VALUES OF THE FOURIER TRANSFORMS OF GREEN FUNCTIONS AT REGULAR ELEMENTS

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1. Introduction

Let G be a connected reductive algebraic group defined over a finite field $\mathbb{F}_q$ whose characteristic p is a very good prime for G and let F be the corresponding Frobenius endomorphism of G. Let $\mathfrak{g}$ be the Lie algebra of G.

In 1976, Springer [28] showed that if $X \in \mathfrak{g}$ is nilpotent and $\mathcal{B}_X^G$ denotes the variety of Borel subgroups of G whose Lie algebras contain X then one can define an action of the Weyl group W of G on the ℓ-adic cohomology groups $H^i(\mathcal{B}_X^G, \overline{\mathbb{Q}}_\ell)$ which will here be called the *Lusztig-Springer* action. For $w \in W$ and $X \in \mathfrak{g}^F$ nilpotent, Springer gave a definition of the Green functions $Q_w(X)$ in terms of certain trigonometric sums and showed that

$$Q_w(X) = \sum_{i \geqslant 0} \operatorname{Tr}(Fw, H^{2i}(\mathcal{B}_X, \overline{\mathbb{Q}}_\ell)). \tag{1.1}$$

It was then shown by Kazhdan [16], Lusztig [22] and Shoji [25] that the Green functions Q_w defined by means of the Deligne-Lusztig representation $R_{T_w}^G(1)$ are also given by equation (1.1) (cf. Shoji [24, §13] and [25, Theorem 5.5]).

In 1981, Lusztig [21] showed how to define an action of W on $H^i(\mathcal{B}_X^G, \overline{\mathbb{Q}}_\ell)$ in the case where X is not necessarily nilpotent. Following Lehrer [19], we may thus define an $\operatorname{Ad} G$-invariant function $\widetilde{Q}_w$ on $\mathfrak{g}^F$ by

$$\widetilde{Q}_w(X) = \sum_{i \geqslant 0} \operatorname{Tr}(Fw, H^{2i}(\mathcal{B}_X, \overline{\mathbb{Q}}_\ell))$$

for any $X \in \mathfrak{g}^F$.

In 1996, Lehrer [18] undertook a study of $\operatorname{Ad} G$-invariant complex valued functions on $\mathfrak{g}^F$ (simply called invariant functions in [18]) and, in particular, studied the operation $\mathcal{F}$ of Fourier transform on such functions. Lehrer showed that if $\mathcal{C}(\mathfrak{g}^F)_{\text{nil}}$ is the space of nilpotently supported invariant functions then Springer's definition [28, 5.1–2] expressed the Green functions in terms of the values at strongly regular semisimple elements of the Fourier transforms of functions in $\mathcal{C}(\mathfrak{g}^F)_{\text{nil}}$. From the Springer correspondence (see, for example, Shoji [24, §6]) an important subspace of $\mathcal{C}(\mathfrak{g}^F)_{\mathrm{nil}}$ is the space spanned by the Green functions. Lehrer [18, Proposition 6.6] has proved that, for good characteristic, $\mathcal{F}Q_w = \varepsilon(w)q^{-r/2}\widetilde{Q}_w$, where r is the rank of G and ε is the alternating character of W. The computation of $\mathcal{F}Q_w$ is thus equivalent to that of $\widetilde{Q}_w$.

The aim of the current paper is to compute the values of the functions $\widetilde{Q}_w$ at regular elements of $\mathfrak{g}^F$. The main result can be stated as follows.

Main Theorem. *Let G be a connected reductive group and suppose the characteristic p is very good for G. If $X \in \mathfrak{g}^F$ is regular and $J \subseteq \Pi$, $x \in G$ and $w_1 \in W$ are such that $C_G(X_s)^\circ = {}^xL_J$, $x^{-1}F(x) = \dot{w}_1$ and $w_1F(J) = J$ then*

$$\widetilde{Q}_w^{\mathfrak{g}}(X) = \mathrm{Ind}_{W_Jw_1F}^{WF} 1(Fw) = |W_J|^{-1}|\{v \in W \mid {}^v(Fw) \in W_Jw_1F\}|.$$

The computation is achieved by generalizing a result of Kato [15] which states (in the case where G is defined over $\mathbb{C}$) that the action of W on $H^i(\mathcal{B}_X^G, \mathbb{C})$ can be obtained by inducing the action of W_H on $H^i(\mathcal{B}_X^H, \mathbb{C})$ where $H = C_G(X_s)^\circ$ and W_H is the Weyl group of H. In section 3, we prove a generalization of this result which relates the actions of W and F on $H^i(\mathcal{B}_X^G, \overline{\mathbb{Q}}_\ell)$ and those of W_H and F on $H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell)$. We take the opportunity here to expand on Kato's proof and fill in some technical details. One of these details relies upon the result that the double coset $L_J\dot{w}P_K$ is closed in G (cf. Proposition 3.18).

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2. Notation and Background

For the general theory of reductive algebraic groups refer to Borel [3], Borel & Tits [4] and Springer [29]. We will often abbreviate the term "reductive algebraic" to merely "reductive". Lower case gothic characters will be used to denote Lie algebras so, for example, $\mathfrak{g}$ (resp. $\mathfrak{h}$, $\mathfrak{b}$ etc.) denotes the Lie algebra of G (resp. H, B etc.). If H is a reductive group then H° will denote the connected component containing the identity and for $X \in \mathfrak{h}$

$$C_H(X) = \{h \in H \mid \mathrm{Ad}\, h(X) = X\}$$
$$C_{\mathfrak{h}}(X) = \{Y \in \mathfrak{h} \mid [Y, X] = 0\}.$$

Let T be a maximal torus of G contained in a Borel subgroup B. Let Φ be the root system of G with respect to T and Π the set of simple roots with respect to the pair (T, B). The coroot system will be denoted by $\Phi^\vee$, the root lattice by $\mathbb{Z}\Phi$ and the weight lattice $\mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$ by Ω. If $\beta = \sum_{\alpha\in\Pi} n_\alpha \alpha$ is a root then we will define the height $\mathrm{ht}(\beta)$ of β to be the integer $\sum_{\alpha\in\Pi} n_\alpha$. For $r \in \Phi$, let X_r be the one-parameter subgroup of G corresponding to root r. If Φ and Π are defined with respect to a rational maximal torus T contained in the rational maximal torus B then the Frobenius endomorphism induces a permutation of Φ, which we shall also denote by F, given by $F(X_r) = X_{F(r)}$. Then F permutes both Φ^+ and Π. We shall let W be the Weyl group $N_G(T)/T$ and denote the set of irreducible characters of W by $\widehat{W}$. We shall denote by $\dot{w}$ a representative in $N = N_G(T)$ of the element w of W. The reflection corresponding to the root r will be denoted by s_r. For $w \in W$, we will denote by U_w the group generated by those X_r for which r is a positive root but $w(r)$ is a negative root. Whenever the Bruhat decomposition is invoked we will have in mind the version of Carter [9, Theorem 2.5.14]. If $w, w' \in W$ we will write ${}^{wF}w' = wF(w')w^{-1}$. If $J \subseteq \Pi$ then the root system Φ_J and groups W_J, P_J, L_J, U_J, B_J, N_J, etc. are as defined in Carter [9, Chapter 2]. A G-conjugate of a group of the form L_J will be called a *Levi subgroup* of G. We denote by $D_J = \{w \in W \mid w(J) \subseteq \Phi^+\}$ the set of distinguished (right) coset representatives of W_J in W (cf. Carter [8, 2.5.9]).

The group G acts on $\mathfrak{g}$ via the adjoint action. A function $f : \mathfrak{g}^F \to \mathbb{C}$ will be called simply an *invariant function* if it is invariant under $\mathrm{Ad}\, G^F$. The space of invariant functions will be denoted by $\mathcal{C}(\mathfrak{g}^F)$. The general properties of invariant functions are given in Lehrer [19].

We shall fix a nontrivial additive character $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ of $\mathbb{F}_q$. We may then define the *Fourier transform* $\mathcal{F}_\mathfrak{g} = \mathcal{F} : \mathcal{C}(\mathfrak{g}^F) \to \mathcal{C}(\mathfrak{g}^F)$ by

$$\mathcal{F}f(X) = |\mathfrak{g}^F|^{-1/2} \sum_{Y\in\mathfrak{g}^F} \psi(\langle X, Y\rangle) f(Y)$$

where $f \in \mathcal{C}(\mathfrak{g}^F)$, $X \in \mathfrak{g}^F$ and $\langle,\rangle : \mathfrak{g} \times \mathfrak{g} \to \overline{\mathbb{F}}_q$ is an $\mathrm{Ad}\, G$-invariant non-degenerate bilinear form defined over $\mathbb{F}_q$. The existence of such a form in general imposes conditions on the characteristic (cf. Springer & Steinberg [30, Lemma I.5.3]).

An element g (resp. subgroup H) of G that is fixed by F will be said to be a *rational* element (resp. subgroup) of G and G^F is the group of rational points of G. We will call an $\mathrm{Ad}\, G^F$-orbit in $\mathfrak{g}^F$ a *rational* orbit.

If F acts on the group H then $a, b \in H$ are said to be F-*conjugate* if there exists $x \in H$ such that $b = x^{-1}aF(x)$. We may form the semidirect product $H \rtimes \langle F\rangle$ by defining $F \cdot h = F(h) \cdot F$ for $h \in H$. Then the elements $a, b \in H$ are F-conjugate if and only if the elements $aF, bF \in H \rtimes \langle F \rangle$ are conjugate by an element of H. We shall denote by $H^1(F, H)$ the set of H-classes of elements of HF.

Let V be an algebraic variety defined over $\mathbb{F}_q$ and assume that an algebraic group G defined over $\mathbb{F}_q$ acts on V by an action defined over $\mathbb{F}_q$. Let $\mathcal{O}$ be an F-stable orbit and X, $\operatorname{Ad} g(X) \in \mathcal{O}^F$. If $c \in \operatorname{Stab}_G(X)/\operatorname{Stab}_G(X)^\circ$ is the image of $g^{-1}F(g)$ under the canonical map then we say that $\operatorname{Ad} g(X)$ is *c-twisted* with respect to X. By Digne & Michel [12, 3.21], c is determined up to F-conjugacy.

If $X \in \mathfrak{g}$, we will denote the semisimple and nilpotent parts of X by X_s and X_n respectively.

Let $\mathcal{B}^G$ be the variety of Borel subgroups of G. For $X \in \mathfrak{g}$ let $\mathcal{B}^G_X = \{B' \in \mathcal{B}^G \mid X \in \operatorname{Lie} B'\}$. We will simply write $\mathcal{B}$ and $\mathcal{B}_X$ if no confusion arises. Let $\mathfrak{g}_{\mathrm{rs}}$ be the dense open subset of $\mathfrak{g}$ consisting of all regular semisimple elements of $\mathfrak{g}$. Define

$$\widetilde{\mathfrak{g}} = \{(X, B') \mid X \in \mathfrak{g}, B' \in \mathcal{B}^G_X\},$$
$$\widetilde{\mathfrak{g}}_{\mathrm{rs}} = \{(X, B') \mid X \in \mathfrak{g}_{\mathrm{rs}}, B' \in \mathcal{B}^G_X\}.$$

Let $p_{\mathfrak{g}} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ and $p_{\mathfrak{g}_{\mathrm{rs}}} : \widetilde{\mathfrak{g}}_{\mathrm{rs}} \to \mathfrak{g}_{\mathrm{rs}}$ be the projections onto their first entries.

If X is an algebraic variety defined over $\mathbb{F}_q$ and ℓ is a prime not equal to p then we can define the ℓ-adic cohomology groups with compact support $H^i_c(X, \overline{\mathbb{Q}}_\ell)$. For details of their definition and proofs of the following facts see, for example, Srinivasan [31, Chapter V] and Digne & Michel [12, Chapter 10]. Any finite morphism $f : X \to X$ induces a linear endomorphism of $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ for any i via the inverse image f^*. In the case of the Frobenius morphism, we will also denote by F the endomorphism of $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ given by $(F^*)^{-1}$.

If X is of dimension n then $H^i_c(X, \overline{\mathbb{Q}}_\ell) = 0$ unless $0 \leqslant i \leqslant 2n$. If X is an affine space of dimension n, the ℓ-adic cohomology groups $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ all vanish except $H^{2n}_c(X, \overline{\mathbb{Q}}_\ell)$, which is one dimensional. The linear endomorphism induced by F on $H^{2n}_c(X, \overline{\mathbb{Q}}_\ell)$ is then simply multiplication by q^n. Let $f : X \to X'$ be a continuous map of $\overline{\mathbb{F}}_q$-schemes and let $\mathcal{S}$ and $\mathcal{S}'$ denote the categories of complexes of $\overline{\mathbb{Q}}_\ell$-sheaves on X and X' respectively. We will consider a $\overline{\mathbb{Q}}_\ell$-sheaf as a complex concentrated in degree zero. Denote the direct and inverse image functors by $f_* : \mathcal{S} \to \mathcal{S}'$ and $f^* : \mathcal{S}' \to \mathcal{S}$ respectively. We will repeatedly use the fact that the inverse image of a constant sheaf is also a constant sheaf. The right derived functor of f will be denoted by $\mathbf{R}^\bullet f_*$. If ${\overline{\mathbb{Q}}_\ell}_X$ denotes the constant sheaf on X then the fibre of $\mathbf{R}^i f_*({\overline{\mathbb{Q}}_\ell}_X)$ at a point $x \in X'$ is $H^i_c(f^{-1}(x), \overline{\mathbb{Q}}_\ell)$. We will frequently omit the subscript X if the context is clear. We will use the term *local system* synonymously with that of locally constant sheaf. We will need the following version of the Base Change Theorem.

The Proper Base Change Theorem. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \Big\downarrow{\scriptstyle g'} & & \Big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a cartesian diagram of morphisms of schemes such that $f : X \to Y$ is a proper morphism. Let S be a torsion $\overline{\mathbb{Q}}_\ell$-sheaf on X. Then the base change morphism

$$g^*(\mathbf{R}^\bullet f_*(S)) \to \mathbf{R}^\bullet f'_*(g'^*S)$$

is an isomorphism for all $g : Y' \to Y$.

We will also need to use the *intersection cohomology functor*. Good accounts are to be found in Shoji [24] and Goresky & Macpherson [13]. Let $\mathcal{D}(X)$ be the derived category of complexes of $\overline{\mathbb{Q}}_\ell$-sheaves on X. For a complex K in $\mathcal{D}(X)$, $\mathcal{H}^iK$ will denote the ith cohomology sheaf of K. We denote by $\mathcal{D}_c^b(X)$ the full subcategory of $\mathcal{D}(X)$ consisting of bounded complexes K such that $\mathcal{H}^iK$ are constructible for any i. For $K \in \mathcal{D}_c^b(X)$, $K[i]$ means the ith shift so that $\mathcal{H}^i(K[j]) = \mathcal{H}^{i+j}K$. We regard a constructible sheaf $\mathcal{T}$ on X as a complex in $\mathcal{D}_c^b(X)$ concentrated in degree 0. Let $D : \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(X)$ be the Verdier dual operator. A complex $K \in \mathcal{D}_c^b(X)$ is said to be a *perverse sheaf* if

(i) $\dim \operatorname{supp} \mathcal{H}^iK \leqslant -i$,
(ii) $\dim \operatorname{supp} \mathcal{H}^iDK \leqslant -i$ for any i.

Let $\mathcal{M}X$ denote the full subcategory of $\mathcal{D}_c^b(X)$ consisting of perverse sheaves (for more details on perverse sheaves see Beilinson, Bernstein and Deligne [2]). If Y is a locally closed smooth subvariety of dimension d of X and $\mathcal{T}$ is a local system on Y then the intersection cohomology complex $\mathrm{IC}(\overline{Y}, \mathcal{T})$ is an object in $\mathcal{D}_c^b(\overline{Y})$. The complex $K = \mathrm{IC}(\overline{Y}, \mathcal{T})[d]$ is characterized by the following properties.

(i) $\mathcal{H}^iK = 0$ if $i < -d$,
(ii) $\mathcal{H}^{-d}K|Y = \mathcal{T}$,
(iii) $\dim \operatorname{supp} \mathcal{H}^iK < -i$ if $i > -d$,
(iv) $\dim \operatorname{supp} \mathcal{H}^iDK < -i$ if $i > -d$.

If $i : \overline{Y} \to X$ is the inclusion then we may consider an object in $\mathcal{M}\overline{Y}$ as lying in $\mathcal{M}X$ by applying i_*. We hence get a functor $\mathrm{IC}(X, \cdot)$ which associates to a local system on Y a perverse sheaf on X.

3. Generalization of a result of Kato

In this section G is an connected reductive group, B is any F-stable Borel subgroup and T is any F-stable maximal torus of B.

Lemma 3.1 (cf. Carter [9, Prop. 5.9.2]). *Suppose p is good for G. If X is a semisimple element of $\mathfrak{g}$ then $C_G(X)^\circ$ is G-conjugate to the standard Levi subgroup L_J for some $J \subseteq \Pi$.*

Proof. We may assume that $X \in \mathfrak{t}$. Since p is good, $\{\alpha \in \Phi \mid \alpha(X) = 0\}$ is a $\mathbb{Q}$-closed root subsystem of Φ (cf. Slodowy [26, 3.5,3.15]) and hence, according to Bourbaki [5, VI, §1, 1.8, Proposition 24], is of the form $\Phi_{J'}$ for some simple system Π' of roots and some $J' \subseteq \Pi'$. For some $w \in W$, $w(\Pi') = \Pi$ and $J = w(J')$. Then ${}^{\dot{w}}C_G(X)^\circ = L_J$. □

Lemma 3.2. *Let $J \subseteq \Pi$. If $w \in D_J^{-1}$ with ${}^{wF}W_J = W_J$ then $wF(J) = J$.*

Proof. If $w \in W$ is such that ${}^{wF}W_J = W_J$ then, for $\alpha \in \Phi_J$, the reflection ${}^{wF}s_\alpha = s_{wF(\alpha)}$ lies in W_J. As is the proof of Digne & Michel [12, Proposition 4.2], this implies that $wF(\alpha) \in \Phi_J$. Hence $wF(\Phi_J) = \Phi_J$. If in addition $w \in D_J^{-1}$ then $w^{-1}(J) \subseteq F(\Phi_J^+)$. Consideration of heights then shows that $w^{-1}(J) = F(J)$. □

Proposition 3.3 (cf. Digne & Michel [12, Prop. 4.3]). *The G^F-classes of rational Levi subgroups of G are parametrized by W-classes of cosets W_JwF where $J \subseteq \Pi$ and wF are such that ${}^{wF}W_J = W_J$. If L is a rational Levi subgroup and W_JwF is an element of the corresponding W-class then there exists $g \in G$ such that $L = {}^gL_J$ and $g^{-1}F(g) = \dot{w}$. According to Lemma 3.2, we may assume that w is chosen so that $wF(J) = J$.*

Definition 3.4. Suppose p is good for G. If $X \in \mathfrak{g}^F$ then, according to Lemma 3.1, $C_G(X_s)^\circ$ is a rational Levi subgroup. By Proposition 3.3, there is a corresponding W-class of cosets W_JwF such that ${}^{wF}W_J = W_J$. We will call the W-class of W_JwF the *genus* of X. We will frequently abuse notation and say simply that X has genus W_JwF.

Notes. There are several points worth making concerning this definition of genus. Carter [7] defines what it means for two rational semisimple elements of the group to be “of the same genus”. We can make a similar definition in the case of rational semisimple elements of the Lie algebra. It then turns out that the use of the term “genus” is consistent with that of Carter in that two rational semisimple elements are "of the same genus" according to Carter if and only if they have the same genus as defined above.

Deriziotis [10] shows that (in the split case which we will assume for simplicity) the equivalence classes of rational semisimple elements (in G) "of the same genus" can be parametrized by pairs (J, C) (which he calls families) where J is a proper subset of $\Pi \cup \{-\delta_0\}$, δ_0 is the highest root and C is a conjugacy class in $N_W(W_J)/W_J$. Note that there is a difference in the behaviours of centralizers of semisimple elements of G and of $\mathfrak{g}$. The situation is illustrated by the group $G = Sp_4(q)$ with p good. Let $\Pi = \{a, b\}$ with a the long root so that $\delta_0 = a + 2b$. The connected centralizer of the semisimple element $s = \mathrm{diag}(-1, 1, 1, -1) \in G$ is a group of type $A_1 \times A_1$ corresponding to $J = \{a, -a - 2b\}$. However, there is no semisimple element of $\mathfrak{g}$ with connected centralizer of this type.

Lehrer [20] uses the phrase "regular semisimple element of type w" instead of "regular semisimple element of genus wF". Under the assumptions that the centralizers of semisimple elements are connected and G is F-split, Lehrer [17] defines a map ω from the set of F-stable semisimple conjugacy classes of G to the set of conjugacy classes of W. The relationship between this map and the notion of genus is given by Lemma 3.6 which follows.

The following definition of the map ω generalizes the definition of Lehrer [17, §5] to the non-split case.

Definition 3.5 (cf. Lehrer [17, §5]). Let G be a connected reductive group and T a maximally split maximal torus of G. If $X \in \mathfrak{g}^F$, let T' be a maximally split maximal torus of $C_G(X_s)^\circ$ which is w-twisted in G with respect to T. Then wF is determined up to W-conjugacy and we may define a map $\omega : \mathfrak{g}^F \to H^1(W, F)$ by sending X to the W-class of wF.

Lemma 3.6. *Suppose p is good for G. Let $\mathcal{G}$ be the set of W-classes of cosets $W_J wF$ where $J \subseteq \Pi$ and ${}^{wF}W_J = W_J$. Let $\alpha : \mathfrak{g}^F \to \mathcal{G}$ be the genus map. If C is a W-class of cosets $W_J wF$ as above then we can choose w so that $wF(J) = J$. If $\nu : \mathcal{G} \to H^1(W, F)$ sends C to the W-class containing wF then the following diagram commutes.*

$$\begin{tikzcd} \mathfrak{g}^F \arrow[r, "\alpha"] \arrow[dr, "\omega"'] & \mathcal{G} \arrow[d, "\nu"] \\ & H^1(W, F) \end{tikzcd}$$

Proof. Let X have genus W_JwF so that $C_G(X_s)^\circ = {}^gL_J$ with $g^{-1}F(g) = \dot{w}$ and $wF(J) = J$. Then ${}^gT$ is an F-stable maximal torus contained in the F-stable Borel subgroup ${}^gB_J$ of ${}^gL_J$ and hence ${}^gT$ is a maximally split maximal torus in ${}^gL_J$ which is w-twisted in G with respect to T. Hence $\omega(X)$ is the W-class containing wF which is just $\nu(W_JwF)$. The proof also shows that ν is well-defined. □

Lemma 3.7 (cf. Carter [9, Prop. 3.5.2]). *Let $X \in \mathfrak{g}$ be semisimple and T a maximal torus of G. Then $X \in \mathfrak{t}$ if and only if $T \subseteq C_G(X)^\circ$.*

Proposition 3.8. *Let p be good for G and X be regular in $\mathfrak{g}$. Then there exist $g \in G$ and $J \subseteq \Pi$ such that $C_G(X_s)^\circ = {}^gL_J$ and $X_n \in \mathrm{Ad}(g)(\mathfrak{b}_J)$ and*

$$\mathcal{B}_X = \{{}^{g\dot{w}}B \mid w \in D_J^{-1}\}.$$

As a consequence $|\mathcal{B}_X| = |W|/|W_J|$.

Proof. By Lemma 3.1, $C_G(X_s)^\circ = {}^{g'}L_J$ for some $J \subseteq \Pi$, $g' \in G$. Now $X_n \in C_{\mathfrak{g}}(X_s) = \mathrm{Lie}(C_G(X_s)^\circ) = \mathrm{Lie}({}^{g'}L_J)$ by Borel [3, 9.1]. According to Borel [3, Prop. 14.25], X_n is contained in a Borel subalgebra of $\mathrm{Ad}\, g'(\mathfrak{l}_J)$. Hence we can find $h \in L_J$ such that $X_n \in \mathrm{Ad}(g'h)(\mathfrak{b}_J)$. Setting $g = g'h$ completes the first part of the proof.

Since X is regular, X_n is regular in $C_{\mathfrak{g}}(X_s) = \mathrm{Ad}\, g(\mathfrak{l}_J)$ (cf. Steinberg [32, Corollary 3.7]) and hence $\mathrm{Ad}(g^{-1})X_n$ is regular in $\mathfrak{l}_J$. By Springer [27, Lemma 5.8], if we write $\mathrm{Ad}(g^{-1})X_n = \sum_{\alpha\in\Phi_J^+} X_\alpha$ with $X_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Phi_J^+$ then we have $X_\alpha \neq 0$ for $\alpha \in J$. Now suppose that ${}^{gx}B \in \mathcal{B}_X$ for some $x \in G$. Then $X \in \mathrm{Lie}({}^{gx}B)$ and hence $X_n \in \mathrm{Lie}({}^{gx}B)$. Write $x = u\dot{w}b$ where $u \in U_{w^{-1}}$, $w \in W$ and $b \in B$. Then $\mathrm{Ad}\, u^{-1}(\mathrm{Ad}\, g^{-1}(X_n)) \in \mathrm{Ad}\, \dot{w}(\mathfrak{b}) = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{w(\alpha)}$ and, by the Chevalley commutator relations,

$$\mathrm{Ad}\, u^{-1}(\mathrm{Ad}\, g^{-1}(X_n)) = \sum_{\alpha\in J} X_\alpha + \sum_{\mathrm{ht}(\alpha)>1} Y_\alpha$$

for some $Y_\alpha \in \mathfrak{g}_\alpha$. Since $X_\alpha \neq 0$ for $\alpha \in J$ we must then have $J \subseteq w(\Phi^+)$ and hence $w \in D_J^{-1}$. Thus $\mathcal{B}_X \subseteq \{{}^{g\dot{w}}B \mid w \in D_J^{-1}\}$.

Conversely, let $w \in D_J^{-1}$. Then ${}^{\dot{w}^{-1}}X_\alpha = X_{w^{-1}(\alpha)} \subseteq B$ for all $\alpha \in \Phi_J^+$. Since $B_J = \langle T, X_\alpha \mid \alpha \in \Phi_J^+\rangle$ and ${}^{\dot{w}^{-1}}T = T$ we thus have $B_J \subseteq {}^{\dot{w}}B$. Now ${}^gT \subseteq {}^gL_J = C_G(X_s)^\circ$ and hence, by Lemma 3.7, $X_s \in \mathrm{Lie}({}^gT)$. Thus

$$X = X_s + X_n \in \mathrm{Lie}({}^gB_J) \subseteq \mathrm{Lie}({}^{g\dot{w}}B)$$

and ${}^{g\dot{w}}B \in \mathcal{B}_X$. The last statement follows since $|D_J| = |W|/|W_J|$. □

Lemma 3.9 (cf. Springer & Steinberg [30, III, 1.8(a)]). *The variety $\widetilde{\mathfrak{g}}$ is irreducible and we have $\dim \widetilde{\mathfrak{g}} = \dim \mathfrak{g}$.*

Proof. Since $N_B(\mathfrak{b}) = B$, we can write $\widetilde{\mathfrak{g}} \cong \{(X, gB) \mid X \in \operatorname{Ad} g(\mathfrak{b})\}$. Its inverse image in $\mathfrak{g} \times G$ is $\{(X, g) \mid X \in \operatorname{Ad} g(\mathfrak{b})\}$. This is the image of the injective morphism $f : \mathfrak{b} \times G \to \mathfrak{g} \times G$ given by $(X, g) \mapsto (\operatorname{Ad} g(X), g)$. The irreducibility of $\widetilde{\mathfrak{g}}$ then follows from that of $\mathfrak{b} \times G$. We also see that

$$\dim \widetilde{\mathfrak{g}} = \dim \operatorname{im} f - \dim B = \dim(\mathfrak{b} \times G) - \dim B = \dim \mathfrak{g}.$$

□

Definition 3.10. Let H be an F-stable Levi subgroup of G. For the rest of this paper $J \subseteq \Pi$, $x \in G$ and $w_1 \in W$ will be such that $H = {}^x L_J$, $x^{-1}F(x) = \dot{w}_1$ and $w_1 F(J) = J$. Define

$$\mathfrak{h}_c = \{X \in \mathfrak{h} \mid C_G(X_s)^0 \subseteq H\}.$$

Lemma 3.11. *If p is a very good prime for G then $\mathfrak{h}_c$ is open dense in $\mathfrak{h}$.*

Proof. Let Φ be the root system in G contained in $X(\mathfrak{t})$, the character group of $\mathfrak{t}$. Let $\overline{\mathbb{F}}_q[\mathfrak{t}]$ be the coordinate ring of $\mathfrak{t}$, identified with the group algebra $\overline{\mathbb{F}}_q[X(\mathfrak{t})]$. Define an element $c_J \in \overline{\mathbb{F}}_q[\mathfrak{t}]$ by

$$c_J = \prod_{\alpha \in \Phi \setminus \Phi_J} \alpha.$$

Then c_J is W_J-invariant and hence $c_J \in \overline{\mathbb{F}}_q[\mathfrak{t}]^{W_J} \cong \overline{\mathbb{F}}_q[\mathfrak{t}/W_J]$. The condition on p implies that p does not divide $|\Omega/\mathbb{Z}\Phi|$ and hence, by Slodowy [26, 3.12], there is a homomorphism

$$\overline{\mathbb{F}}_q[\mathfrak{t}/W_J] \cong \overline{\mathbb{F}}_q[\mathfrak{t}]^{W_J} \xrightarrow{\sim} \overline{\mathbb{F}}_q[\mathfrak{l}_J]^{L_J} \hookrightarrow \overline{\mathbb{F}}_q[\mathfrak{l}_J].$$

which induces a morphism $\chi_J : \mathfrak{l}_J \to \mathfrak{t}/W_J$ taking an element $Y \in \mathfrak{l}_J$ to $\mathfrak{t} \cap \operatorname{Ad} L_J(Y_s)$. Then $X \in \mathfrak{h}_c$ if and only if $X \in \mathfrak{h}$ and $C_G(\operatorname{Ad} x^{-1}(X_s))^0 \subseteq L_J$; that is, if and only if $c_J(\chi_J(\operatorname{Ad} x^{-1}(X))) \neq 0$. Hence

$$\mathfrak{h}_c = \operatorname{Ad} x(\chi_J^{-1}(\{X \in \mathfrak{t}/W_J \mid c_J(X) \neq 0\})).$$

Since p is very good this is a nonempty open subset of $\mathfrak{h}$ (cf. Slodowy [26, 3.14]). Since $\mathfrak{h}$ is an irreducible algebraic variety this implies that $\mathfrak{h}_c$ is also dense in $\mathfrak{h}$. □

Lemma 3.12. *We have $\{B' \in \mathcal{B}^G \mid T \subset B'\} = \{{}^{\dot{w}}B \mid w \in W\}$.*

Proof. Suppose that B' is a Borel subgroup of G containing T. Then $B' = {}^g B$ for some $g \in G$. Since $T \subseteq B'$, ${}^{g^{-1}}T$ is a maximal torus of B, and hence ${}^{g^{-1}}T = {}^b T$ for some $b \in B$. Thus $gb \in N$ and $gb = \dot{w}t$ for some $w \in W$, $t \in T$. Then $B' = {}^g B = {}^{gb} B = {}^{\dot{w}t} B = {}^{\dot{w}} B$. □

Lemma 3.13. *We have* $D_J^{-1} = \{w \in W \mid {}^{\dot w}B \cap L_J = B_J\}$.

Proof. We have

$$w \in D_J^{-1} \iff J \subseteq w(\Phi^+) \iff \Phi_J^+ \subseteq w(\Phi^+) \iff w(\Phi^+) \cap \Phi_J = \Phi_J^+.$$

If $L_J \cap {}^{\dot w}B = B_J$ then $B_J \subseteq {}^{\dot w}B$ implying that $\Phi_J^+ \subseteq w(\Phi^+)$ and hence that $w \in D_J^{-1}$.

Conversely, suppose that $w \in D_J^{-1}$ so that $w(\Phi^+) \cap \Phi_J = \Phi_J^+$. If $\alpha \in \Phi_J^+$ then $\alpha \in \Phi_J$, $\alpha \in w(\Phi^+)$ and hence $X_\alpha \in L_J \cap {}^{\dot w}B$. Since $T \subseteq L_J \cap {}^{\dot w}B$, we thus have $B_J \subseteq L_J \cap {}^{\dot w}B$. On the other hand, suppose that $\dot w b \dot w^{-1} \in L_J \cap {}^{\dot w}B$ with $b \in B$. By the Bruhat decomposition in L_J, write $\dot w b \dot w^{-1} = b_1 \dot w_1 b_2$ with $b_1, b_2 \in B_J$ and $w_1 \in W_J$. Then

$$(\dot w^{-1} b_1 \dot w)(\dot w^{-1} \dot w_1 \dot w)(\dot w^{-1} b_2 \dot w) = b.$$

Now both $\dot w^{-1} b_1 \dot w$ and $\dot w^{-1} b_2 \dot w$ lie in B since $w^{-1}(\Phi_J^+) = \Phi^+ \cap w^{-1}(\Phi_J)$. Thus, by the Bruhat decomposition in G, we must have $w^{-1} w_1 w = 1$ and hence $w_1 = 1$ implying that $\dot w b \dot w^{-1} \in B_J$. Thus $L_J \cap {}^{\dot w}B \subseteq B_J$. $\square$

Lemma 3.14. *If* $(X, B') \in p_{\mathfrak g}^{-1}(\mathfrak h_c)$ *then* $B' = {}^{xm\dot w}B$ *for some* $m \in L_J$ *and* $w \in D_J^{-1}$. *In this expression,* w *and the coset* mB_J *are uniquely determined.*

Proof. If $(X, B') \in p_{\mathfrak g}^{-1}(\mathfrak h_c)$ then $X \in \mathfrak h_c$ and $X_s \in \mathfrak t' = \mathrm{Lie}(T')$ for some maximal torus T' of B'. According to Lemma 3.7, this implies that $T' \subseteq C_G(X_s)^0 \subseteq H$ and hence that ${}^{x^{-1}}T' \subseteq L_J$. Thus, for some $m \in L_J$, ${}^{m^{-1}x^{-1}}T' = T$ and hence ${}^{m^{-1}x^{-1}}B' \supseteq T$. Lemma 3.12 then implies that ${}^{m^{-1}x^{-1}}B' = {}^{\dot w}B$ for some $w \in W$ which by choice of m we may assume to lie in D_J^{-1}.

Now suppose that ${}^{xm_1\dot w_1}B = {}^{xm_2\dot w_2}B$ for some $m_1, m_2 \in L_J$, $w_1, w_2 \in D_J^{-1}$. If we let $m = m_1^{-1}m_2$ then $\dot w_1^{-1} m \dot w_2 \in B$. This implies that $\dot w_1 \in L_J \dot w_2 B$ and hence that $B\dot w_1 B \subseteq BL_J\dot w_2 B$. Now $L_J = B_J N_J B_J \subseteq BN_J B$. Because of the BN-pair relation

$$B\dot w_\alpha B \cdot B\dot w B \subseteq B\dot w B \cup B\dot w_\alpha \dot w B$$

we must have

$$BL_J\dot w_2 B \subseteq \bigcup_{w \in W_J} B\dot w \dot w_2 B.$$

By the Bruhat decomposition we then have $w_1 w_2^{-1} \in W_J$ and hence $w_1 = w_2$. Thus $m = m_1^{-1} m_2 \in {}^{\dot w_1}B \cap L_J = B_J$ by Lemma 3.13 and $m_1 B_J = m_2 B_J$. $\square$

Corollary 3.15. *If $(X, B') \in p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)$ and $B' = {}^{xm\dot{w}}B$ for some $m \in L_J$ and $w \in D_J^{-1}$ then $B' \cap H = {}^{xm}B_J \in \mathcal{B}^H$.*

Proof. We have $B' \cap H = {}^{xm\dot{w}}B \cap H = {}^{xm}({}^{\dot{w}}B \cap L_J) = {}^{xm}B_J$ by Lemma 3.13 and this is a Borel subgroup of H. □

Proposition 3.16 (cf. Kato [15, 2.6]). *With notation as above, define a map*

$$\phi : p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c) \times W_J\backslash W \to p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)$$

by

$$\phi((X, {}^{xm}B_J), W_J w) = (X, {}^{xm\dot{w}}B)$$

where $m \in L_J$ and $w \in D_J^{-1}$. Then ϕ is well-defined and is an H-equivariant isomorphism where H acts on $p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c)$ and $p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)$ (resp. $W_J\backslash W$) by conjugation (resp. the trivial action).

Proof. It is routine to show that ϕ is well-defined and H-equivariant. To show that ϕ is an isomorphism we explicitly construct an inverse

$$\psi : p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c) \to p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c) \times W_J\backslash W.$$

Let $(X, B') \in p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)$. By Lemma 3.14, we have $B' = {}^{xm\dot{w}}B$ for some $m \in L_J$ and $w \in D_J^{-1}$ and in this expression w and the coset mB_J are uniquely determined. By Corollary 3.15, $B' \cap H = {}^{xm}B_J$ is a Borel subgroup of H. Hence we may define ψ by

$$\psi(X, {}^{xm\dot{w}}B) = ((X, {}^{xm}B_J), W_J w).$$

It is clear that ϕ and ψ are mutually inverse. □

Corollary 3.17. *Let the notation be as above. For $w \in D_J^{-1}$ let*

$$Y_w = \{(X, {}^{xm\dot{w}}B) \mid X \in \mathfrak{h}_c, m \in L_J, X \in \mathrm{Ad}(xm\dot{w})(\mathfrak{b})\}.$$

Then $p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c) = \coprod_{w \in D_J^{-1}} Y_w$ is a decomposition into irreducible components and for each $w \in D_J^{-1}$ the variety $Y_w \cong p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c)$.

Proof. All the statements follow immediately from Proposition 3.16 except that the decomposition is into irreducible components. Since the decomposition is a finite disjoint union, it suffices to show that Y_w is closed in $p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)$.

Consider the map $\pi : p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c) \to G/B$ taking $(X, {}^{xm\dot{w}}B)$ to $m\dot{w}B$ where $m \in L_J$. Then $Y_w = \pi^{-1}(L_J\dot{w}B/B)$. By Proposition 3.18 (the statement of which follows), $L_J\dot{w}B$ is closed in G and hence $L_J\dot{w}B/B$ is closed in G/B. Thus Y_w is closed in $p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)$ as required. □

Proposition 3.18. *If $J, K \subseteq \Pi$ and $w \in W$ then $L_J\dot{w}P_K$ is closed in G.*

Proof. We may assume that w is a distinguished coset representative in $W_J\backslash W/W_K$, that is, $w \in D_J^{-1} \cap D_K$. If $b \in B_J$ then $\dot{w}^{-1}b\dot{w} \in B \subseteq P_K$ and so $lb\dot{w}P_K = l\dot{w}P_K$ for all $l \in L_J$, $b \in B_J$. Thus the morphism $L_J \to L_J\dot{w}P_K/P_K$ sending l to $l\dot{w}P_K$ factors through $L_J \to L_J/B_J$, so $lB_J \mapsto l\dot{w}P_K$ is a well-defined morphism of varieties $L_J/B_J \to G/P_K$. Since L_J/B_J is complete, its image $L_J\dot{w}P_K/P_K$ is closed in G/P_K whence the result. □

Proposition 3.19. *If p is very good for G then the variety $p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)$ is smooth and* $\dim p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c) = \dim \mathfrak{h}$.

Proof. Let Y_w be as in the statement of Corollary 3.17. Since $\mathfrak{h}_c$ is open in $\mathfrak{h}$, $p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c)$ is open in $\widetilde{\mathfrak{h}}$. Since $\widetilde{\mathfrak{h}}$ is smooth this implies that $p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c)$ and hence Y_w is smooth for each $w \in D_J^{-1}$. By Corollary 3.17, we thus find that $p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)$ is smooth.

Since the Y_w are all isomorphic to $p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c)$ we have $\dim p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c) = \dim p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c)$. Now $p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c)$ is open in $\widetilde{\mathfrak{h}}$ and since $\widetilde{\mathfrak{h}}$ is irreducible this implies that $p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c)$ is dense in $\widetilde{\mathfrak{h}}$. Hence $\dim p_{\mathfrak{h}}^{-1}(\mathfrak{h}_c) = \dim \widetilde{\mathfrak{h}} = \dim \mathfrak{h}$. □

Lemma 3.20 (cf. Kato [15, §2]). *Let $(X, B') \in \widetilde{\mathfrak{g}}_{\mathrm{rs}}$. Then there exists a unique element (Y, gT) of $\mathfrak{t}_{\mathrm{reg}} \times G/T$ satisfying $(\mathrm{Ad}\, g(Y), {}^gB) = (X, B')$ where $\mathfrak{t}_{\mathrm{reg}} = \mathfrak{t} \cap \mathfrak{g}_{\mathrm{rs}}$.*

Proof. Choose $g' \in G$ so that ${}^{g'^{-1}}B' = B$. Then there exists $b \in B$ such that $Y = \mathrm{Ad}(bg'^{-1})(X) \in \mathfrak{t}$. Hence, if $g = g'b^{-1}$, $(\mathrm{Ad}\, g(Y), {}^gB) = (X, B')$ and we have thus proved existence. To prove uniqueness suppose that $\mathrm{Ad}\, g(Y) = \mathrm{Ad}\, g'(Y') = X$ and ${}^gB = {}^{g'}B = B'$ with $Y, Y' \in \mathfrak{t}$. Then $g^{-1}g' \in N_G(B) = B$ and, by Springer-Steinberg [30, II, 3.16], $Y' = \mathrm{Ad}\, \dot{w}(Y)$ for some $w \in W$. Hence $\mathrm{Ad}(g^{-1}g'\dot{w})(Y) = Y$ and $g^{-1}g'\dot{w} \in C_G(Y)$. Since X is regular semisimple, $C_G(Y) \subseteq N$ and hence $g^{-1}g' \in B \cap N = T$. Thus $gT = g'T$ and $\dot{w} \in C_G(Y)$ implying that $Y' = Y$. □

The Weyl group $W = N_G(T)/T$ acts on $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$ from the right via $(\mathrm{Ad}\, g(Y), {}^gB) \cdot w = (\mathrm{Ad}\, g(Y), {}^{g\dot{w}}B)$. Define the local system $\mathcal{T}^{\mathfrak{g}} = p_{\mathfrak{g}_{rs}*}\overline{\mathbb{Q}}_\ell$. The right action of W on $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$ induces a left action of W on $\mathcal{T}^{\mathfrak{g}}$. A result of Lusztig (cf. Shoji [24, Prop. 4.2] and Lusztig [21, §3]) states that $\mathrm{IC}(\mathfrak{g}, \mathcal{T}^{\mathfrak{g}}) \cong p_{\mathfrak{g}*}\overline{\mathbb{Q}}_\ell$ in $\mathcal{M}\mathfrak{g}$. Via the intersection cohomology functor, the action of W on $\mathcal{T}^{\mathfrak{g}}$ induces an action of W on $p_{\mathfrak{g}*}\overline{\mathbb{Q}}_\ell$ which in turn induces an action of W on the fibre of $p_{\mathfrak{g}*}\overline{\mathbb{Q}}_\ell$ above $X \in \mathfrak{g}_{\mathrm{rs}}$. But this fibre is isomorphic to $H^*(p_{\mathfrak{g}}^{-1}(X), \overline{\mathbb{Q}}_\ell) = H^*(\mathcal{B}_X, \overline{\mathbb{Q}}_\ell)$. The action of W on the ℓ-adic cohomology groups $H^*(\mathcal{B}_X, \overline{\mathbb{Q}}_\ell)$ defined in this way will be called the *Lusztig-Springer action* of W.

The analogous result to Lemma 3.20 for $\widetilde{\mathfrak{h}}$ states that if $(X, B') \in \widetilde{\mathfrak{h}}$ then there exists $m \in L_J$ such that $B' = {}^{xm}B_J$ and $X \in \mathrm{Ad}(xm)(\mathfrak{t})$. Thus W_J acts on $p_{\mathfrak{h}_{\mathrm{rs}}}^{-1}(\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h})$ from the right via

$$(X, {}^{xm}B_J) \cdot w = (X, {}^{xm\dot{w}}B_J).$$

This induces a left action of W_J on $\mathcal{T}^{\mathfrak{h}}$. For reasons that will be apparent later on we will henceforth denote the Frobenius morphisms on $\mathfrak{h}$ and $p_{\mathfrak{h}_{\mathrm{rs}}}^{-1}(\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h})$ and the induced action on $\mathcal{T}^{\mathfrak{h}}$ by F'.

The proofs of the following two lemmas follow by some simple calculations.

Lemma 3.21. *We have the following commutative diagram*

$$\begin{array}{ccc} p_{\mathfrak{h}_{rs}}^{-1}(\mathfrak{g}_{rs} \cap \mathfrak{h}) \times W_J\backslash W & \xrightarrow{\phi} & p_{\mathfrak{g}_{rs}}^{-1}(\mathfrak{g}_{rs} \cap \mathfrak{h}) \\ \Big\downarrow {\scriptstyle F' \times w_1 F} & & \Big\downarrow {\scriptstyle F} \\ p_{\mathfrak{h}_{rs}}^{-1}(\mathfrak{g}_{rs} \cap \mathfrak{h}) \times W_J\backslash W & \xrightarrow{\phi} & p_{\mathfrak{g}_{rs}}^{-1}(\mathfrak{g}_{rs} \cap \mathfrak{h}) \end{array}$$

Lemma 3.22. *Let $w \in W$, $w' \in D_J^{-1}$ and suppose that $w'w = w_2 w_3$ with $w_2 \in W_J$ and $w_3 \in D_J^{-1}$. Then the following diagram commutes*

$$\begin{array}{ccc} p_{\mathfrak{h}_{rs}}^{-1}(\mathfrak{g}_{rs} \cap \mathfrak{h}) \times \{W_J w'\} & \xrightarrow{\phi} & p_{\mathfrak{g}_{rs}}^{-1}(\mathfrak{g}_{rs} \cap \mathfrak{h}) \\ \Big\downarrow {\scriptstyle w_2 \times w} & & \Big\downarrow {\scriptstyle w} \\ p_{\mathfrak{h}_{rs}}^{-1}(\mathfrak{g}_{rs} \cap \mathfrak{h}) \times \{W_J w_3\} & \xrightarrow{\phi} & p_{\mathfrak{g}_{rs}}^{-1}(\mathfrak{g}_{rs} \cap \mathfrak{h}) \end{array}$$

Let $\widetilde{W} = W \rtimes \langle F \rangle$ and $\widetilde{W}_J$ be the semidirect product $W_J \rtimes \langle F' \rangle$ with $F' \cdot w = {}^{w_1 F}w \cdot F'$ for $w \in W_J$. We can define an injective group homomorphism $\iota : \widetilde{W}_J \hookrightarrow \widetilde{W}$ by $\iota(w) = w$ for $w \in W_J$ and $\iota(F') = w_1 F$. We may thus regard $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$ as a right $\overline{\mathbb{Q}}_\ell[\widetilde{W}_J]$-module via ι. It is easily seen that for the right action of W on $p_{\mathfrak{g}_{\mathrm{rs}}}^{-1}(\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h})$ we have the equation $w \cdot F = F \cdot F(w)$. Since the induced action of F on $\mathcal{T}^{\mathfrak{g}}|_{\mathfrak{g}_{rs} \cap \mathfrak{h}}$ is given by $(F^*)^{-1} = (F^{-1})^*$, the left action of $\widetilde{W}$ on $\mathcal{T}^{\mathfrak{g}}|_{\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h}}$ satisfies the equation $F \cdot w = F(w) \cdot F$. Similarly the left action of $\widetilde{W}_J$ on $\mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h}}$ satisfies the equation $F' \cdot w = {}^{w_1 F}w \cdot F'$. Hence $\mathcal{T}^{\mathfrak{g}}|_{\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h}}$ has the structure of a left $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$-module and $\mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h}}$ has the structure of a left $\overline{\mathbb{Q}}_\ell[\widetilde{W}_J]$-module. The relationship between these two modules is given as follows.

Proposition 3.23. *Let the notation be as above. Then*

$$\mathcal{T}^{\mathfrak{g}}|_{\mathfrak{g}_{rs} \cap \mathfrak{h}} \cong \overline{\mathbb{Q}}_\ell[\widetilde{W}] \otimes_{\overline{\mathbb{Q}}_\ell[\widetilde{W}_J]} \mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{rs} \cap \mathfrak{h}}$$

as left $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$-modules.

Proof. We have the following commutative diagram in which π_1 is projection onto the first entry and each square is Cartesian.

$$\begin{tikzcd}
p_{\mathfrak{h}_{\mathrm{rs}}}^{-1}(\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h}) \times W_J\backslash W \arrow[r, "\phi"] \arrow[d, "p_{\mathfrak{h}_{\mathrm{rs}}} \circ \pi_1"] & p_{\mathfrak{g}_{\mathrm{rs}}}^{-1}(\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h}) \arrow[r, hook, "i_1"] \arrow[d, "p_{\mathfrak{g}_{\mathrm{rs}}}"] & \widetilde{\mathfrak{g}}_{\mathrm{rs}} \arrow[d, "p_{\mathfrak{g}_{\mathrm{rs}}}"] \\
\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h} \arrow[r, "\mathrm{id}"] & \mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h} \arrow[r, hook, "i_2"] & \mathfrak{g}_{\mathrm{rs}}
\end{tikzcd}$$

Then disregarding the actions of W and Frobenius, we have

$$\begin{aligned}
\mathcal{T}^{\mathfrak{g}}\big|_{\mathfrak{g}_{\mathrm{rs}}\cap\mathfrak{h}} &= i_2^* p_{\mathfrak{g}_{\mathrm{rs}}*}\overline{\mathbb{Q}}_\ell \\
&\cong (\mathrm{id} \circ i_2)^* p_{\mathfrak{g}_{\mathrm{rs}}*}\overline{\mathbb{Q}}_\ell \\
&\cong (p_{\mathfrak{h}_{\mathrm{rs}}} \circ \pi_1)_*(i_1 \circ \phi)^*\overline{\mathbb{Q}}_\ell \qquad \text{by change of base} \\
&\cong p_{\mathfrak{h}_{\mathrm{rs}}*}\pi_{1*}\overline{\mathbb{Q}}_\ell \\
&\cong p_{\mathfrak{h}_{\mathrm{rs}}*}\Big(\bigoplus_{wW_J \in W/W_J} [\overline{\mathbb{Q}}_{\ell\, p_{\mathfrak{h}_{\mathrm{rs}}}^{-1}(\mathfrak{g}_{\mathrm{rs}}\cap\mathfrak{h})}]_{wW_J}\Big) \quad \text{by definition of } \pi_{1*} \\
&\cong \bigoplus_{wW_J \in W/W_J} [p_{\mathfrak{h}_{\mathrm{rs}}*}\overline{\mathbb{Q}}_\ell]_{wW_J}.
\end{aligned}$$

We have used W/W_J in the summation since the right action of W on $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$ induces a left action on the corresponding local system. By applying base change to

$$\begin{tikzcd}
p_{\mathfrak{h}_{\mathrm{rs}}}^{-1}(\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h}) \arrow[r, hook] \arrow[d, "p_{\mathfrak{h}_{\mathrm{rs}}}"] & \widetilde{\mathfrak{h}}_{\mathrm{rs}} \arrow[d, "p_{\mathfrak{h}_{\mathrm{rs}}}"] \\
\mathfrak{g}_{\mathrm{rs}} \cap \mathfrak{h} \arrow[r, hook] & \mathfrak{h}_{\mathrm{rs}}
\end{tikzcd}$$

we deduce that

$$\mathcal{T}^{\mathfrak{g}}|_{\mathfrak{g}_{\mathrm{rs}}\cap\mathfrak{h}} \cong \bigoplus_{wW_J \in W/W_J} [\mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{\mathrm{rs}}\cap\mathfrak{h}}]_{wW_J}. \tag{3.1}$$

Let $w' \in D_J$ and $w \in W$. Write $ww_3 = w'w_2$ with $w_2 \in W_J$ and $w_3 \in D_J$. Under the bijection $W_J\backslash W \cong W/W_J$ which takes W_Jw to $w^{-1}W_J$, the map $W_Jw'^{-1} \mapsto W_Jw_3^{-1}$ given by right multiplication by w corresponds to the map $w'W_J \leftarrow\!\shortmid w_3W_J$ given by left multiplication by w. Lemma 3.22 thus implies that the action of w on the summands in Equation (3.1) is given by

$$[\mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{\mathrm{rs}}\cap\mathfrak{h}}]_{w_3W_J} \to [\mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{\mathrm{rs}}\cap\mathfrak{h}}]_{w_3W_J} \to [\mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{\mathrm{rs}}\cap\mathfrak{h}}]_{w'W_J}$$

where the first map is that induced by the action of w_2 and the second map is the canonical isomorphism.

Similarly Lemma 3.21 implies that the action of F on $\mathcal{T}^{\mathfrak{g}}|_{\mathfrak{g}_{rs}\cap\mathfrak{h}}$ is determined by its action on the summands,

$$[\mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{rs}\cap\mathfrak{h}}]_{w'W_J} \to [\mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{rs}\cap\mathfrak{h}}]_{w'W_J} \to [\mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{rs}\cap\mathfrak{h}}]_{F(w')w_1^{-1}W_J}$$

where the first map is that induced by F' and the second map is the canonical isomorphism. We are then done by virtue of the relation $Fw' = F(w')w_1^{-1}\iota(F')$ in $\widetilde{W}$. □

Proposition 3.24. *Let the notation be as above. If p is very good for G then* $\mathrm{IC}(\mathfrak{g},\mathcal{T}^{\mathfrak{g}})|_{\mathfrak{h}_c} \cong \mathrm{IC}(\mathfrak{h}_c,\mathcal{T}^{\mathfrak{g}}|_{\mathfrak{g}_{rs}\cap\mathfrak{h}})$ *and* $\mathrm{IC}(\mathfrak{h},\mathcal{T}^{\mathfrak{h}})|_{\mathfrak{h}_c} \cong \mathrm{IC}(\mathfrak{h}_c,\mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{rs}\cap\mathfrak{h}})$.

Proof. We will prove the existence of the first isomorphism. The proof of the existence of the second is entirely analogous. We will follow the proof of Shoji [24, Proposition 4.2]. Let $d = \dim\mathfrak{h}_c$ and $K = \mathrm{IC}(\mathfrak{g},\mathcal{T}^{\mathfrak{g}})|_{\mathfrak{h}_c}[d]$. We have to show that this complex satisfies the following four conditions (cf. Shoji [24, 3.3.1]):

(i) $\mathcal{H}^iK = 0$ if $i < -d$,
(ii) $\mathcal{H}^{-d}K|Y = \mathcal{T}$,
(iii) $\dim\operatorname{supp}\mathcal{H}^iK < -i$ if $i > -d$,
(iv) $\dim\operatorname{supp}\mathcal{H}^iDK < -i$ if $i > -d$.

From the isomorphism $\mathrm{IC}(\mathfrak{g},\mathcal{T}^{\mathfrak{g}}) \cong p_{\mathfrak{g}*}\overline{\mathbb{Q}}_\ell$ in $\mathcal{M}\mathfrak{g}$ and using the proper base change theorem for

$$\begin{array}{ccc} p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c) & \hookrightarrow & \widetilde{\mathfrak{g}} \\ \downarrow{\scriptstyle p_{\mathfrak{g}}} & & \downarrow{\scriptstyle p_{\mathfrak{g}}} \\ \mathfrak{h}_c & \hookrightarrow & \mathfrak{g} \end{array}$$

we have $K \cong p_{\mathfrak{g}*}\overline{\mathbb{Q}}_\ell[d]$. Hence $\mathcal{H}^iK \cong \mathbf{R}^ip_{\mathfrak{g}*}\overline{\mathbb{Q}}_\ell[d]$. Thus (i) follows from the fact that the stalk $(\mathcal{H}^iK)_X$ at $X \in \mathfrak{h}_c$ is isomorphic to $H^{i+d}(p_{\mathfrak{g}}^{-1}(X),\overline{\mathbb{Q}}_\ell)$ and is hence zero for $i < -d$. By using the proper base change theorem for

$$\begin{array}{ccc} p_{\mathfrak{g}_{rs}}^{-1}(\mathfrak{g}_{rs}\cap\mathfrak{h}) & \hookrightarrow & p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c) \\ \downarrow{\scriptstyle p_{\mathfrak{g}_{rs}}} & & \downarrow{\scriptstyle p_{\mathfrak{g}}} \\ \mathfrak{g}_{rs}\cap\mathfrak{h} & \hookrightarrow & \mathfrak{h}_c \end{array}$$

and

$$\begin{array}{ccc} p_{\mathfrak{g}_{rs}}^{-1}(\mathfrak{g}_{rs}\cap\mathfrak{h}) & \hookrightarrow & \widetilde{\mathfrak{g}}_{rs} \\ \downarrow{\scriptstyle p_{\mathfrak{g}_{rs}}} & & \downarrow{\scriptstyle p_{\mathfrak{g}_{rs}}} \\ \mathfrak{g}_{rs}\cap\mathfrak{h} & \hookrightarrow & \mathfrak{g}_{rs} \end{array}$$

we see that $\mathcal{H}^{-d}K|_{\mathfrak{g}_{rs}\cap\mathfrak{h}} = p_{\mathfrak{g}_{rs}*}\overline{\mathbb{Q}}_\ell = \mathcal{T}^{\mathfrak{g}}|_{\mathfrak{g}_{rs}\cap\mathfrak{h}}$ verifying (ii). From Proposition 3.19, $p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)$ is smooth and $\dim p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c) = d$. Hence $D(\overline{\mathbb{Q}}_{\ell_{p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)}}[d]) \cong \overline{\mathbb{Q}}_{\ell_{p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c)}}[d]$. Since $\widetilde{\mathfrak{g}}$ is closed in $\mathfrak{g}\times\mathcal{B}^G$, $p_{\mathfrak{g}} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is proper and the diagram

$$\begin{array}{ccc} p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c) & \hookrightarrow & \widetilde{\mathfrak{g}} \\ \downarrow{\scriptstyle p_{\mathfrak{g}}} & & \downarrow{\scriptstyle p_{\mathfrak{g}}} \\ \mathfrak{h}_c & \hookrightarrow & \mathfrak{g} \end{array}$$

shows that $p_{\mathfrak{g}} : p_{\mathfrak{g}}^{-1}(\mathfrak{h}_c) \to \mathfrak{h}_c$ is also proper (see Mumford [23, II. 7(iii)]). This implies that $DK \cong K$ since the Verdier dual operator commutes with the direct image for a proper map. It is hence enough to show (iii). According to Proposition 3.16, $p_{\mathfrak{g}}^{-1}(X) \cong \mathcal{B}_X^{\mathfrak{h}} \times W_J\backslash W$ for $X \in \mathfrak{h}_c$ and hence the condition $\mathbf{R}^i p_{\mathfrak{g}*}(\overline{\mathbb{Q}}_\ell)_X \neq 0$ implies that $H^i(\mathcal{B}_X^{\mathfrak{h}}, \overline{\mathbb{Q}}_\ell) \neq 0$ and thus $i \leqslant 2\dim\mathcal{B}_X^{\mathfrak{h}}$. Hence the support of $\mathcal{H}^iK$ is contained in the set $\{X \in \mathfrak{h} \mid i + d \leqslant 2\dim\mathcal{B}_X^{\mathfrak{h}}\}$ which has dimension less than $-i$ for $i > -d$ by Shoji [24, Corollary 1.7(i)], thus proving (iii). □

The W (resp. W_J) and F actions on $\mathcal{T}^{\mathfrak{g}}$ (resp. $\mathcal{T}^{\mathfrak{h}}$) induce corresponding actions on $\mathrm{IC}(\mathfrak{g}, \mathcal{T}^{\mathfrak{g}})|_{\mathfrak{h}_c}$ (resp. $\mathrm{IC}(\mathfrak{h}, \mathcal{T}^{\mathfrak{h}})|_{\mathfrak{h}_c}$) by means of the intersection cohomology functor.

Corollary 3.25. *Let the notation be as above and suppose p is very good for G. Then*

$$\mathrm{IC}(\mathfrak{g}, \mathcal{T}^{\mathfrak{g}})|_{\mathfrak{h}_c} \cong \overline{\mathbb{Q}}_\ell[\widetilde{W}] \otimes_{\overline{\mathbb{Q}}_\ell[\widetilde{W}_J]} \mathrm{IC}(\mathfrak{h}, \mathcal{T}^{\mathfrak{h}})|_{\mathfrak{h}_c}$$

as left $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$-modules.

Proof. By Proposition 3.23, $\mathcal{T}^{\mathfrak{g}}|_{\mathfrak{g}_{rs}\cap\mathfrak{h}}$ and $\overline{\mathbb{Q}}_\ell[\widetilde{W}] \otimes_{\overline{\mathbb{Q}}_\ell[\widetilde{W}_J]} \mathcal{T}^{\mathfrak{h}}|_{\mathfrak{g}_{rs}\cap\mathfrak{h}}$ are isomorphic left $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$-modules. The result then follows from Proposition 3.24 and the uniqueness and functoriality of the intersection cohomology functor. □

For the following theorem we recall that G is an arbitrary connected reductive algebraic group over the field $\mathbb{F}_q$ of very good characteristic p. Let $H = {}^xL_J$ where $J \subseteq \Pi$, $x^{-1}F(x) = \dot{w}_1$ with $w_1F(J) = J$. Let $X \in \mathfrak{h}^F$ with $C_G(X_s)^\circ \subseteq H$. Let W (resp. W_J) act on $H^i(\mathcal{B}_X^G, \overline{\mathbb{Q}}_\ell)$ (resp. $H^i(\mathcal{B}^{L_J}_{\mathrm{Ad}\,x^{-1}(X)}, \overline{\mathbb{Q}}_\ell)$) via the Lusztig-Springer action and let W_J act on $H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell)$ by transport of structure via the isomorphism

$$H^i(\mathcal{B}^{L_J}_{\mathrm{Ad}\,x^{-1}(X)}, \overline{\mathbb{Q}}_\ell) \cong H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell).$$

Let the F and F'-actions on $H^i(\mathcal{B}_X^G, \overline{\mathbb{Q}}_\ell)$ and $H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell)$, respectively, be those induced by $F : \mathcal{B}_X^G \to \mathcal{B}_X^G$ and $F' : \mathcal{B}_X^H \to \mathcal{B}_X^H$, respectively. Let $\widetilde{W} = W \rtimes \langle F\rangle$ with $F\cdot w = F(w)\cdot F$ and $\widetilde{W}_J = W_J \rtimes \langle F'\rangle$ with $F'\cdot w = {}^{w_1F}w \cdot F'$ for $w \in W_J$.

Define $\iota : \widetilde{W}_J \hookrightarrow \widetilde{W}$ by $\iota(w) = w$ for $w \in W_J$ and $\iota(F') = w_1 \cdot F$. Regard $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$ as a right $\overline{\mathbb{Q}}_\ell[\widetilde{W}_J]$-module via ι.

Theorem 3.26. *With notation as above*

$$H^i(\mathcal{B}_X^G, \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell[\widetilde{W}] \otimes_{\overline{\mathbb{Q}}_\ell[\widetilde{W}_J]} H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell)$$

as left $\overline{\mathbb{Q}}_\ell[\widetilde{W}]$-modules.

Proof. The fibres of $\mathbf{R}^i p_{\mathfrak{g}*}\overline{\mathbb{Q}}_\ell$ and $\mathbf{R}^i p_{\mathfrak{h}*}\overline{\mathbb{Q}}_\ell$ at $X \in \mathfrak{h}_c^F$ are

$$H^i(p_{\mathfrak{g}}^{-1}(X), \overline{\mathbb{Q}}_\ell) = H^i(\mathcal{B}_X^G, \overline{\mathbb{Q}}_\ell)$$

and

$$H^i(p_{\mathfrak{h}}^{-1}(X), \overline{\mathbb{Q}}_\ell) = H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell).$$

The result follows by taking fibres at X of the isomorphism of Corollary 3.25. □

This result may easily be expressed in terms of "coset induction" (cf. Broué, Malle & Michel [6, §1A] and Aubert [1, *Rappels sur les fonctions de F-classes*, pp. 35–36]). Let $\mathcal{C}(W/F)$ denote the space of complex valued functions on WF which are restrictions of class functions on $\widetilde{W}$. We may similarly define $\mathcal{C}(W_J/F') = \mathcal{C}(W_J/w_1F)$ upon identifying F' with its image $\iota(F')$ in $\widetilde{W}$. If $\alpha \in \mathcal{C}(W_J/w_1F)$ then we define $\operatorname{Ind}_{W_Jw_1F}^{WF} \alpha \in \mathcal{C}(W/F)$ by

$$\operatorname{Ind}_{W_Jw_1F}^{WF} \alpha(wF) = \frac{1}{|W_J|} \sum_{\substack{v \in W \\ {}^v(wF) \in W_Jw_1F}} \alpha({}^v(wF)).$$

Corollary 3.27. *With notation as above, if $X \in \mathfrak{h}_c^F$ and p is very good then*

$$H^i(\mathcal{B}_X^G, \overline{\mathbb{Q}}_\ell) = \operatorname{Ind}_{W_Jw_1F}^{WF} H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell)$$

and

$$\operatorname{Tr}(Fw, H^i(\mathcal{B}_X^G, \overline{\mathbb{Q}}_\ell)) = \frac{1}{|W_J|} \sum_{\substack{v \in W \\ {}^v(Fw) \in W_Jw_1F}} \operatorname{Tr}(vF(wv^{-1})w_1^{-1} \cdot F', H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell)).$$

Proof. According to Theorem 3.26, if $w_3 \in D_J$ and $ww_3 = w'w_2$ with $w' \in D_J$, $w_2 \in W_J$ then the action of Fw on $H^i(\mathcal{B}_X^G, \overline{\mathbb{Q}}_\ell)$ is determined by the map

$$[H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell)]_{w_3W_J} \to [H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell)]_{w_3W_J} \to [H^i(\mathcal{B}_X^H, \overline{\mathbb{Q}}_\ell)]_{F(w')w_1^{-1}W_J}$$

where the first map is that induced by $F'w_2$ and the second map is the canonical map. We thus get a contribution to the trace of Fw from its action on

$[H^i(\mathcal{B}_X^H,\overline{\mathbb{Q}}_\ell)]_{w_3W_J}$ only if $w_3 = F(w')w_1^{-1}$, that is, only if $w_2 = F^{-1}(w_1^{-1}w_3^{-1})ww_3 \in W_J$. Hence

$$\begin{aligned}\operatorname{Tr}(Fw, H^i(\mathcal{B}_X^G,\overline{\mathbb{Q}}_\ell)) &= \sum_{\substack{w_3\in D_J\\ F^{-1}(w_1^{-1}w_3^{-1})ww_3\in W_J}} \operatorname{Tr}(F'\cdot F^{-1}(w_1^{-1}w_3^{-1})ww_3, H^i(\mathcal{B}_X^H,\overline{\mathbb{Q}}_\ell))\\ &= \sum_{\substack{w_3\in D_J\\ w_3^{-1}F(ww_3)w_1^{-1}\in W_J}} \operatorname{Tr}(w_3^{-1}F(ww_3)w_1^{-1}\cdot F', H^i(\mathcal{B}_X^H,\overline{\mathbb{Q}}_\ell)).\end{aligned}$$

The result follows once we realize that, for any $u \in W_J$, $w_3^{-1}F(ww_3)w_1^{-1} \in W_J$ if and only if $w_3^{-1}(Fw)w_3 \in W_Jw_1F$ if and only if $(w_3u)^{-1}(Fw)(w_3u) \in W_Jw_1F$. □

Corollary 3.28. *Let G be a connected reductive group and suppose the characteristic p is very good for G. If X is regular of genus W_Jw_1F in $\mathfrak{g}^F$ then*

$$\widetilde{Q}_w^{\mathfrak{g}}(X) = \operatorname{Ind}_{W_Jw_1F}^{WF} 1(Fw) = |W_J|^{-1}|\{v\in W \mid {}^v(Fw)\in W_Jw_1F\}|.$$

Proof. Let $H = C_G(X_s)^\circ$. Then $X \in \mathfrak{h}_c^F$ and according to Proposition 3.8, $\mathcal{B}_X^H$ is a single point. Hence, by Digne and Michel [12, Proposition 10.11], $H^i(\mathcal{B}_X^H,\overline{\mathbb{Q}}_\ell)$ is zero dimensional for $i \neq 0$ and 1-dimensional when $i = 0$. Moreover, F' acts trivially on $H^0(\mathcal{B}_X^H,\overline{\mathbb{Q}}_\ell)$. We also have $\mathcal{B}_X^H \cong \mathcal{B}_{X_n}^H$ since X_s is central in $\mathfrak{h}$. Thus, by Springer [28, Lemma 7.4] and the fact that the definition of the Lusztig-Springer action differs from Springer's own by the sign representation of W, W_J also acts trivially on $H^0(\mathcal{B}_X^H,\overline{\mathbb{Q}}_\ell)$. The result then follows from Corollary 3.27. □

Definition 3.29. If $\chi\in\widehat{W}$ is F-stable then denote by $\tilde{\chi}$ an arbitrary (but fixed) extension of χ to $\widetilde{W}$. Let $\mathcal{C}(W/F)_{\mathrm{irr}}$ be the set of such $\tilde{\chi}$. For $\tilde{\chi}\in\mathcal{C}(W/F)_{\mathrm{irr}}$ define

$$T(\tilde{\chi};W_Jw_1F) = |W_J|^{-1}\sum_{w\in W_J}\tilde{\chi}(ww_1F).$$

and

$$\widetilde{Q}_\chi = |W|^{-1}\sum_{w\in W}\tilde{\chi}(w)\widetilde{Q}_w.$$

Theorem 3.30. *Let G be a connected reductive group and suppose the characteristic p is very good for G. If $X\in\mathfrak{g}^F$ is regular of genus W_Jw_1F and $\chi\in\widehat{W}$ is F-stable then*

$$\widetilde{Q}_\chi^{\mathfrak{g}}(X) = T(\tilde{\chi};W_Jw_1F).$$

Proof. From Corollary 3.28,

$$\begin{aligned}
\widetilde{Q}^{\mathfrak{g}}_{\tilde\chi}(X) &= |W|^{-1}\sum_{w\in W}\tilde\chi(Fw)\widetilde{Q}^{\mathfrak{g}}_{w}(X)\\
&= |W|^{-1}|W_J|^{-1}\sum_{\substack{w,v\in W\\ {}^{v}(Fw)\in W_J w_1 F}}\tilde\chi(Fw)\\
&= |W|^{-1}|W_J|^{-1}\sum_{v\in W}\sum_{w_2\in W_J}\tilde\chi(v^{-1}(w_2w_1F)v)\\
&= |W|^{-1}|W_J|^{-1}\sum_{v\in W}\sum_{w_2\in W_J}\tilde\chi(w_2w_1F)\\
&= T(\tilde\chi; W_Jw_1F).
\end{aligned}$$

□

Theorem 3.31. *Let G be a connected reductive group and suppose the characteristic p is very good for G. If $X\in\mathfrak{g}^F$ is regular of genus W_Jw_1F then*

$$\widetilde{Q}^{\mathfrak{g}}_{w}(X) = \sum_{\tilde\chi\in\mathcal{C}(W/F)_{\mathrm{irr}}}\overline{T(\tilde\chi; W_Jw_1F)}\tilde\chi(Fw).$$

Proof. From Theorem 3.30, we find that

$$\begin{aligned}
\sum_{\tilde\chi\in\mathcal{C}(W/F)_{\mathrm{irr}}}\overline{T(\tilde\chi; W_Jw_1F)}\tilde\chi(Fw) &= \sum_{\tilde\chi\in\mathcal{C}(W/F)_{\mathrm{irr}}}\tilde\chi(Fw)\overline{\widetilde{Q}^{\mathfrak{g}}_{\tilde\chi}(X)}\\
&= \sum_{\tilde\chi\in\mathcal{C}(W/F)_{\mathrm{irr}}}\tilde\chi(Fw)|W|^{-1}\sum_{v\in W}\overline{\tilde\chi(Fv)}\widetilde{Q}^{\mathfrak{g}}_{v}(X)\\
&= \sum_{v\in W}\widetilde{Q}^{\mathfrak{g}}_{v}(X)|W|^{-1}\sum_{\tilde\chi\in\mathcal{C}(W/F)_{\mathrm{irr}}}\tilde\chi(Fw)\overline{\tilde\chi(Fv)}\\
&= \widetilde{Q}^{\mathfrak{g}}_{w}(X).
\end{aligned}$$

In the second line we have used the fact that $\widetilde{Q}^{\mathfrak{g}}_{w}(X)$ is a rational number (cf. Lehrer [18, Corollary 6.9]). The fourth line follows from the relations

$$\sum_{\tilde\chi\in\mathcal{C}(W/F)_{\mathrm{irr}}}\tilde\chi(Fw)\overline{\tilde\chi(Fv)} = \begin{cases}|W|^{-1}|C_{\widetilde{W}}(Fw)| & \text{if } Fw\sim_{\widetilde{W}}Fv,\\ 0 & \text{otherwise,}\end{cases}$$

which, since the number of F-conjugacy classes of W equals the number of F-stable conjugacy classes, in turn follow from the following relations (cf. Digne & Michel [11, §0.4])

$$|W|^{-1}\sum_{w\in W}\tilde{\chi}(Fw)\overline{\tilde{\phi}(Fw)}=\begin{cases}1 & \text{if } \chi=\phi,\\ 0 & \text{otherwise,}\end{cases}$$

where $\tilde{\chi}$, $\tilde{\phi}\in\mathcal{C}(W/F)_{\mathrm{irr}}$. □

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