3t_2z - 3t_1z^2 + t_0z^3)e_{3a+b}\end{aligned}$$

and hence $x_a(z)$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -z & 1 & 0 & 0 \\ z^2 & -2z & 1 & 0 \\ z^3 & -3z^2 & 3z & 1 \end{pmatrix}.$$

We have

$$\operatorname{Ad}\omega_a(t_0e_b + t_1e_{a+b} + t_2e_{2a+b} + t_3e_{3a+b}) = -t_3e_b - t_2e_{a+b} + t_1e_{2a+b} + t_0e_{3a+b}$$

and hence ω_a is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Note. The analogous formula of Chang [5, (3.4)] is incorrect and should read

$$(t_0, t_1, t_2, t_3)\omega_a = (t_3, t_2, -t_1, -t_0).$$

Finally, if $ia+b \in \Phi$ then $\operatorname{Ad} h_a(z) e_{ia+b} = z^{2i-3} e_{ia+b}$ and hence $h_a(z)$ is represented by the matrix

$$\begin{pmatrix} z^{-3} & 0 & 0 & 0 \\ 0 & z^{-1} & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^3 \end{pmatrix}.$$

If $\gamma \neq 0$, $\phi_a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha\gamma^{-1} & 1 & 0 & 0 \\ \alpha^2\gamma^{-2} & -2\alpha\gamma^{-1} & 1 & 0 \\ \alpha^3\gamma^{-3} & -3\alpha^2\gamma^{-2} & 3\alpha\gamma^{-1} & 1 \end{pmatrix} \begin{pmatrix} -\gamma^3 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma^{-1} & 0 \\ 0 & 0 & 0 & -\gamma^{-3} \end{pmatrix}.$$

$$\cdot \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\gamma^{-1}\delta & 1 & 0 & 0 \\ -\gamma^{-2}\delta^2 & -2\gamma^{-1}\delta & 1 & 0 \\ \gamma^{-3}\delta^3 & -3\gamma^{-2}\delta^2 & 3\gamma^{-1}\delta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \delta^3 & -3\gamma\delta^2 & 3\gamma^2\delta & \gamma^3 \\ -\beta\delta^2 & \delta(\alpha\delta+2\beta\gamma) & -\gamma(2\alpha\delta+\beta\gamma) & -\alpha\gamma^2 \\ \beta^2\delta & -\beta(2\alpha\delta+\beta\gamma) & \alpha(\alpha\delta+2\beta\gamma) & \alpha^2\gamma \\ \beta^3 & -3\alpha\beta^2 & 3\alpha^2\beta & \alpha^3 \end{pmatrix}.$$

If $\gamma = 0$, $\phi_a \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$ is represented by the matrix

$$\begin{pmatrix} \alpha^{-3} & 0 & 0 & 0 \\ 0 & \alpha^{-1} & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta\alpha^{-1} & 1 & 0 & 0 \\ \beta^2\alpha^{-2} & -2\beta\alpha^{-1} & 1 & 0 \\ \beta^3\alpha^{-3} & -3\beta^2\alpha^{-2} & 3\beta\alpha^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^{-3} & 0 & 0 & 0 \\ -\beta\alpha^{-2} & \alpha^{-1} & 0 & 0 \\ \beta^2\alpha^{-1} & -2\beta & \alpha & 0 \\ \beta^3 & -3\beta^2\alpha & 3\beta\alpha^2 & \alpha^3 \end{pmatrix}.$$

Hence for all $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2$, $\phi_a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is represented by the matrix

$$\begin{pmatrix} \delta^3 & -3\gamma\delta^2 & 3\gamma^2\delta & \gamma^3 \\ -\beta\delta^2 & \delta(\alpha\delta+2\beta\gamma) & -\gamma(2\alpha\delta+\beta\gamma) & -\alpha\gamma^2 \\ \beta^2\delta & -\beta(2\alpha\delta+\beta\gamma) & \alpha(\alpha\delta+2\beta\gamma) & \alpha^2\gamma \\ \beta^3 & -3\alpha\beta^2 & 3\alpha^2\beta & \alpha^3 \end{pmatrix}.$$

□

Lemma 23. $\omega_a h(z_1, z_2, z_3) = h(z_3^{-1}, z_2^{-1}, z_1^{-1})\omega_a$.

Proof. Since ω_a normalizes T we know that

$$\omega_a h(z_1, z_2, z_3)\omega_a^{-1} = h(z_1', z_2', z_3')$$

for some z_1', z_2', z_3'. Then

$$\begin{aligned} z_i' &= \kappa_i(h(z_1', z_2', z_3')) \\ &= \kappa_i(\omega_a h(z_1, z_2, z_3)\omega_a^{-1}) \\ &= w_a^{-1}(\kappa_i)(h(z_1, z_2, z_3)). \end{aligned}$$

The result then follows from the relation $w_a(\kappa_i) = -\kappa_{4-i}$. □

Lemma 24. *If* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2$, $t \in T$ *then*

(i) $\omega_a \phi_a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \phi_a \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \omega_a$

(ii) $t\phi_a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \phi_a \begin{pmatrix} \alpha & a(t)\beta \\ a(t)^{-1}\gamma & \delta \end{pmatrix} t.$

Proof. (i) follows from the relations $\omega_a = \phi_a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As for (ii), if $t = h(z_1, z_2, z_3)$ then

$$\begin{aligned} {}^t\omega_a &= h(z_1, z_2, z_3)\omega_a h(z_1^{-1}, z_2^{-1}, z_3^{-1}) \\ &= h(z_1, z_2, z_3)h(z_3, z_2, z_1)\omega_a \text{ by Lemma 23} \\ &= h(z_2^{-1}, z_2^2, z_2^{-1})\omega_a \text{ since } z_1z_2z_3 = 1 \\ &= h_a(z_2)\omega_a \\ &= h_a(a(t))\omega_a. \end{aligned}$$

Hence, if $\gamma \neq 0$, then according to Equation (7.1),

$$\begin{aligned} t\phi_a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} t^{-1} &= t x_a(\alpha\gamma^{-1}) h_a(-\gamma^{-1}) \omega_a x_a(\gamma^{-1}\delta) t^{-1} \\ &= {}^t x_a(\alpha\gamma^{-1}) \cdot {}^t h_a(-\gamma^{-1}) \cdot {}^t\omega_a \cdot {}^t x_a(\gamma^{-1}\delta) \\ &= x_a(a(t)\alpha\gamma^{-1}) h_a(-\gamma^{-1}) h_a(a(t)) \omega_a x_a(a(t)\gamma^{-1}\delta) \\ &= x_a(\alpha(a(t)^{-1}\gamma)^{-1}) h_a(-(a(t)^{-1}\gamma)^{-1}) \omega_a x_a((a(t)^{-1}\gamma)^{-1}\delta) \\ &= \phi_a \begin{pmatrix} \alpha & a(t)\beta \\ a(t)^{-1}\gamma & \delta \end{pmatrix} \end{aligned}$$

again by Equation (7.1).

If $\gamma = 0$,

$$\begin{aligned} t\phi_a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} t^{-1} &= t h_a(\alpha) x_a(\beta\alpha^{-1}) t^{-1} \\ &= {}^t h_a(\alpha) \cdot {}^t x_a(\beta\alpha^{-1}) \\ &= h_a(\alpha) x_a(a(t)\beta\alpha^{-1}) \\ &= \phi_a \begin{pmatrix} \alpha & a(t)\beta \\ a(t)^{-1}\gamma & \delta \end{pmatrix}. \end{aligned}$$

□

Lemma 25. *The map*
$$\phi_a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} h_b(z) \mapsto \begin{pmatrix} \alpha & z\beta \\ \gamma & z\delta \end{pmatrix}$$
is an isomorphism $G_0 \cong \mathrm{GL}_2$.

Proof. Since $G_0 = \langle T, X_a, X_{-a}\rangle$, we may deduce from Lemma 24(ii) and the relation $h(z_1, z_2, z_3) = h_a(z_1 z_2) h_b(z_1^2 z_2)$ that every element of G_0 can be expressed in the form $\phi_a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} h_b(z)$. That such an expression is unique follows from the fact that $\langle X_a, X_{-a}\rangle \cap H_b = H_a \cap H_b = \{e\}$.

It is clear that e gets sent to I so let $g_i = \phi_a \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} h_b(z_i)$ with $\alpha_i\delta_i - \beta_i\gamma_i = 1$ for $i = 1, 2$. Then, using Lemma 24(ii),
$$g_1^{-1} g_2 = \phi_a \begin{pmatrix} \delta_1\alpha_2 - \beta_1\gamma_2 & z_1(\delta_1\beta_2 - \beta_1\delta_2) \\ z_1^{-1}(-\gamma_1\alpha_2 + \alpha_1\gamma_2) & -\gamma_1\beta_2 + \alpha_1\delta_2 \end{pmatrix} h_b(z_1^{-1} z_2)$$
gets mapped to
$$\begin{pmatrix} \delta_1\alpha_2 - \beta_1\gamma_2 & z_2(\delta_1\beta_2 - \beta_1\delta_2) \\ z_1^{-1}(-\gamma_1\alpha_2 + \alpha_1\gamma_2) & z_1^{-1} z_2(-\gamma_1\beta_2 + \alpha_1\delta_2) \end{pmatrix} = \begin{pmatrix} \alpha_1 & z_1\beta_1 \\ \gamma_1 & z_1\delta_1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_2 & z_2\beta_2 \\ \gamma_2 & z_2\delta_2 \end{pmatrix}.$$
Hence the map is a group homomorphism, which, being clearly surjective, is thus an isomorphism. □

Corollary. *The map* $\psi_a : \mathrm{GL}_2 \to G_0$, *defined by*
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \phi_a \begin{pmatrix} \alpha & D^{-1}\beta \\ \gamma & D^{-1}\delta \end{pmatrix} h_b(D), \tag{7.2}$$
where $D = \alpha\delta - \beta\gamma$, *is an isomorphism.*

Lemma 26. *The action of* $h_b(z)$ *on* $\mathfrak{g}_2$ *is given by the matrix*
$$\begin{pmatrix} z^2 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-1} \end{pmatrix}.$$

Proof. This follows from the relation $\mathrm{Ad}\, h_b(z) e_{ia+b} = z^{2-i} e_{ia+b}$. □

Proposition 27. *The action of* $\psi_a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_0$ *on* $\mathfrak{g}_2$ *is given by the matrix*
$$D^{-1} \begin{pmatrix} \delta^3 & -3\gamma\delta^2 & 3\gamma^2\delta & \gamma^3 \\ -\beta\delta^2 & \delta(\alpha\delta + 2\beta\gamma) & -\gamma(2\alpha\delta + \beta\gamma) & -\alpha\gamma^2 \\ \beta^2\delta & -\beta(2\alpha\delta + \beta\gamma) & \alpha(\alpha\delta + 2\beta\gamma) & \alpha^2\gamma \\ \beta^3 & -3\alpha\beta^2 & 3\alpha^2\beta & \alpha^3 \end{pmatrix}$$
where $D = \alpha\delta - \beta\gamma$.

Proof. This follows from Equation (7.2), Proposition 22 and Lemma 26 by multiplying out the matrices. □

Lemma 28 (cf. Chang [5, Equation 3.6]). *Let $z, t \in \overline{\mathbb{F}}_q$ be such that $t \neq \pm 1$ and $z = t + t^{-1}$. Then*

$$\psi_a \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ t^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ z \end{pmatrix}.$$

Proof. By Proposition 27, the left hand side is given by

$$\frac{1}{1-t^2} \begin{pmatrix} 1 & 3t & 3t^2 & -t^3 \\ t & 1+2t^2 & t(2+t^2) & -t^2 \\ t^2 & t(2+t^2) & 1+2t^2 & -t \\ -t^3 & -3t^2 & -3t & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ t^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ z \end{pmatrix}$$

□

Lemma 29. *Let $r, s, t, u \in \overline{\mathbb{F}}_q$ be such that $u, r \neq 0$, $r^3 = t^{-1}$ and $s^2 = u$. Then*

$$\psi_a \begin{pmatrix} \frac{r^2}{2u} & \frac{r^2}{2s} \\ -\frac{r}{2u} & \frac{r}{2s} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ u \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ t^{-1} \end{pmatrix} \tag{7.3}$$

Proof. Writing

$$\psi_a \begin{pmatrix} \frac{r^2}{2u} & \frac{r^2}{2s} \\ -\frac{r}{2u} & \frac{r}{2s} \end{pmatrix} = \psi_a \left(\frac{r}{2u} \begin{pmatrix} r & rs \\ -1 & s \end{pmatrix} \right)$$

and using Proposition 27, we see that the left hand side of Equation (7.3) is given by

$$2ts^3 \cdot \frac{r^3}{8u^3} \begin{pmatrix} s^3 & 3u & 3s & -1 \\ -rs^3 & -ru & rs & -r \\ r^2s^3 & -r^2u & -r^2s & -r^2 \\ r^3s^3 & -3r^3u & 3r^3s & r^3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ u \\ 0 \end{pmatrix} = \frac{tr^3}{4s^3} \begin{pmatrix} 4s^3 \\ 0 \\ 0 \\ 4r^3s^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ t^{-1} \end{pmatrix}$$

since $r^3 = t^{-1}$. □

7.3. **Classification of nilpotent orbits.** It is useful to define the number ϵ by

$$\epsilon = \begin{cases} 1 & \text{if } q \equiv 1 \mod 3, \\ -1 & \text{if } q \equiv -1 \mod 3. \end{cases}$$

Let λ be a fixed nonsquare in $\mathbb{F}_q$, μ a fixed noncube in $\mathbb{F}_q$ if $q \equiv 1 \mod 3$ and $\zeta \in \mathbb{F}_q$ such that $x^3 - 3x - \zeta$ is irreducible over $\mathbb{F}_q$ if $q \equiv -1 \mod 3$. It is useful to record here the following result proved in [9].

Proposition 30. [9, Lemma 2, Corollary 3] *Let $q \equiv -1 \bmod 3$. Then the polynomial $x^3 - 3x - \zeta$ is irreducible over $\mathbb{F}_q$ if and only if $\zeta = t + t^{-1}$ where t is a noncube in $\mathbb{F}_{q^2}$ such that $t^{q+1} = 1$.*

We define the following nilpotent elements of $\mathfrak{g}^F$.

$$
\begin{aligned}
e_0 &= e_a + e_b \\
e_1 &= \begin{cases} e_b + e_{2a+b} & \text{if } q \equiv 1 \mod 3, \\ e_b + \lambda e_{2a+b} & \text{if } q \equiv -1 \mod 3 \end{cases} \\
e_2 &= \begin{cases} e_b + \lambda e_{2a+b} & \text{if } q \equiv 1 \mod 3, \\ e_b + e_{2a+b} & \text{if } q \equiv -1 \mod 3 \end{cases} \\
e_3 &= \begin{cases} e_b + \mu e_{3a+b} & \text{if } q \equiv 1 \mod 3, \\ e_b - e_{2a+b} + \zeta e_{3a+b} & \text{if } q \equiv -1 \mod 3 \end{cases} \\
e_4 &= e_{2a+b} \\
e_5 &= e_{3a+2b} \\
e_6 &= 0.
\end{aligned}
$$

For $i = 1, 2$ it will be convenient to write $e_i = e_b + \lambda_i e_{2a+b}$ with

$$
\lambda_1 = \begin{cases} 1 & \text{if } q \equiv 1 \mod 3, \\ \lambda & \text{if } q \equiv -1 \mod 3 \end{cases}
$$

and

$$
\begin{aligned}
\lambda_2 &= \begin{cases} \lambda & \text{if } q \equiv 1 \mod 3, \\ 1 & \text{if } q \equiv -1 \mod 3 \end{cases} \\
&= \lambda^{\epsilon} \lambda_1.
\end{aligned}
$$

Theorem 31 (cf. Stuhler [17, Satz 1′]). *Let G be the connected reductive group of type G_2 over the field $\mathbb{F}_q$ of characteristic $p \neq 2, 3$. Let e_i be as defined above. The set $\{e_0, e_1, e_4, e_5, e_6\}$ is a complete set of representatives for the nilpotent* $\operatorname{Ad} G$*-orbits in $\mathfrak{g}$.*

Proof. We will show that every nilpotent element of $\mathfrak{g}$ can be transformed into one of e_0, e_1, e_4, e_5 or e_6 by applying some element of G. Computation of the dimensions of their centralizers carried out later in the chapter will then show that these elements of $\mathfrak{g}$ lie in different $\operatorname{Ad} G$-orbits.

Firstly, every nilpotent element X of $\mathfrak{g}$ can be transformed into an element of $\mathfrak{u}$ so we may suppose that $X \in \mathfrak{u}$. Write $X = \sum_{\alpha \in \Phi^+} t_\alpha e_\alpha$. If $t_a, t_b \neq 0$ then X is regular and by Proposition 18 it is conjugate to e_0. Now suppose that $t_a = 0$. If $t_b = t_{a+b} = t_{2a+b} = t_{3a+b} = 0$ then after conjugating by an element of T if necessary we see that X is conjugate to e_5 or e_6. So suppose that at least one of $t_{ia+b} \neq 0$ for $0 \leqslant i \leqslant 3$. Then, conjugating by a suitable element of $X_{(3-i)a+b}$, we may suppose $t_{3a+2b} = 0$.

Case 1. $t_b \neq 0$. In this case we may first conjugate by an element of T to get $t_b = 1$. If $t_{a+b} \neq 0$ then, conjugating by a suitable element of X_α, we may assume that $t_{a+b} = 0$. If now $t_{2a+b} = t_{3a+b} = 0$ then X is conjugate to e_b, which, by conjugating by $h_b(-1)\omega_{3a+b}$, is seen to be conjugate to e_5. If $t_{2a+b} = 0$ but $t_{3a+b} \neq 0$ then, conjugating by a suitable element of T, we see that X is conjugate to $e_b + e_{3a+b}$, which, by Lemma 29, is conjugate to e_1. If $t_{2a+b} \neq 0$ and $t_{3a+b} = 0$ then similarly we see that X is conjugate to e_1. Lastly, if $t_{2a+b} \neq 0$ and $t_{3a+b} \neq 0$, we see that X is conjugate to $e_b - e_{2a+b} + ze_{3a+b}$ with $z \neq 0$, which, by Lemmas 28 and 29, is conjugate to e_1.

Case 2. $t_b = t_{a+b} = t_{2a+b} = 0$, $t_{3a+b} \neq 0$. Conjugating by ω_a transforms X into $-t_{3a+b}e_b$ and we are back in Case 1.

Case 3. $t_b = t_{a+b} = 0, t_{2a+b} \neq 0$. We can conjugate by elements of T and X_a to transform X to $e_4 = e_{2a+b}$.

Case 4. $t_b = 0, t_{a+b} \neq 0$. Conjugating by elements of T and X_a we can reduce X to the form $e_{a+b} + t_{3a+b}e_{3a+b}$. Conjugating by ω_a now transforms this to $-t_{3a+b}e_b + e_{2a+b}$, bringing us back to Case 1 or 3.

Case 5. $t_b = 0, t_a \neq 0$. In this case we may conjugate by elements of T, X_b, X_{a+b} and X_{2a+b} in turn to transform X into the form $e_a + t_{3a+2b}e_{3a+2b}$. Conjugating by ω_6^{-1} then transforms this to $-t_{3a+2b}e_b - e_{2a+b}$ and we are back to Case 1 or 3 above. □

Note. It is worth pointing out that Stuhler's proof of this result uses a Springer morphism f from the unipotent variety to the nilpotent variety but omits some details. If we assume that the regular unipotent element $x_a(1)x_b(1)$ gets sent to the regular nilpotent element $e_a + e_b$ under f then f is completely determined. By letting $t \to 0$ in the formulae

$$\begin{aligned}
\operatorname{Ad} h_a(t^2)h_b(t^3)(e_a + e_b) &= te_a + e_b \\
\operatorname{Int} h_a(t^2)h_b(t^3)(x_a(1)x_b(1)) &= x_a(t)x_b(1) \\
\operatorname{Ad} h_a(t)h_b(t^2)(e_a + e_b) &= e_a + te_b \\
\operatorname{Int} h_a(t)h_b(t^2)(x_a(1)x_b(1)) &= x_a(1)x_b(t) \\
\operatorname{Ad} h_a(t^4)h_b(t^6)x_b(t^{-1})x_{a+b}(\tfrac{1}{2}t^{-4})x_{3a+b}(\tfrac{3}{4}t^{-8})(e_a + e_b) &= t^2e_a + e_b + te_{a+b} + e_{2a+b}
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Int} h_a(t^4)h_b(t^6)x_b(t^{-1})x_{a+b}(\tfrac{1}{2}t^{-4})x_{3a+b}(\tfrac{3}{4}t^{-8})(x_a(1)x_b(1)) \\
&\quad = x_a(t^2)x_b(1)x_{a+b}(t)x_{2a+b}(1+t^3)x_{3a+b}(-t^5 - \tfrac{3}{2}t^2)x_{3a+2b}(\tfrac{3}{2}t^2 - 2t^4 - \tfrac{3}{2}t + t^5)
\end{aligned}$$

we see that $f(x_b(1)) = e_b$, $f(x_a(1)) = e_a$ and $f(x_b(1)x_{2a+b}(1)) = e_b + e_{2a+b}$ which leads to the set of representatives given by Stuhler.

Theorem 32 (cf. Chang [5, 3.9]). *Let G be the connected reductive group of type G_2 over the field $\mathbb{F}_q$ of characteristic $p \neq 2, 3$. Let e_i be as defined above. The set $\{e_i \mid 0 \leqslant i \leqslant 6\}$ is a complete set of representatives for the $\operatorname{Ad} G^F$-orbits on $\mathfrak{g}^F$ and the values of $\dim C_G(e_i)$ and $|C_{G^F}(e_i)|$ are given in the following table.*

representative e_i	$\dim C_G(e_i)$	$\lvert C_{G^F}(e_i)\rvert$
e_0	2	q^2
e_1	4	$6q^4$
e_2	4	$2q^4$
e_3	4	$3q^4$
e_4	6	$q^4(q^2-1)$
e_5	8	$q^6(q^2-1)$
e_6	14	$q^6(q^2-1)(q^6-1)$

Proof. The proof of this will proceed in several stages. Firstly, according to Proposition 18, every regular nilpotent element of $\mathfrak{g}^F$ is $\mathrm{Ad}\, G^F$-conjugate to e_0. Also, if $X = 0$ then $X = e_6$. Then, according to Theorem 31 and Proposition 18 every nonzero irregular nilpotent element of $\mathfrak{g}^F$ is conjugate under $\mathrm{Ad}\, G$ to one of e_1, e_4, or e_5. Now, according to Digne & Michel [6, 3.21], if $\mathcal{O}_i$ is the $\mathrm{Ad}\, G$-orbit of e_i then $\mathcal{O}_i^F$ is a union of $\mathrm{Ad}\, G^F$-orbits in bijective correspondence with the F-conjugacy classes of $C_G(e_i)/C_G(e_i)^\circ$. We will show in Proposition 38 that the centralizers of e_4 and e_5 are connected. Hence if $X \in \mathfrak{g}^F$ is $\mathrm{Ad}\, G$-conjugate to one of e_4 or e_5 then it is in fact $\mathrm{Ad}\, G^F$-conjugate to e_4 or e_5. We will also show in Proposition 39 that $C_G(e_1)/C_G(e_1)^\circ \cong S_3$ with trivial F-action. Hence $\mathcal{O}_1^F$ is a union of three $\mathrm{Ad}\, G^F$-orbits. The three elements e_1, e_2 and e_3 are $\mathrm{Ad}\, G$-conjugate to e_1 and if $e_i = \mathrm{Ad}\, g_i(e_1)$ for $i = 1, 2, 3$ it will be shown that the images of $g_i^{-1}F(g_i)$ in $C_G(e_1)/C_G(e_1)^\circ$ lie in different F-conjugacy classes and thus that e_1, e_2 and e_3 are representatives of the three $\mathrm{Ad}\, G^F$-orbits in $\mathcal{O}_1^F$. Hence if $X \in \mathfrak{g}^F$ is $\mathrm{Ad}\, G$-conjugate to e_1 then it is $\mathrm{Ad}\, G^F$-conjugate to one of e_1, e_2 or e_3. The set $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6\}$ is thus a complete set of representatives for the nilpotent $\mathrm{Ad}\, G^F$-orbits on $\mathfrak{g}^F$. The dimensions and orders of the centralizers will be computed in the next section. □

7.4. **Centralizers of nilpotent elements.** We will now prove the statements about the dimensions, orders and connectedness of the centralizers.

First of all, since e_0 is regular we have $\dim C_G(e_0) = 2$ and since $Z(G) = 1$, Proposition 18 shows that $|C_{G^F}(e_0)| = q^2$. It is also clear that $C_G(e_6) = G$ and hence $\dim C_G(e_6) = 14$ and $|C_{G^F}(e_6)| = q^6(q^2-1)(q^6-1)$. We will thus concentrate on the elements e_i for $1 \leqslant i \leqslant 5$.

Lemma 33. *We have $C_U(e_5) = U$, $C_U(e_4) = \langle X_b, X_{2a+b}, X_{3a+b}, X_{3a+2b}\rangle$ and $C_U(e_1) = \langle X_b, X_{2a+b}, X_{3a+2b}, V\rangle$, where $V = \{x_{3a+b}(3\lambda_1 t)x_{a+b}(t) \mid t \in \overline{\mathbb{F}}_q\}$. Moreover, for $i = 1, 2, 3$, $C_U(e_i) \subseteq G_{\geqslant 2}$, $\dim C_U(e_i) = 4$ and $|C_{U^F}(e_i)| = q^4$.*

Proof. Using the fact that $\mathrm{Ad}\, x_\alpha(t)(e_\beta) = e_\beta$ if $\alpha + \beta$ is not a root we see that $C_U(e_i) \supseteq \langle X_\alpha \mid \alpha \in R_i\rangle$ where $R_1 = R_2 = \{b, 2a+b, 3a+2b\}$,

$$R_3 = \begin{cases} \{a+b, 2a+b, 3a+2b\} & \text{if } q \equiv 1 \mod 3, \\ \{2a+b, 3a+2b\} & \text{if } q \equiv -1 \mod 3, \end{cases}$$

$R_4 = \{b, 2a+b, 3a+b, 3a+2b\}$ and $R_5 = \Phi^+$. For each i write $u \in U$ in the form

$$u = \prod_{\beta \in \Phi^+ - R_i} x_\beta(t_\beta) \prod_{\alpha \in R_i} x_\alpha(t_\alpha)$$

where the product $\prod x_\beta(t_\beta)$ is taken in decreasing order of $\beta \in \Phi^+ - R_i$. Then, using the formulae in the Appendix, $\mathrm{Ad}\, u(e_5) = e_5$ and

$$\mathrm{Ad}\, u(e_4) = e_4 + 3t_a e_{3a+b} + 3t_{a+b}e_{3a+2b}. \tag{7.4}$$

For $i = 1, 2$,

$$\begin{aligned} \mathrm{Ad}\, u(e_i) = e_i - t_a e_{a+b} + t_a^2 e_{2a+b} + (t_a^3 + 3\lambda_i t_a)e_{3a+b} \\ + (3\lambda_i t_{a+b} + 3t_a^2 t_{a+b} - t_{3a+b})e_{3a+2b}. \end{aligned} \tag{7.5}$$

If $q \equiv 1 \mod 3$ then

$$\mathrm{Ad}\, u(e_3) = e_3 - t_a e_{a+b} + t_a^2 e_{2a+b} + t_a^3 e_{3a+b} + (t_b\mu + t_a^3 t_b - t_{3a+b})e_{3a+2b}. \tag{7.6}$$

If $q \equiv -1 \mod 3$ then

$$\begin{aligned} \operatorname{Ad} u(e_3) &= e_3 - t_a e_{a+b} + t_a^2 e_{2a+b} + (t_a^3 - 3t_a)e_{3a+b} \\ &\quad + (3t_{a+b}(t_a^2-1) + t_b(\zeta - 3t_a + t_a^3) - t_{3a+b})e_{3a+2b}. \end{aligned} \tag{7.7}$$

From the above, we see that $C_U(e_5) = U$, $C_U(e_4) = \langle X_b, X_{2a+b}, X_{3a+b}, X_{3a+2b}\rangle$ and $C_U(e_1) = \langle X_b, X_{2a+b}, X_{3a+2b}, V\rangle$ where $V = \{x_{3a+b}(3\lambda_1 t)x_{a+b}(t) \mid t \in \overline{\mathbb{F}}_q\}$. We also see that if $u \in C_U(e_i)$ for $1 \leqslant i \leqslant 3$ then $t_a = 0$ and hence $C_U(e_i) \subseteq G_{\geqslant 2}$. Inspection of the above formulae shows that, for $1 \leqslant i \leqslant 3$, $\dim \operatorname{Ad} U(e_i) = 2$, $|\operatorname{Ad} U^F(e_i)| = q^2$ and hence that $\dim C_U(e_i) = \dim U - \dim \operatorname{Ad} U(e_i) = 4$ and $|C_{U^F}(e_i)| = |U^F|/|\operatorname{Ad} U^F(e_i)| = q^4$. □

Lemma 34. *For $1 \leqslant i \leqslant 5$, $C_B(e_i) = C_U(e_i)C_T(e_i)$.*

Proof. If $u \in U$ and $t \in T$, then $ut = t \cdot t^{-1}ut$ and $t^{-1}ut \in U$. It is thus sufficient to show that $C_B(e_i) = C_T(e_i)C_U(e_i)$. So let $tu \in C_B(e_i)$ with $t \in T$, $u \in U$. Let S_i be the set of roots α such that the coefficient of e_α in e_i does not vanish. Then $\operatorname{Ad} u(e_i)$ is of the form

$$\operatorname{Ad} u(e_i) = e_i + \sum_{\alpha \in S_i} c_\alpha e_\alpha + \sum_{\alpha \notin S_i} c_\alpha e_\alpha$$

and hence

$$\operatorname{Ad} tu(e_i) = \operatorname{Ad} t(e_i) + \sum_{\alpha \in S_i} c_\alpha \alpha(t) e_\alpha + \sum_{\alpha \notin S_i} c_\alpha \alpha(t) e_\alpha.$$

Since $\alpha(t) \neq 0$ and $tu \in C_B(e_i)$ we must have $c_\alpha = 0$ for $\alpha \notin S_i$. From Equations (7.4)–(7.7) we see that in all cases this implies that $c_\alpha = 0$ also for $\alpha \in S_i$ and hence $u \in C_U(e_i)$ implying also that $t \in C_T(e_i)$. □

Lemma 35. *We have $C_{G_0}(e_1) = \{\psi_a(\Omega^i \Sigma^j) \mid 0 \leqslant i \leqslant 2, 0 \leqslant j \leqslant 1\} \cong S_3$ where*

$$\Omega = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}(-3\lambda_1)^{\frac{1}{2}} \\ -\frac{1}{2\lambda_1}(-3\lambda_1)^{\frac{1}{2}} & -\frac{1}{2} \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and an isomorphism is given by

$$\begin{array}{ccc} e \leftrightarrow 1 & \psi_a(\Sigma) \leftrightarrow (12) & \psi_a(\Omega\Sigma) \leftrightarrow (23) \\ \psi_a(\Omega^2\Sigma) \leftrightarrow (13) & \psi_a(\Omega^2) \leftrightarrow (123) & \psi_a(\Omega) \leftrightarrow (132). \end{array}$$

Moreover, for $1 \leqslant i \leqslant 3$, there exist $g_i \in G_0$ such that $e_i = \operatorname{Ad} g_i(e_1)$ with $g_1^{-1}F(g_1) = 1$, $g_2^{-1}F(g_2) = \psi_a(\Sigma)$ and $g_3^{-1}F(g_3) = \psi_a(\Omega^k)$ ($k = 1$ or 2).

Proof. Let $g = \psi_a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then Proposition 27 shows that $g \in C_{G_0}(e_1)$ if and only if

$$D^{-1} \begin{pmatrix} \delta^3 & -3\gamma\delta^2 & 3\gamma^2\delta & \gamma^3 \\ -\beta\delta^2 & \delta(\alpha\delta + 2\beta\gamma) & -\gamma(2\alpha\delta + \beta\gamma) & -\alpha\gamma^2 \\ \beta^2\delta & -\beta(2\alpha\delta + \beta\gamma) & \alpha(\alpha\delta + 2\beta\gamma) & \alpha^2\gamma \\ \beta^3 & -3\alpha\beta^2 & 3\alpha^2\beta & \alpha^3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \lambda_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \lambda_1 \\ 0 \end{pmatrix},$$

that is,

$$\delta(\delta^2 + 3\lambda_1\gamma^2) = D \tag{7.8}$$

$$-\beta\delta^2 - \lambda_1\gamma(2\alpha\delta + \beta\gamma) = 0 \tag{7.9}$$

$$\beta^2\delta + \lambda_1\alpha(\alpha\delta + 2\beta\gamma) = \lambda_1 D \tag{7.10}$$

and

$$\beta(\beta^2 + 3\lambda_1\alpha^2) = 0 \tag{7.11}$$

where

$$\alpha\delta - \beta\gamma = D. \tag{7.12}$$

First suppose that $\beta = 0$. Then Equation (7.9) implies that $\alpha\gamma\delta = 0$ and Equation (7.12) implies that $\alpha\delta = D$. Equation (7.10) then implies that $\alpha = 1$, $\delta = D$, $\beta = 0$ and $\gamma = 0$. But then Equation (7.8) implies that $D^2 = 1$. Hence $D = \pm 1 = (-1)^j$, say, and

$$g = \psi_a \begin{pmatrix} 1 & 0 \\ 0 & (-1)^j \end{pmatrix} = \psi_a(\Sigma^j).$$

Now suppose that $\beta \neq 0$. Then Equation (7.11) implies that $\beta^2 + 3\lambda_1\alpha^2 = 0$. Hence $\beta = \pm(-3\lambda_1)^{\frac{1}{2}}\alpha = -\varepsilon_1(-3\lambda_1)^{\frac{1}{2}}\alpha$ with $\varepsilon_1 = \pm 1$ and we have fixed the choice of square root. Equation (7.12) then implies that $\alpha\delta + \varepsilon_1(-3\lambda_1)^{\frac{1}{2}}\alpha\gamma = D \neq 0$. Hence $\alpha \neq 0$ and

$$\delta + \varepsilon_1(-3\lambda_1)^{\frac{1}{2}}\gamma = \alpha^{-1}D. \tag{7.13}$$

Equation (7.10) now implies that $-3\lambda_1\alpha^2\delta + \lambda_1\alpha^2\delta + 2\lambda_1\alpha\beta\gamma = \lambda_1 D$, that is, $2\alpha(\beta\gamma - \alpha\delta) = D$ and hence that $\alpha = -\frac{1}{2}$. Hence $\beta = \frac{1}{2}\varepsilon_1(-3\lambda_1)^{\frac{1}{2}}$ and from Equation (7.13),

$$\delta = -2D - \varepsilon_1(-3\lambda_1)^{\frac{1}{2}}\gamma. \tag{7.14}$$

Adding δ times Equation (7.10) to β times Equation (7.9) gives

$$\delta D = \alpha\delta(\alpha\delta + 2\beta\gamma) - \beta\gamma(2\alpha\delta + \beta\gamma) = \alpha^2\delta^2 - \beta^2\gamma^2$$

and hence

$$\delta = \alpha\delta + \beta\gamma = 2\beta\gamma + D = D + \varepsilon_1(-3\lambda_1)^{\frac{1}{2}}\gamma. \tag{7.15}$$

Subtracting Equations 7.15 and 7.14 then gives $3D + 2\varepsilon_1(-3\lambda_1)^{\frac{1}{2}}\gamma = 0$ and hence

$$\gamma = -\frac{3D}{2\varepsilon_1(-3\lambda_1)^{\frac{1}{2}}}.$$

Substituting this into Equation (7.15) gives $\delta = -\frac{D}{2}$. Substituting the values of α, β, γ and δ found so far into Equation (7.8) now gives

$$-\frac{D}{2}\left(\frac{D^2}{4} - 3\lambda_1\frac{D^2 3\lambda_1}{4\lambda_1^2}\right) = D$$

and hence $D = \pm 1$. Thus

$$\begin{aligned} g &= \psi_a \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\varepsilon_1(-3\lambda_1)^{\frac{1}{2}} \\ \pm\frac{1}{2\lambda_1}\varepsilon_1(-3\lambda_1)^{\frac{1}{2}} & \mp\frac{1}{2} \end{pmatrix} \\ &= \psi_a \left(\begin{pmatrix} -\frac{1}{2} & \pm\frac{1}{2}\varepsilon_1(-3\lambda_1)^{\frac{1}{2}} \\ \pm\frac{1}{2\lambda_1}\varepsilon_1(-3\lambda_1)^{\frac{1}{2}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right) \\ &= \psi_a(\Omega^i \Sigma^j) \end{aligned}$$

for some $i = 1$ or 2, $j = 0$ or 1. Hence

$$C_{G_0}(e_1) = \{\psi_a(\Omega^i\Sigma^j) \mid 0 \leqslant i \leqslant 2, 0 \leqslant j \leqslant 1\}.$$

Calculations in GL_2 show that $\Omega^3 = I$, $\Sigma^2 = I$ and $\Sigma\Omega = \Omega^2\Sigma$ and hence

$$\psi_a(\Omega^i\Sigma^j)\psi_a(\Omega^k\Sigma^l) = \begin{cases} \psi_a(\Omega^{i+k}\Sigma^{j+l}) & \text{if } j \text{ is even,} \\ \psi_a(\Omega^{i+2k}\Sigma^{j+l}) & \text{if } j \text{ is odd} \end{cases} \tag{7.16}$$

Now $S_3 \cong \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^3 = 1\rangle$ and we can make the identification $s_1 = (12)$, $s_2 = (23)$. Equation (7.16) implies that $\psi_a(\Sigma)^2 = I$, $\psi_a(\Omega\Sigma)^2 = I$ and $(\psi_a(\Sigma)\psi_a(\Omega\Sigma))^3 = \psi_a(\Omega^2)^3 = 1$. But since $\psi_a(\Omega^i\Sigma^j) = (\psi_a(\Omega\Sigma)\psi_a(\Sigma))^i\psi_a(\Sigma)^j$, we see that $C_{G_0}(e_1)$ is generated by $\psi_a(\Sigma)$ and $\psi_a(\Omega\Sigma)$. Hence there is a homomorphism from $C_{G_0}(e_1)$ to S_3 sending $\psi_a(\Sigma)$ to (12) and $\psi_a(\Omega\Sigma)$ to (23). Since $|C_{G_0}(e_1)| = |S_3| = 6$ we must then have $C_{G_0}(e_1) \cong S_3$. Writing the elements of S_3 in terms of the generators (12) and (23), replacing (12) and (23) by $\psi_a(\Sigma)$ and $\psi_a(\Omega\Sigma)$ and using Equation (7.16) then gives the isomorphism explicitly.

We will now find $g_i \in G_0$ such that $e_i = \operatorname{Ad} g_i(e_1)$ for $1 \leqslant i \leqslant 3$. Since the e_i are F-stable, this also implies that $g_i^{-1}F(g_i) \in C_{G_0}(e_1)$. We may clearly take $g_1 = 1$. Now

$$\begin{aligned} e_2 &= e_b + \lambda_2 e_{2a+b} \\ &= e_b + \lambda^\epsilon \lambda_1 e_{2a+b} \\ &= \operatorname{Ad}(h_a(\lambda^\epsilon)h_b(\lambda^{\frac{3\epsilon}{2}}))(e_b + \lambda_1 e_{2a+b}). \end{aligned}$$

Hence let $g_2 = h_a(\lambda^\epsilon)h_b(\lambda^{\frac{3\epsilon}{2}})$. Then $F(g_2) = h_a(\lambda^\epsilon)h_b(-\lambda^{\frac{3\epsilon}{2}})$ and hence

$$g_2^{-1}F(g_2) = h_b(-1) = \psi_a(\Sigma).$$

If $q \equiv 1 \mod 3$, let $g_3 = \psi_a \begin{pmatrix} \frac{1}{2}\mu^{\frac{2}{3}} & \frac{1}{2}\mu^{\frac{2}{3}} \\ -\frac{1}{2}\mu^{\frac{1}{3}} & \frac{1}{2}\mu^{\frac{1}{3}} \end{pmatrix}$ where we make a fixed choice of cube root. Since $\lambda_1 = 1$ in this case, by substituting $t = \mu^{-1}$, $s = u = 1$ and $r = \mu^{\frac{1}{3}}$ into Lemma 29 we see that $e_3 = \operatorname{Ad} g_3(e_1)$. Now $F(\mu^{\frac{1}{3}}) = \omega\mu^{\frac{1}{3}}$ where $\omega = \frac{1}{2}(-1 \pm (-3)^{\frac{1}{2}})$

is a primitive cube root of unity. Then

$$\begin{aligned}
g_3^{-1}F(g_3) &= \psi_a\left(\begin{pmatrix}\mu^{-\frac{2}{3}} & -\mu^{-\frac{1}{3}}\\ \mu^{-\frac{2}{3}} & \mu^{-\frac{1}{3}}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\omega^2\mu^{\frac{2}{3}} & \frac{1}{2}\omega^2\mu^{\frac{2}{3}}\\ -\frac{1}{2}\omega\mu^{\frac{1}{3}} & \frac{1}{2}\omega\mu^{\frac{1}{3}}\end{pmatrix}\right)\\
&= \psi_a\begin{pmatrix}\frac{1}{2}(\omega+\omega^2) & \frac{1}{2}(\omega^2-\omega)\\ \frac{1}{2}(\omega^2-\omega) & \frac{1}{2}(\omega+\omega^2)\end{pmatrix}\\
&= \psi_a\begin{pmatrix}-\frac{1}{2} & \mp\frac{1}{2}(-3)^{\frac{1}{2}}\\ \mp\frac{1}{2}(-3)^{\frac{1}{2}} & -\frac{1}{2}\end{pmatrix}\\
&= \psi_a(\Omega^k)
\end{aligned}$$

where $k = 1$ or 2.

If $q \equiv -1 \mod 3$ then, according to Proposition 30, $\zeta = t+t^{-1}$ where t is a noncube in $\mathbb{F}_{q^2}$ such that $t^{q+1} = 1$. Let $\theta \in \mathbb{F}_{q^6}$ be a cube root of t^{-1}, $g_3' = \psi_a\begin{pmatrix}1 & -t\\ -t & 1\end{pmatrix}$ and $g_3'' = \psi_a\begin{pmatrix}\frac{\theta^2}{2\lambda} & \frac{\theta^2}{2\lambda^{\frac{1}{2}}}\\ -\frac{\theta}{2\lambda} & \frac{\theta}{2\lambda^{\frac{1}{2}}}\end{pmatrix}$ where we make a fixed choice of square root of λ. Lemma 28 shows that $e_3 = \mathrm{Ad}\, g_3'(e_b + t^{-1}e_{3a+b})$ and Lemma 29 shows that $e_b + t^{-1}e_{3a+b} = \mathrm{Ad}\, g_3''(e_1)$. Hence, if we let $g_3 = g_3'g_3'' = \psi_a(M)$ with

$$M = \begin{pmatrix}\frac{\theta^2+\theta^{-2}}{2\lambda} & \frac{\theta^2-\theta^{-2}}{2\lambda^{\frac{1}{2}}}\\ -\frac{\theta+\theta^{-1}}{2\lambda} & \frac{\theta-\theta^{-1}}{2\lambda^{\frac{1}{2}}}\end{pmatrix},$$

then $e_3 = \mathrm{Ad}\, g_3(e_1)$. Now $F(t) = t^{-1}$, $F(\lambda^{\frac{1}{2}}) = -\lambda^{\frac{1}{2}}$ and $F(\theta) = \omega\theta^{-1}$ and hence

$$F(g_3) = \psi_a\begin{pmatrix}\frac{\omega\theta^2+\omega^2\theta^{-2}}{2\lambda} & \frac{\omega\theta^2-\omega^2\theta^{-2}}{2\lambda^{\frac{1}{2}}}\\ -\frac{\omega^2\theta+\omega\theta^{-1}}{2\lambda} & \frac{\omega^2\theta-\omega\theta^{-1}}{2\lambda^{\frac{1}{2}}}\end{pmatrix}.$$

Since $\det M = \frac{\theta^3-\theta^{-3}}{2\lambda^{\frac{3}{2}}} = \frac{t^{-1}-t}{2\lambda^{\frac{3}{2}}}$, we have

$$\begin{aligned}
g_3^{-1}F(g_3) &= \psi_a\left(\frac{2\lambda^{\frac{3}{2}}}{t^{-1}-t}\begin{pmatrix}\frac{\theta-\theta^{-1}}{2\lambda^{\frac{1}{2}}} & -\frac{\theta^2-\theta^{-2}}{2\lambda^{\frac{1}{2}}}\\ \frac{\theta+\theta^{-1}}{2\lambda} & \frac{\theta^2+\theta^{-2}}{2\lambda}\end{pmatrix}\begin{pmatrix}\frac{\omega\theta^2+\omega^2\theta^{-2}}{2\lambda} & \frac{\omega\theta^2-\omega^2\theta^{-2}}{2\lambda^{\frac{1}{2}}}\\ -\frac{\omega^2\theta+\omega\theta^{-1}}{2\lambda} & \frac{\omega^2\theta-\omega\theta^{-1}}{2\lambda^{\frac{1}{2}}}\end{pmatrix}\right)\\
&= \psi_a\left(\frac{2\lambda^{\frac{3}{2}}}{t^{-1}-t}\begin{pmatrix}\frac{(\omega+\omega^2)(\theta^3-\theta^{-3})}{4\lambda^{\frac{3}{2}}} & \frac{(\omega-\omega^2)(\theta^3-\theta^{-3})}{4\lambda}\\ \frac{(\omega-\omega^2)(\theta^3-\theta^{-3})}{4\lambda^2} & \frac{(\omega+\omega^2)(\theta^3-\theta^{-3})}{4\lambda^{\frac{3}{2}}}\end{pmatrix}\right)\\
&= \psi_a\begin{pmatrix}-\frac{1}{2} & \pm\frac{1}{2}(-3\lambda)^{\frac{1}{2}}\\ \pm\frac{1}{2\lambda}(-3\lambda)^{\frac{1}{2}} & -\frac{1}{2}\end{pmatrix}\\
&= \psi_a(\Omega^k)
\end{aligned}$$

for $k = 1$ or 2. $\square$

Lemma 36. *Let* $X = t_\alpha e_\alpha + \sum_{\beta>\alpha} t_\beta e_\beta$ *and* $X' = s_{\alpha'} e_{\alpha'} + \sum_{\beta'>\alpha'} s_{\beta'} e_{\beta'}$ *be two elements of* $\mathfrak{u}$ *with* $t_\alpha, s_{\alpha'} \neq 0$. *If* $\operatorname{Ad}\dot{w}(X) = X'$ *for some* $w \in W$ *then* $w(\alpha) \geqslant \alpha'$ *and* $w^{-1}(\alpha') \geqslant \alpha$.

Proof. We have proved earlier that $\operatorname{Ad}\dot{w}(e_\alpha) = c_\alpha e_{w(\alpha)}$ for some $c_\alpha \neq 0$. Hence

$$\operatorname{Ad}\dot{w}(X) = t_\alpha c_\alpha e_{w(\alpha)} + \sum_{\beta>\alpha} t_\beta c_\beta e_{w(\beta)}.$$

If $\operatorname{Ad}\dot{w}(X) = X'$ then $t_\alpha c_\alpha e_{w(\alpha)}$ must be one of the terms in the expansion of X' and hence $w(\alpha) \geqslant \alpha'$. Similarly, since $\operatorname{Ad}\dot{w}^{-1}(X') = X$, we have $w^{-1}(\alpha') \geqslant \alpha$. $\square$

Proposition 37. *We have* $C_G(e_i) \subseteq G_{\geqslant 0} = B \cup B\omega_a X_a$ *for* $i = 1, 2, 3, 5$ *and* $C_G(e_4) \subseteq B \cup B\omega_b X_b$.

Proof. Let $g = b\dot{w}v \in C_G(e_5)$ with $b \in B$, $w \in W$ and $v \in X_w$. It then follows that $\operatorname{Ad}(\dot{w}^{-1}b^{-1})(e_{3a+2b}) = \operatorname{Ad}v(e_{3a+2b})$ and hence that $w(3a+2b) = 3a+2b$. Thus $w = 1$ or w_a and hence $C_G(e_5) \subseteq B \cup B\omega_a X_a = G_{\geqslant 0}$.

Now let $g = b\dot{w}v \in C_G(e_4)$ with $b \in B$, $w \in W$ and $v \in U_w$. Then we have $\operatorname{Ad}(\dot{w}^{-1}b^{-1})(e_{2a+b}) = \operatorname{Ad}v(e_{2a+b})$. Now $\operatorname{Ad}b^{-1}(e_{2a+b})$ and $\operatorname{Ad}v(e_{2a+b})$ are both of the form $t_{2a+b}e_{2a+b} + \sum_{\beta>2a+b} t_\beta e_\beta$ with $t_{2a+b} \neq 0$ and hence Lemma 36 implies that $w^{-1}(2a+b) \geqslant 2a+b$ and $w(2a+b) \geqslant 2a+b$ which implies that $w = 1$ or w_b. Thus $C_G(e_4) \subseteq B \cup B\omega_b X_b$.

For $C_G(e_1)$ it is convenient to consider $C_G(X)$ where X is the $\operatorname{Ad}G^F$-conjugate $\operatorname{Ad}\omega_a(e_1) = e_{3a+b} - \lambda_1 e_{a+b}$ of e_1. If we are able to show that $C_G(X) \subseteq G_{\geqslant 0}$ then $C_G(e_1) \subseteq {}^{\omega_a^{-1}}G_{\geqslant 0} = G_{\geqslant 0}$. Since $e_i = \operatorname{Ad}g_i(e_1)$ $(i = 1,2,3)$ with $g_i \in G_0$, we then have $C_G(e_i) = {}^{g_i}C_G(e_1) \subseteq {}^{g_i}G_{\geqslant 0} = G_{\geqslant 0}$. Let $g = ut\dot{w}v \in C_G(X)$ with $u \in U$, $t \in T$, $w \in W$ and $v \in U_w$. Then $\operatorname{Ad}(\dot{w}^{-1}(ut)^{-1})(e_{3a+b} - \lambda_1 e_{a+b}) = \operatorname{Ad}v(e_{3a+b} - \lambda_1 e_{a+b})$ and hence by Lemma 36, $w^{-1}(a+b) \geqslant a+b$ and $w(a+b) \geqslant a+b$. Hence $w = 1$, w_a or w_{3a+b}. Suppose that $w = w_{3a+b}$. If

$$u = x_{a+b}(t_{a+b})x_{3a+b}(t_{3a+b})x_{3a+2b}(t_{3a+2b})x_a(t_a)x_b(t_b)x_{2a+b}(t_{2a+b}),$$

then, since $\alpha + a + b$ and $\alpha + 3a + b$ are not roots for $\alpha \in \{a+b, 3a+b, 3a+2b\}$,

$$\begin{aligned}\operatorname{Ad}(\dot{w}^{-1}(ut)^{-1})(X) = \operatorname{Ad}(\omega_{3a+b}^{-1}t^{-1})(&-\lambda_1 e_{a+b} - 2\lambda_1 t_a e_{2a+b} + (1 + 3\lambda_1 t_a^2)e_{3a+b} \\ &- (3\lambda_1 t_{2a+b} + t_b + 3\lambda_1 t_a^2 t_b)e_{3a+2b}) \\ = \operatorname{Ad}v(X) \in \mathfrak{u}.&\end{aligned}$$

But $w_{3a+b}^{-1}(2a+b) < 0$ and $w_{3a+b}^{-1}(3a+b) < 0$ and hence $-2\lambda_1 t_a = 1 + 3\lambda_1 t_a^2 = 0$, a contradiction. Hence $C_G(X) \subseteq B \cup B\omega_a X_a = G_{\geqslant 0}$. $\square$

Proposition 38. *We have* $C_G(e_5) = G_{\geqslant 2} \cdot \langle X_a, X_{-a} \rangle$ *and*

$$C_G(e_4) = \langle X_{2a+b}, X_{3a+b}, X_{3a+2b} \rangle \cdot \langle X_b, X_{-b} \rangle$$

as direct products. Hence $\dim C_G(e_5) = 8$, $\dim C_G(e_4) = 6$, $|C_{G^F}(e_5)| = q^6(q^2-1)$, $|C_{G^F}(e_4)| = q^4(q^2-1)$ *and* $C_G(e_5)$ *and* $C_G(e_4)$ *are connected.*

Proof. First let $t = h(z_1, z_2, z_3) \in C_T(e_5)$. Then

$$\operatorname{Ad}t(e_{3a+2b}) = (3a+2b)(t)e_{3a+2b} = e_{3a+2b}$$

if and only if $(3a+2b)(t) = 1$. But $3a+2b = \kappa_1 - \kappa_3$ and hence this occurs if and only if $z_1 = z_3$, that is, $t \in H_a$. Hence $C_T(e_5) = H_a$. Similarly $t = h(z_1, z_2, z_3) \in C_T(e_4)$ if and only if $(2a+b)(t) = 1$, that is, if and only if $z_3 = 1$ and hence $C_T(e_4) = H_b$. Moreover, $\omega_a \in C_G(e_5)$, $\omega_b \in C_G(e_4)$, $X_a \subseteq C_G(e_5)$ and $X_b \subseteq C_G(e_4)$. Hence $b\omega_a u \in C_G(e_5)$ with $b \in B$, $u \in X_a$ if and only if $b \in C_G(e_5)$ and hence

$$\begin{aligned} C_G(e_5) &= C_B(e_5) \cup C_B(e_5)\omega_a X_a \\ &= C_U(e_5)C_T(e_5) \cup C_U(e_5)C_T(e_5)\omega_a X_a \quad \text{by Lemma 34} \\ &= UH_a \cup UH_a\omega_a X_a \quad \text{by Lemma 33} \\ &= G_{\geqslant 2} \cdot (X_a H_a \cup X_a H_a \omega_a X_a) \quad \text{as a direct product} \\ &= G_{\geqslant 2} \cdot \langle X_a, X_{-a} \rangle. \end{aligned}$$

Hence $\dim C_G(e_5) = 8$ and $|C_{G^F}(e_5)| = |G_{\geqslant 2}^F||\langle X_a, X_{-a}\rangle^F| = q^6(q^2-1)$.

Similarly

$$\begin{aligned} C_G(e_4) &= C_B(e_4) \cup C_B(e_4)\omega_b X_b \\ &= C_U(e_4)C_T(e_4) \cup C_U(e_4)C_T(e_4)\omega_b X_b \\ &= \langle X_b, X_{2a+b}, X_{3a+b}, X_{3a+2b}\rangle (H_b \cup H_b \omega_b X_b) \\ &= \langle X_{2a+b}, X_{3a+b}, X_{3a+2b}\rangle (X_b H_b \cup X_b H_b \omega_b X_b) \\ &= \langle X_{2a+b}, X_{3a+b}, X_{3a+2b}\rangle \cdot \langle X_b, X_{-b}\rangle \end{aligned}$$

as a direct product. Thus $\dim C_G(e_4) = 6$ and $|C_{G^F}(e_4)| = q^4(q^2-1)$. ☐

Proposition 39. *For* $1 \leqslant i \leqslant 3$, $\dim C_G(e_i) = 4$ *and*

$$C_G(e_i)/C_G(e_i)^\circ \cong S_3.$$

Moreover F *acts trivially on* $C_G(e_1)/C_G(e_1)^\circ$. *We have*

$$|C_{G^F}(e_1)| = 6q^4, |C_{G^F}(e_2)| = 2q^4 \text{ and } |C_{G^F}(e_3)| = 3q^4.$$

Proof. First we show that

$$C_G(e_i) = C_{G_0}(e_i) \cdot C_{G_{\geqslant 2}}(e_i). \tag{7.17}$$

By Proposition 37, $C_G(e_i) \subseteq G_{\geqslant 0} = G_0 \cdot G_{\geqslant 2}$. So let $gh \in C_G(e_i)$ with $g \in G_0$, $h \in G_{\geqslant 2}$. If $\alpha \geqslant b$ and $\beta \geqslant b$ are roots such that $\alpha + \beta$ is a root then $\alpha + \beta = 3a + 2b$ and hence $\operatorname{Ad} h(e_i) \equiv e_i \mod \mathfrak{g}_{3a+2b}$. Now $\operatorname{Ad} g(\mathfrak{g}_{3a+2b}) = \mathfrak{g}_{3a+2b}$ and hence

$$e_i = \operatorname{Ad}(gh)(e_i) \equiv \operatorname{Ad} g(e_i) \mod \mathfrak{g}_{3a+2b}.$$

But $\operatorname{Ad} g(\mathfrak{g}_2) \subseteq \mathfrak{g}_2$ and hence $g \in C_{G_0}(e_i)$, $h \in C_{G_{\geqslant 2}}(e_i)$ and Equation (7.17) holds.

By Lemma 21(iii), $C_{G_{\geqslant 2}}(e_i)$ is a normal subgroup of $C_G(e_i)$. Moreover, $C_{G_0}(e_i) \cap C_{G_{\geqslant 2}}(e_i) \subseteq G_0 \cap G_{\geqslant 2} = \{1\}$ and the product in Equation (7.17) is semidirect. The index of $C_{G_{\geqslant 2}}(e_i)$ in $C_G(e_i)$ is hence $|C_{G_0}(e_i)| = 6$ by Lemma 35. According to Borel [1, Prop. 1.7(b)], $C_{G_{\geqslant 2}}(e_i)$ is closed in $G_{\geqslant 2}$. But $G_{\geqslant 2}$ is itself closed in G and hence $C_{G_{\geqslant 2}}(e_i)$ is a closed subgroup of $C_G(e_i)$. By Lemma 33, $C_{G_{\geqslant 2}}(e_1) = \langle X_b, X_{2a+b}, X_{3a+2b}, V\rangle$ with $V = \{x_{3a+b}(3\lambda_1 t)x_{a+b}(t) \mid t \in \overline{\mathbb{F}}_q\}$ and, being generated by closed connected subgroups, $C_{G_{\geqslant 2}}(e_1)$ is itself connected (see Humphreys [7, Cor. 7.5]). Thus $C_{G_{\geqslant 2}}(e_i) = {}^{g_i}C_{G_{\geqslant 2}}(e_1)$ is also connected. We deduce that $C_{G_{\geqslant 2}}(e_i)$ is a closed connected normal subgroup of finite index in $C_G(e_i)$ and hence that

$C_{G_{\geqslant 2}}(e_i) = C_G(e_i)^\circ$. Thus $\dim C_G(e_i) = \dim C_G(e_i)^\circ = \dim C_{G_{\geqslant 2}}(e_i) = 4$ according to Lemma 33. Equation (7.17) implies that the elements of $C_{G_0}(e_i)$ form a set of coset representatives of $C_{G_{\geqslant 2}}(e_i)$ in $C_G(e_i)$ and hence that

$$C_G(e_i)/C_G(e_i)^\circ \cong C_{G_0}(e_i) \cong S_3.$$

Now $\left(\frac{\lambda_1}{q}\right) = \epsilon$ and it is well known that $\left(\frac{-3}{q}\right) = \epsilon$. Hence $-3\lambda_1$ is a square in $\mathbb{F}_q$ and, by Lemma 35, F acts trivially on $C_{G_0}(e_1)$. Thus F also acts trivially on $C_G(e_1)/C_G(e_1)^\circ$.

Finally, according to Lemma 33,

$$|C_{G^F}(e_i)| = |C_{G_0^F}(e_i)||C_{G_{\geqslant 2}^F}(e_i)| = q^4|C_{G_0^F}(e_i)|.$$

If $e_i = \operatorname{Ad} g_i(e_1)$ according to Lemma 35 then $C_{G_0}(e_i) = {}^{g_i}C_{G_0}(e_1)$. So let ${}^{g_i}x$ be an element of $C_{G_0^F}(e_i)$ with $x \in C_{G_0}(e_1)$. Then

$${}^{g_i}x = F({}^{g_i}x) = {}^{F(g_i)}F(x) = {}^{F(g_i)}x,$$

since F acts trivially on $C_{G_0}(e_1)$, and hence ${}^{g_i^{-1}F(g_i)}x = x$. Thus $|C_{G_0^F}(e_i)|$ is the number of fixed points of $g_i^{-1}F(g_i)$ acting by conjugation on $C_{G_0}(e_1) \cong S_3$. According to Lemma 35 we then have $|C_{G_0^F}(e_1)| = 6$, $|C_{G_0^F}(e_2)| = 2$ and $|C_{G_0^F}(e_3)| = 3$ giving the desired values of $|C_{G^F}(e_i)|$. $\square$

8. Summary of adjoint orbits (type G_2)

Let λ be a fixed nonsquare in $\mathbb{F}_q$, μ a fixed noncube in $\mathbb{F}_q$ if $q \equiv 1 \mod 3$ and $\zeta \in \mathbb{F}_q$ such that $x^3 - 3x - \zeta$ is irreducible over $\mathbb{F}_q$ if $q \equiv -1 \mod 3$. Define

$$e_0 = e_a + e_b$$

$$e_1 = \begin{cases} e_b + e_{2a+b} & \text{if } q \equiv 1 \mod 3, \\ e_b + \lambda e_{2a+b} & \text{if } q \equiv -1 \mod 3 \end{cases}$$

$$e_2 = \begin{cases} e_b + \lambda e_{2a+b} & \text{if } q \equiv 1 \mod 3, \\ e_b + e_{2a+b} & \text{if } q \equiv -1 \mod 3 \end{cases}$$

$$e_3 = \begin{cases} e_b + \mu e_{3a+b} & \text{if } q \equiv 1 \mod 3, \\ e_b - e_{2a+b} + \zeta e_{3a+b} & \text{if } q \equiv -1 \mod 3 \end{cases}$$

$$e_4 = e_{2a+b}$$

$$e_5 = e_{3a+2b}$$

$$e_6 = 0.$$

Then $\{e_0, e_1, e_4, e_5, e_6\}$ is a set of representatives for the nilpotent $\mathrm{Ad}\, G$-orbits on $\mathfrak{g}$. We have $C_G(e_1)/C_G(e_1)^\circ \cong S_3$ and for $i = 0, 4, 5, 6$ the group $C_G(e_i)/C_G(e_i)^\circ$ is trivial. The dimensions of their centralizers are given by the following table (for further information on the structure of the centralizers refer to Lemmas 33 and 35 and Propositions 37 and 38).

representative e_i	$\dim C_G(e_i)$
e_0	2
e_1	4
e_4	6
e_5	8
e_6	14

If $\lambda_1, \lambda_2, \lambda_3 \in \overline{\mathbb{F}}_q$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$ then let $(\lambda_1, \lambda_2, \lambda_3)$ denote the element $\frac{1}{3}(\lambda_1 h_{a+b} + \lambda_2 h_a + \lambda_3 h_{-2a-b})$ of $\mathrm{Lie}\, T$. If X is a rational element of $\mathfrak{g}$ then the $\mathrm{Ad}\, G^F$-orbit of X in $\mathfrak{g}^F$ is uniquely determined by giving a representative for the $\mathrm{Ad}\, G$-orbit of X in $\mathfrak{g}$. If $X \in \mathfrak{g}^F$ then it can be shown (see, for example, [10, Proposition 3.3]) that $C_G(X_s)^\circ$ is of the form ${}^g L_J$ for some $J \subseteq \Pi$ and some $g \in G$ where $g^{-1}F(g) = \dot{w}$ and $wF(J) = J$. The *genus* of X is defined to be the W-class of the coset $W_J w$ (see [10] for further discussion).

A complete set of representatives for these orbits and the orders of their centralizers in G^F are given in the following table. The table also gives the genus of X (in the form of a pair (J, w) for which $W_J w$ is a representative of the genus of X). For further information, refer to Theorems 14, 16, 20, 31 and 32.

representative of $\mathrm{Ad}\, G$-orbit of X	(J, w)	$\lvert C_{G^F}(X)\rvert$	no. of $\mathrm{Ad}\, G^F$-orbits of given form
$(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i^{q-1} = 1$, $\lambda_i \neq \lambda_j \quad (i \neq j)$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$	$(\emptyset, 1)$	$(q-1)^2$	$\frac{(q-1)(q-5)}{12}$
$(\lambda, \lambda^q - \lambda, -\lambda^q)$, $\lambda^{q^2-1} = 1$, $\lambda^{q-1} \neq \pm 1$	$(\emptyset, w_a)$	$q^2 - 1$	$\frac{(q-1)^2}{4}$
$(\lambda, \lambda^q, -\lambda^q - \lambda)$, $\lambda^{q^2-1} = 1$, $\lambda^{q-1} \neq \pm 1$	$(\emptyset, w_b)$	$q^2 - 1$	$\frac{(q-1)^2}{4}$
$(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i^{q-1} = -1$, $\lambda_i \neq \lambda_j \quad (i \neq j)$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$	$(\emptyset, w_6^3)$	$(q+1)^2$	$\frac{(q-1)(q-5)}{12}$
$(\lambda, \lambda^q, -\lambda^q - \lambda)$, $\lambda^{q^2-1} + \lambda^{q-1} + 1 = 0$	$(\emptyset, w_6^4)$	$q^2 + q + 1$	$\frac{q^2-1}{6}$
$(\lambda, -\lambda^q, -\lambda^q - \lambda)$, $\lambda^{q^2-1} - \lambda^{q-1} + 1 = 0$	$(\emptyset, w_6)$	$q^2 - q + 1$	$\frac{q^2-1}{6}$
$(\lambda, \lambda, -2\lambda), \lambda^{q-1} = 1$	$(\{b\}, 1)$	$q(q-1)^2(q+1)$	$\frac{q-1}{2}$
$(\lambda, \lambda, -2\lambda) + e_b, \lambda^{q-1} = 1$	$(\{b\}, 1)$	$q(q-1)$	$\frac{q-1}{2}$
$(\lambda, \lambda, -2\lambda), \lambda^{q-1} = -1$	$(\{b\}, w_{2a+b})$	$q(q-1)(q+1)^2$	$\frac{q-1}{2}$
$(\lambda, \lambda, -2\lambda) + e_b, \lambda^{q-1} = -1$	$(\{b\}, w_{2a+b})$	$q(q+1)$	$\frac{q-1}{2}$
$(\lambda, 0, -\lambda), \lambda^{q-1} = 1$	$(\{a\}, 1)$	$q(q-1)^2(q+1)$	$\frac{q-1}{2}$
$(\lambda, 0, -\lambda) + e_a, \lambda^{q-1} = 1$	$(\{a\}, 1)$	$q(q-1)$	$\frac{q-1}{2}$
$(\lambda, 0, -\lambda), \lambda^{q-1} = -1$	$(\{a\}, w_{3a+2b})$	$q(q-1)(q+1)^2$	$\frac{q-1}{2}$
$(\lambda, 0, -\lambda) + e_a, \lambda^{q-1} = -1$	$(\{a\}, w_{3a+2b})$	$q(q+1)$	$\frac{q-1}{2}$
e_0	$(\{a, b\}, 1)$	q^2	1
e_1	$(\{a, b\}, 1)$	$6q^4$	1
e_2	$(\{a, b\}, 1)$	$2q^4$	1
e_3	$(\{a, b\}, 1)$	$3q^4$	1
e_4	$(\{a, b\}, 1)$	$q^4(q^2-1)$	1
e_5	$(\{a, b\}, 1)$	$q^6(q^2-1)$	1
e_6	$(\{a, b\}, 1)$	$q^6(q^2-1)(q^6-1)$	1

Appendix. Tables Relating to Chevalley Groups of Type G_2

In this appendix we give explicit tables for several different quantities associated to a Chevalley group of type G_2.

We have the following relations:

$$
\begin{aligned}
[e_\alpha, e_\beta] &= N_{\alpha,\beta} e_{\alpha+\beta} \quad (\alpha \neq -\beta, \alpha+\beta \in \Phi) \\
[e_\alpha, e_{-\alpha}] &= h_\alpha \\
[h_\alpha, e_\beta] &= \beta(h_\alpha) e_\beta = A_{\alpha\beta} e_\beta \\
[h_\alpha, h_\beta] &= 0 \\
\operatorname{Ad} x_\alpha(t) e_\beta &= \sum_i M_{\alpha,\beta,i} t^i e_{i\alpha+\beta} \quad (\alpha \neq -\beta) \\
\operatorname{Ad} x_\alpha(t) e_{-\alpha} &= e_{-\alpha} + t h_\alpha - t^2 e_\alpha \\
\operatorname{Ad} x_\alpha(t) h_\beta &= h_\beta - \alpha(h_\beta) t e_\alpha = h_\beta - A_{\beta\alpha} t e_\alpha \\
\operatorname{Ad} h_\alpha(t) e_\beta &= t^{\beta(h_\alpha)} e_\beta = t^{A_{\alpha\beta}} e_\beta \\
\operatorname{Ad} h_\alpha(t) h_\beta &= h_\beta \\
\operatorname{Ad} n_\alpha e_\beta &= \eta_{\alpha,\beta} e_{w_\alpha(\beta)} \\
\operatorname{Ad} n_\alpha h_\beta &= h_{w_\alpha(\beta)}.
\end{aligned}
$$

We will use the choice of structure constants $N_{\alpha\beta}$ given in [3, p. 211]. The values are given in the following table in which the rows are indexed by α, the columns by β.

	a	b	$a+b$	$2a+b$	$3a+b$	$3a+2b$	$-a$	$-b$	$-a-b$	$-2a-b$	$-3a-b$	$-3a-2b$
a		-1	-2	3					3	2	-1	
b	1				1				-1			-1
$a+b$	2			3			3	-1		-2		-1
$2a+b$	-3		-3				2		-2		1	1
$3a+b$		-1					-1			1		1
$3a+2b$								-1	-1	1	1	
$-a$			-3	-2	1			1	2	-3		
$-b$			1			1	-1				-1	
$-a-b$	-3	1		2		1	-2			-3		
$-2a-b$	-2		2		-1	-1	3		3			
$-3a-b$	1			-1		-1		1				
$-3a-2b$		1	1	-1	-1							

The values of $A_{\alpha\beta}$ for $\alpha, \beta \in \Phi^+$ are given in the following table in which the rows are indexed by α, the columns by β and the remaining values are given by

$A_{-\alpha,\beta} = A_{\alpha,-\beta} = -A_{\alpha\beta}$.

	a	b	$a+b$	$2a+b$	$3a+b$	$3a+2b$
a	2	-3	-1	1	3	0
b	-1	2	1	0	-1	1
$a+b$	-1	3	2	1	0	3
$2a+b$	1	0	1	2	3	3
$3a+b$	1	-1	0	1	2	1
$3a+2b$	0	1	1	1	1	2

The values of $\eta_{\alpha,\beta}$ are given in [12]. The values of $\eta_{\alpha,\beta}$ for $\alpha \in \Phi^+$ are given by the following table, the rows and columns being labelled as before. The remaining values are given by $\eta_{\alpha,-\beta} = \eta_{\alpha,\beta}$.

	a	b	$a+b$	$2a+b$	$3a+b$	$3a+2b$
a	-1	1	1	-1	-1	1
b	1	-1	-1	1	1	-1
$a+b$	-1	-1	-1	1	1	1
$2a+b$	-1	1	1	-1	-1	1
$3a+b$	-1	-1	1	1	-1	1
$3a+2b$	1	-1	-1	1	1	-1
$-a$	-1	-1	-1	1	1	1
$-b$	-1	-1	1	1	-1	1
$-a-b$	1	1	-1	-1	1	-1
$-2a-b$	1	1	-1	-1	1	-1
$-3a-b$	1	1	1	-1	-1	-1
$-3a-2b$	1	1	1	-1	-1	-1

It will also be convenient to have a table of $w(\beta)$ for $w \in W$, $\beta \in \Phi$. The values of $w(\beta)$ for $\beta \in \Phi^+$ are given by the following table in which the rows are labelled by w and the columns by β.

	a	b	$a+b$	$2a+b$	$3a+b$	$3a+2b$
w_a	$-a$	$3a+b$	$2a+b$	$a+b$	b	$3a+2b$
w_b	$a+b$	$-b$	a	$2a+b$	$3a+2b$	$3a+b$
w_{a+b}	$2a+b$	$-3a-2b$	$-a-b$	a	$3a+b$	$-b$
w_{2a+b}	$-a-b$	b	$-a$	$-2a-b$	$-3a-2b$	$-3a-b$
w_{3a+b}	$-2a-b$	$3a+2b$	$a+b$	$-a$	$-3a-b$	b
w_{3a+2b}	a	$-3a-b$	$-2a-b$	$-a-b$	$-b$	$-3a-2b$
w_6	$2a+b$	$-3a-b$	$-a$	$a+b$	$3a+2b$	b
w_6^2	$a+b$	$-3a-2b$	$-2a-b$	$-a$	b	$-3a-b$
$w_2 = w_6^3$	$-a$	$-b$	$-a-b$	$-2a-b$	$-3a-b$	$-3a-2b$
$w_3 = w_6^4$	$-2a-b$	$3a+b$	a	$-a-b$	$-3a-2b$	$-b$
w_6^5	$-a-b$	$3a+2b$	$2a+b$	a	$-b$	$3a+b$

The Chevalley relations are explicitly given as follows:

$$\begin{array}{lcl}
\mathrm{Ad}\, x_a(t)e_a &=& e_a \\
\mathrm{Ad}\, x_a(t)e_b &=& e_b - te_{a+b} + t^2e_{2a+b} + t^3e_{3a+b} \\
\mathrm{Ad}\, x_a(t)e_{a+b} &=& e_{a+b} - 2te_{2a+b} - 3t^2e_{3a+b} \\
\mathrm{Ad}\, x_a(t)e_{2a+b} &=& e_{2a+b} + 3te_{3a+b} \\
\mathrm{Ad}\, x_a(t)e_{3a+b} &=& e_{3a+b} \\
\mathrm{Ad}\, x_a(t)e_{3a+2b} &=& e_{3a+2b} \\
\mathrm{Ad}\, x_a(t)e_{-a} &=& e_{-a} + th_a - t^2e_a \\
\mathrm{Ad}\, x_a(t)e_{-b} &=& e_{-b} \\
\mathrm{Ad}\, x_a(t)e_{-a-b} &=& e_{-a-b} + 3te_{-b} \\
\mathrm{Ad}\, x_a(t)e_{-2a-b} &=& e_{-2a-b} + 2te_{-a-b} + 3t^2e_{-b} \\
\mathrm{Ad}\, x_a(t)e_{-3a-b} &=& e_{-3a-b} - te_{-2a-b} - t^2e_{-a-b} - t^3e_{-b} \\
\mathrm{Ad}\, x_a(t)e_{-3a-2b} &=& e_{-3a-2b} \\
\mathrm{Ad}\, x_a(t)h_a &=& h_a - 2te_a \\
\mathrm{Ad}\, x_a(t)h_b &=& h_b + te_a
\end{array}$$

$$\begin{array}{lcl}
\mathrm{Ad}\, x_b(t)e_a &=& e_a + te_{a+b} \\
\mathrm{Ad}\, x_b(t)e_b &=& e_b \\
\mathrm{Ad}\, x_b(t)e_{a+b} &=& e_{a+b} \\
\mathrm{Ad}\, x_b(t)e_{2a+b} &=& e_{2a+b} \\
\mathrm{Ad}\, x_b(t)e_{3a+b} &=& e_{3a+b} + te_{3a+2b} \\
\mathrm{Ad}\, x_b(t)e_{3a+2b} &=& e_{3a+2b} \\
\mathrm{Ad}\, x_b(t)e_{-a} &=& e_{-a} \\
\mathrm{Ad}\, x_b(t)e_{-b} &=& e_{-b} + th_b - t^2e_b \\
\mathrm{Ad}\, x_b(t)e_{-a-b} &=& e_{-a-b} - te_{-a} \\
\mathrm{Ad}\, x_b(t)e_{-2a-b} &=& e_{-2a-b} \\
\mathrm{Ad}\, x_b(t)e_{-3a-b} &=& e_{-3a-b} \\
\mathrm{Ad}\, x_b(t)e_{-3a-2b} &=& e_{-3a-2b} - te_{-3a-b} \\
\mathrm{Ad}\, x_b(t)h_a &=& h_a + 3te_b \\
\mathrm{Ad}\, x_b(t)h_b &=& h_b - 2te_b
\end{array}$$

$$
\begin{aligned}
\operatorname{Ad} x_{a+b}(t)e_a &= e_a + 2te_{2a+b} + 3t^2e_{3a+2b}\\
\operatorname{Ad} x_{a+b}(t)e_b &= e_b\\
\operatorname{Ad} x_{a+b}(t)e_{a+b} &= e_{a+b}\\
\operatorname{Ad} x_{a+b}(t)e_{2a+b} &= e_{2a+b} + 3te_{3a+2b}\\
\operatorname{Ad} x_{a+b}(t)e_{3a+b} &= e_{3a+b}\\
\operatorname{Ad} x_{a+b}(t)e_{3a+2b} &= e_{3a+2b}\\
\operatorname{Ad} x_{a+b}(t)e_{-a} &= e_{-a} + 3te_b\\
\operatorname{Ad} x_{a+b}(t)e_{-b} &= e_{-b} - te_a - t^2e_{2a+b} - t^3e_{3a+2b}\\
\operatorname{Ad} x_{a+b}(t)e_{-a-b} &= e_{-a-b} + th_a + 3th_b - t^2e_{a+b}\\
\operatorname{Ad} x_{a+b}(t)e_{-2a-b} &= e_{-2a-b} - 2te_{-a} - 3t^2e_b\\
\operatorname{Ad} x_{a+b}(t)e_{-3a-b} &= e_{-3a-b}\\
\operatorname{Ad} x_{a+b}(t)e_{-3a-2b} &= e_{-3a-2b} - te_{-2a-b} + t^2e_{-a} + t^3e_b\\
\operatorname{Ad} x_{a+b}(t)h_a &= h_a + te_{a+b}\\
\operatorname{Ad} x_{a+b}(t)h_b &= h_b - te_{a+b}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Ad} x_{2a+b}(t)e_a &= e_a - 3te_{3a+b}\\
\operatorname{Ad} x_{2a+b}(t)e_b &= e_b\\
\operatorname{Ad} x_{2a+b}(t)e_{a+b} &= e_{a+b} - 3te_{3a+2b}\\
\operatorname{Ad} x_{2a+b}(t)e_{2a+b} &= e_{2a+b}\\
\operatorname{Ad} x_{2a+b}(t)e_{3a+b} &= e_{3a+b}\\
\operatorname{Ad} x_{2a+b}(t)e_{3a+2b} &= e_{3a+2b}\\
\operatorname{Ad} x_{2a+b}(t)e_{-a} &= e_{-a} + 2te_{a+b} - 3t^2e_{3a+2b}\\
\operatorname{Ad} x_{2a+b}(t)e_{-b} &= e_{-b}\\
\operatorname{Ad} x_{2a+b}(t)e_{-a-b} &= e_{-a-b} - 2te_a + 3t^2e_{3a+b}\\
\operatorname{Ad} x_{2a+b}(t)e_{-2a-b} &= e_{-2a-b} + 2th_a + 3th_b - t^2e_{2a+b}\\
\operatorname{Ad} x_{2a+b}(t)e_{-3a-b} &= e_{-3a-b} + te_{-a} + t^2e_{a+b} - t^3e_{3a+2b}\\
\operatorname{Ad} x_{2a+b}(t)e_{-3a-2b} &= e_{-3a-2b} + te_{-a-b} - t^2e_a + t^3e_{3a+b}\\
\operatorname{Ad} x_{2a+b}(t)h_a &= h_a - te_{2a+b}\\
\operatorname{Ad} x_{2a+b}(t)h_b &= h_b
\end{aligned}
$$

$$\begin{aligned}
\mathrm{Ad}\, x_{3a+b}(t)e_a &= e_a\\
\mathrm{Ad}\, x_{3a+b}(t)e_b &= e_b - te_{3a+2b}\\
\mathrm{Ad}\, x_{3a+b}(t)e_{a+b} &= e_{a+b}\\
\mathrm{Ad}\, x_{3a+b}(t)e_{2a+b} &= e_{2a+b}\\
\mathrm{Ad}\, x_{3a+b}(t)e_{3a+b} &= e_{3a+b}\\
\mathrm{Ad}\, x_{3a+b}(t)e_{3a+2b} &= e_{3a+2b}\\
\mathrm{Ad}\, x_{3a+b}(t)e_{-a} &= e_{-a} - te_{2a+b}\\
\mathrm{Ad}\, x_{3a+b}(t)e_{-b} &= e_{-b}\\
\mathrm{Ad}\, x_{3a+b}(t)e_{-a-b} &= e_{-a-b}\\
\mathrm{Ad}\, x_{3a+b}(t)e_{-2a-b} &= e_{-2a-b} + te_a\\
\mathrm{Ad}\, x_{3a+b}(t)e_{-3a-b} &= e_{-3a-b} + th_a + th_b - t^2e_{3a+b}\\
\mathrm{Ad}\, x_{3a+b}(t)e_{-3a-2b} &= e_{-3a-2b} + te_{-b}\\
\mathrm{Ad}\, x_{3a+b}(t)h_a &= h_a - 3te_{3a+b}\\
\mathrm{Ad}\, x_{3a+b}(t)h_b &= h_b + te_{3a+b}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ad}\, x_{3a+2b}(t)e_a &= e_a\\
\mathrm{Ad}\, x_{3a+2b}(t)e_b &= e_b\\
\mathrm{Ad}\, x_{3a+2b}(t)e_{a+b} &= e_{a+b}\\
\mathrm{Ad}\, x_{3a+2b}(t)e_{2a+b} &= e_{2a+b}\\
\mathrm{Ad}\, x_{3a+2b}(t)e_{3a+b} &= e_{3a+b}\\
\mathrm{Ad}\, x_{3a+2b}(t)e_{3a+2b} &= e_{3a+2b}\\
\mathrm{Ad}\, x_{3a+2b}(t)e_{-a} &= e_{-a}\\
\mathrm{Ad}\, x_{3a+2b}(t)e_{-b} &= e_{-b} - te_{3a+b}\\
\mathrm{Ad}\, x_{3a+2b}(t)e_{-a-b} &= e_{-a-b} - te_{2a+b}\\
\mathrm{Ad}\, x_{3a+2b}(t)e_{-2a-b} &= e_{-2a-b} + te_{a+b}\\
\mathrm{Ad}\, x_{3a+2b}(t)e_{-3a-b} &= e_{-3a-b} + te_b\\
\mathrm{Ad}\, x_{3a+2b}(t)e_{-3a-2b} &= e_{-3a-2b} + th_a + 2th_b - t^2e_{3a+2b}\\
\mathrm{Ad}\, x_{3a+2b}(t)h_a &= h_a\\
\mathrm{Ad}\, x_{3a+2b}(t)h_b &= h_b - te_{3a+2b}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Ad} x_{-a}(t)e_a &= e_a - th_a - t^2e_{-a}\\
\operatorname{Ad} x_{-a}(t)e_b &= e_b\\
\operatorname{Ad} x_{-a}(t)e_{a+b} &= e_{a+b} - 3te_b\\
\operatorname{Ad} x_{-a}(t)e_{2a+b} &= e_{2a+b} - 2te_{a+b} + 3t^2e_b\\
\operatorname{Ad} x_{-a}(t)e_{3a+b} &= e_{3a+b} + te_{2a+b} - t^2e_{a+b} + t^3e_b\\
\operatorname{Ad} x_{-a}(t)e_{3a+2b} &= e_{3a+2b}\\
\operatorname{Ad} x_{-a}(t)e_{-a} &= e_{-a}\\
\operatorname{Ad} x_{-a}(t)e_{-b} &= e_{-b} + te_{-a-b} + t^2e_{-2a-b} - t^3e_{-3a-b}\\
\operatorname{Ad} x_{-a}(t)e_{-a-b} &= e_{-a-b} + 2te_{-2a-b} - 3t^2e_{-3a-b}\\
\operatorname{Ad} x_{-a}(t)e_{-2a-b} &= e_{-2a-b} - 3te_{-3a-b}\\
\operatorname{Ad} x_{-a}(t)e_{-3a-b} &= e_{-3a-b}\\
\operatorname{Ad} x_{-a}(t)e_{-3a-2b} &= e_{-3a-2b}\\
\operatorname{Ad} x_{-a}(t)h_a &= h_a + 2te_{-a}\\
\operatorname{Ad} x_{-a}(t)h_b &= h_b - te_{-a}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Ad} x_{-b}(t)e_a &= e_a\\
\operatorname{Ad} x_{-b}(t)e_b &= e_b - th_b - t^2e_{-b}\\
\operatorname{Ad} x_{-b}(t)e_{a+b} &= e_{a+b} + te_a\\
\operatorname{Ad} x_{-b}(t)e_{2a+b} &= e_{2a+b}\\
\operatorname{Ad} x_{-b}(t)e_{3a+b} &= e_{3a+b}\\
\operatorname{Ad} x_{-b}(t)e_{3a+2b} &= e_{3a+2b} + te_{3a+b}\\
\operatorname{Ad} x_{-b}(t)e_{-a} &= e_{-a} - te_{-a-b}\\
\operatorname{Ad} x_{-b}(t)e_{-b} &= e_{-b}\\
\operatorname{Ad} x_{-b}(t)e_{-a-b} &= e_{-a-b}\\
\operatorname{Ad} x_{-b}(t)e_{-2a-b} &= e_{-2a-b}\\
\operatorname{Ad} x_{-b}(t)e_{-3a-b} &= e_{-3a-b} - te_{-3a-2b}\\
\operatorname{Ad} x_{-b}(t)e_{-3a-2b} &= e_{-3a-2b}\\
\operatorname{Ad} x_{-b}(t)h_a &= h_a - 3te_{-b}\\
\operatorname{Ad} x_{-b}(t)h_b &= h_b + 2te_{-b}
\end{aligned}$$

$$
\begin{aligned}
\operatorname{Ad} x_{-a-b}(t)e_a &= e_a - 3te_{-b}\\
\operatorname{Ad} x_{-a-b}(t)e_b &= e_b + te_{-a} - t^2e_{-2a-b} + t^3e_{-3a-2b}\\
\operatorname{Ad} x_{-a-b}(t)e_{a+b} &= e_{a+b} - th_a - 3th_b - t^2e_{-a-b}\\
\operatorname{Ad} x_{-a-b}(t)e_{2a+b} &= e_{2a+b} + 2te_a - 3t^2e_{-b}\\
\operatorname{Ad} x_{-a-b}(t)e_{3a+b} &= e_{3a+b}\\
\operatorname{Ad} x_{-a-b}(t)e_{3a+2b} &= e_{3a+2b} + te_{2a+b} + t^2e_a - t^3e_{-b}\\
\operatorname{Ad} x_{-a-b}(t)e_{-a} &= e_{-a} - 2te_{-2a-b} + 3t^2e_{-3a-2b}\\
\operatorname{Ad} x_{-a-b}(t)e_{-b} &= e_{-b}\\
\operatorname{Ad} x_{-a-b}(t)e_{-a-b} &= e_{-a-b}\\
\operatorname{Ad} x_{-a-b}(t)e_{-2a-b} &= e_{-2a-b} - 3te_{-3a-2b}\\
\operatorname{Ad} x_{-a-b}(t)e_{-3a-b} &= e_{-3a-b}\\
\operatorname{Ad} x_{-a-b}(t)e_{-3a-2b} &= e_{-3a-2b}\\
\operatorname{Ad} x_{-a-b}(t)h_a &= h_a - te_{-a-b}\\
\operatorname{Ad} x_{-a-b}(t)h_b &= h_b + te_{-a-b}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Ad} x_{-2a-b}(t)e_a &= e_a - 2te_{-a-b} - 3t^2e_{-3a-2b}\\
\operatorname{Ad} x_{-2a-b}(t)e_b &= e_b\\
\operatorname{Ad} x_{-2a-b}(t)e_{a+b} &= e_{a+b} + 2te_{-a} + 3t^2e_{-3a-b}\\
\operatorname{Ad} x_{-2a-b}(t)e_{2a+b} &= e_{2a+b} - 2th_a - 3th_b - t^2e_{-2a-b}\\
\operatorname{Ad} x_{-2a-b}(t)e_{3a+b} &= e_{3a+b} - te_a + t^2e_{-a-b} + t^3e_{-3a-2b}\\
\operatorname{Ad} x_{-2a-b}(t)e_{3a+2b} &= e_{3a+2b} - te_{a+b} - t^2e_{-a} - t^3e_{-3a-b}\\
\operatorname{Ad} x_{-2a-b}(t)e_{-a} &= e_{-a} + 3te_{-3a-b}\\
\operatorname{Ad} x_{-2a-b}(t)e_{-b} &= e_{-b}\\
\operatorname{Ad} x_{-2a-b}(t)e_{-a-b} &= e_{-a-b} + 3te_{-3a-2b}\\
\operatorname{Ad} x_{-2a-b}(t)e_{-2a-b} &= e_{-2a-b}\\
\operatorname{Ad} x_{-2a-b}(t)e_{-3a-b} &= e_{-3a-b}\\
\operatorname{Ad} x_{-2a-b}(t)e_{-3a-2b} &= e_{-3a-2b}\\
\operatorname{Ad} x_{-2a-b}(t)h_a &= h_a + te_{-2a-b}\\
\operatorname{Ad} x_{-2a-b}(t)h_b &= h_b
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Ad} x_{-3a-b}(t)e_a &= e_a + te_{-2a-b}\\
\operatorname{Ad} x_{-3a-b}(t)e_b &= e_b\\
\operatorname{Ad} x_{-3a-b}(t)e_{a+b} &= e_{a+b}\\
\operatorname{Ad} x_{-3a-b}(t)e_{2a+b} &= e_{2a+b} - te_{-a}\\
\operatorname{Ad} x_{-3a-b}(t)e_{3a+b} &= e_{3a+b} - th_a - th_b - t^2e_{-3a-b}\\
\operatorname{Ad} x_{-3a-b}(t)e_{3a+2b} &= e_{3a+2b} - te_b\\
\operatorname{Ad} x_{-3a-b}(t)e_{-a} &= e_{-a}\\
\operatorname{Ad} x_{-3a-b}(t)e_{-b} &= e_{-b} + te_{-3a-2b}\\
\operatorname{Ad} x_{-3a-b}(t)e_{-a-b} &= e_{-a-b}\\
\operatorname{Ad} x_{-3a-b}(t)e_{-2a-b} &= e_{-2a-b}\\
\operatorname{Ad} x_{-3a-b}(t)e_{-3a-b} &= e_{-3a-b}\\
\operatorname{Ad} x_{-3a-b}(t)e_{-3a-2b} &= e_{-3a-2b}\\
\operatorname{Ad} x_{-3a-b}(t)h_a &= h_a + 3te_{-3a-b}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Ad} x_{-3a-2b}(t)e_a &= e_a\\
\operatorname{Ad} x_{-3a-2b}(t)e_b &= e_b + te_{-3a-b}\\
\operatorname{Ad} x_{-3a-2b}(t)e_{a+b} &= e_{a+b} + te_{-2a-b}\\
\operatorname{Ad} x_{-3a-2b}(t)e_{2a+b} &= e_{2a+b} - te_{-a-b}\\
\operatorname{Ad} x_{-3a-2b}(t)e_{3a+b} &= e_{3a+b} - te_{-b}\\
\operatorname{Ad} x_{-3a-2b}(t)e_{3a+2b} &= e_{3a+2b} - th_a - 2th_b - t^2e_{-3a-2b}\\
\operatorname{Ad} x_{-3a-2b}(t)e_{-a} &= e_{-a}\\
\operatorname{Ad} x_{-3a-2b}(t)e_{-b} &= e_{-b}\\
\operatorname{Ad} x_{-3a-2b}(t)e_{-a-b} &= e_{-a-b}\\
\operatorname{Ad} x_{-3a-2b}(t)e_{-2a-b} &= e_{-2a-b}\\
\operatorname{Ad} x_{-3a-2b}(t)e_{-3a-b} &= e_{-3a-b}\\
\operatorname{Ad} x_{-3a-2b}(t)e_{-3a-2b} &= e_{-3a-2b}\\
\operatorname{Ad} x_{-3a-2b}(t)h_a &= h_a\\
\operatorname{Ad} x_{-3a-2b}(t)h_b &= h_b + te_{-3a-2b}
\end{aligned}
$$

$$\begin{array}{lcl}
\mathrm{Ad}\, h_a(t)e_a & = & t^2 e_a \\
\mathrm{Ad}\, h_a(t)e_b & = & t^{-3} e_b \\
\mathrm{Ad}\, h_a(t)e_{a+b} & = & t^{-1} e_{a+b} \\
\mathrm{Ad}\, h_a(t)e_{2a+b} & = & t e_{2a+b} \\
\mathrm{Ad}\, h_a(t)e_{3a+b} & = & t^3 e_{3a+b} \\
\mathrm{Ad}\, h_a(t)e_{3a+2b} & = & e_{3a+2b} \\
\mathrm{Ad}\, h_a(t)e_{-a} & = & t^{-2} e_{-a} \\
\mathrm{Ad}\, h_a(t)e_{-b} & = & t^3 e_{-b} \\
\mathrm{Ad}\, h_a(t)e_{-a-b} & = & t e_{-a-b} \\
\mathrm{Ad}\, h_a(t)e_{-2a-b} & = & t^{-1} e_{-2a-b} \\
\mathrm{Ad}\, h_a(t)e_{-3a-b} & = & t^{-3} e_{-3a-b} \\
\mathrm{Ad}\, h_a(t)e_{-3a-2b} & = & e_{-3a-2b} \\
\mathrm{Ad}\, h_a(t)h_a & = & h_a \\
\mathrm{Ad}\, h_a(t)h_b & = & h_b
\end{array}$$

$$\begin{array}{lcl}
\mathrm{Ad}\, h_b(t)e_a & = & t^{-1} e_a \\
\mathrm{Ad}\, h_b(t)e_b & = & t^2 e_b \\
\mathrm{Ad}\, h_b(t)e_{a+b} & = & t e_{a+b} \\
\mathrm{Ad}\, h_b(t)e_{2a+b} & = & e_{2a+b} \\
\mathrm{Ad}\, h_b(t)e_{3a+b} & = & t^{-1} e_{3a+b} \\
\mathrm{Ad}\, h_b(t)e_{3a+2b} & = & t e_{3a+2b} \\
\mathrm{Ad}\, h_b(t)e_{-a} & = & t e_{-a} \\
\mathrm{Ad}\, h_b(t)e_{-b} & = & t^{-2} e_{-b} \\
\mathrm{Ad}\, h_b(t)e_{-a-b} & = & t^{-1} e_{-a-b} \\
\mathrm{Ad}\, h_b(t)e_{-2a-b} & = & e_{-2a-b} \\
\mathrm{Ad}\, h_b(t)e_{-3a-b} & = & t e_{-3a-b} \\
\mathrm{Ad}\, h_b(t)e_{-3a-2b} & = & t^{-1} e_{-3a-2b} \\
\mathrm{Ad}\, h_b(t)h_a & = & h_a \\
\mathrm{Ad}\, h_b(t)h_b & = & h_b
\end{array}$$

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