Polynomial Identities of Finite Dimensional Lie Algebras

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Preface

During a long time one of the central problem in the theory of associative algebras satisfying a polynomial identity (PI algebras) was Specht's conjecture [**36**] saying that any associative algebra over a field of characteristic 0 has a finite basis of polynomial identities. A positive solution was given by A.R.Kemer in 1987, see [**19**]. His proof is based on a geometric approach due to Amitsur [**2**] and accumulates ideas and results of many specialists on PI rings and algebras, see, e.g., [**27, 6, 33, 34**].

The underlying idea is that free generators of a relatively free algebra are regarded as independent (vector) variables taking value in an algebra A (generic elements of A), so noncommutative polynomials are contained in the algebra of mappings in many varibales from A to A with pointwise operations. In fact, A.Kemer proved a more general statement saying, in particular, that a relatively free PI algebra of finite rank k is canonically isomorphic to the algebra of k generic elements of a *finite dimensional* algebra. It is a far-reaching generalization of a basic fact that over an infinite field the algebra of usual polynomials in k variables is isomorphic to the algebra of regular functions of the k-dimensional affine space.

The success in the associative case stimulated attempts to get a similar statement in varieties of nonassociative algebras. Thus, A.Ya.Vais and E.I.Zel'manov [**38**] proved an analog of Kemer's theorem on finite basis of polynomial identities for finitely generated Jordan PI algebras. In the class of Lie algebras the first example of a finite dimensional algebra which has no a finite basis of identities was constructed by M.Vaughan-Lee [**39**]; it was essential that the ground field was infinite and of characteristic 2. Later V.Drensky generalized it to the case of an infinite field of arbitrary positive characteristic [**12**]. On the other hand, by a theorem due to Ju.A.Bakhturin and A.Ju.Ol'shanskii [**4**], any finite dimensional Lie algebra over a finite field has a finite basis of identities; that was an analog of a well-known theorem in Group Theory proven by S.Oates and M.B.Powell [**28, 26**]. So, the case of characteristic zero was especially interesting, see [**10**, p.43], [**33**], [**14**], [**6**, p.188]; some important particular results were obtained in [**33, 22**].

Using Kemer's approach, the author gave an affirmative answer to this conjecture [**16**]. Moreover, the way used in that work forced to prove a similar statement in a more general context, in the class of PI representations of finitely generated Lie algebras. The aim of this notes is to give the proof of that result in a revised form. For convenience of the reader we also include Kemer's theorem about representability of an associative relatively free algebra of finite rank in terms of generic elements of a finite dimensional algebra; in the main, the proof is similar to the original, however we have changed notations and some technical arguments to prepare the reader to the Lie algebra case.

Although the analog of Kemer's theorem about finite basis of identities of a finite dimensional Lie algebra takes place, it is still unknown whether the relatively free Lie algebra of finite rank, whose multiplication algebra satisfies a polynomial identity, is the algebra of generic elements of a finite dimensional Lie algebra. A similar question is still open in varieties of alternative and Jordan algebras. It would be also interesting to outline the class of algebras where the problem of finite basis of identities in the finite dimensional case can be resolved in a similar way. Probably, the class of structurable algebras [**1**], which includes Jordan and alternative (with involution) algebras and is deeply related to Lie algebras, is an appropriate candidate, cf.[**15**].

Some familiarity with basic facts on associative, Lie algebras (e.g., [**34, 35, 17**]) and varieties of algebraic systems is assumed; although we recall some necessary facts about free algebras, identities, linearizations, etc., for a detailed exposition the reader is referred to [**5, 37, 34, 33, 25**].

The ground field F will be always of characteristic 0; we will also use the following notations:

- ∘ $\mathrm{vect}_F(X) = \mathrm{vect}(X)$ is the span of the subset X in a vector space over F;
- ∘ $\mathrm{End}(V) = \mathrm{End}_F(V)$ is the algebra of linear transformations of a vector space over F;
- ∘ $F[V]$ is the algebra of F-valued polynomial functions on a vector space V;
- ∘ $\mathrm{Pol}(V, W) \simeq F[V] \otimes_F W$ is the space of polynomial mappings from a vectors space V to a vector space W;
- ∘ kV is the direct sum of k copies of a vector space V; $\vec{V}$ is the direct sum of a countable number of V;
- ∘ If A, B are subspaces of an algebra, then $A \cdot B$ denotes the span of all $a \cdot b$, where $a \in A$ and $b \in B$;
- ∘ $A^{(-)}$ is the Lie algebra of an associative algebra A with multiplication $[a, b] = ab - ba$;
- ∘ $A^{\#}$ is an associative algebra A with the adjoint identity element, $A^{\#} = F \cdot 1 \dot{+} A$;
- ∘ $\mathbb{N}$ is the set of naturals; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;
- ∘ $\mathrm{Sym}(M)$ is the symmetric group of a set M.

CHAPTER 1

Kemer's theorem

This chapter gives a proof of Kemer's theorem about representability of associative relatively free PI algebras in the case of finite rank. Throughout this chapter "algebra" means an associative algebra (unless specified); we do not assume that A has the identity element.

1. Algebras of generic elements

We start with some basic definitions , cf. [**34, 37**]. Let $\mathrm{A}\langle X\rangle$ denote the free algebra with the set of free generators X (variables); it means that every mapping π of X to an algebra A extends to an algebra homomorphism from $\mathrm{A}\langle X\rangle$ to A. The cardinality of X is called the rank of $\mathrm{A}\langle X\rangle$; unless specified, we assume that X is a countable set.

In particular, every algebra endomorphism of $\mathrm{A}\langle X\rangle$ is defined by a mapping from X to $\mathrm{A}\langle X\rangle$. A subspace of $\mathrm{A}\langle X\rangle$ is said to be fully characteristic if it is invariant under all algebra endomorphisms of $\mathrm{A}\langle X\rangle$ (i.e., it is invariant under substitution of any polynomials in place of variables).

A polynomial $f(x_1, \dots , x_n) \in \mathrm{A}\langle X\rangle$ is called *an identity of an algebra A*, if it vanishes under substitution of any elements in A in place of variables. The set of all identities of A forms a fully characteristic ideal of $\mathrm{A}\langle X\rangle$, i.e., a *T-ideal.* Conversely, a T-ideal $\Gamma \subseteq \mathrm{A}\langle X\rangle$ is equal to $T(A)$, where $A = \mathrm{A}\langle X\rangle/\Gamma$. The class of all algebras satisfying a fixed set of identities S is a subvariety $\mathfrak{M}$; let Γ be the T-ideal generated by S, i.e., $\Gamma = (S)_{\mathsf{T}}$, then $\mathrm{A}\langle X\rangle/\Gamma$ is the free algebra $\mathrm{A}_{\mathfrak{M}}\langle X\rangle$ of the variety $\mathfrak{M}$; we also call it a *relatively free algebra.*

Recall that A is said to be a PI-algebra if it satisfies an nontrivial identity; in other words, it means that its ideal of identities $T(A) \triangleleft \mathrm{A}\langle X\rangle$ is not equal to zero [**34, 37**]. The main objective of this chapter is to prove the following statement due to A.Kemer [**20**].

THEOREM 1.1. *Let A be a finitely generated associative PI-algebra. Then there exists a finite dimensional algebra B such that $T(A) = T(B)$.*

This fact can rewritten in terms of generic elements.

Let A be an algebra over a field F. Recall that kA is the direct sum of k copies of A; let y_i denote the projection of kA onto the i-th summand, i.e., y_i is a mapping which takes a vector $a \in kA$ to its i-th coordinate, i.e., an element in A. So y_i is a function from kA to A; the set of all functions from kA to A (defined at all points) is an algebra with respect to pointwise operations. Let $\mathcal{F}_k(A)$ denote the subalgebra generated by $y_1, \dots , y_k$, we call it *the algebra of generic elements of A* (of rank k).

For example, if A is the ground field F (which is infinite), then $\mathcal{F}_k(A)$ is isomorphic to the algebra of usual polynomials without the absolute term. Next, suppose A is finite dimensional, then y_i belongs to the algebra of polynomial functions $\mathrm{Pol}(kA, A)$. Let y_{ij} denote the coordinates of y_i with respect to a basis $e_1, \ldots, e_n$ of A, then $\mathrm{Pol}(kA, A)$ is isomorphic to the scalar extension of A:

$$\mathrm{Pol}(kA, A) \simeq F[kA] \otimes_F A$$

where $F[kA]$ is the algebra polynomial functions from the vector space kA to A that is isomorphic to the algebra of polynomials $F[y_{ij}]$.

In fact, we need an infinite number of variables, i.e., generic elements; let $\vec{A}$ denote the direct sum of a countable number of copies of A, we consider it as the direct limit of kA, so the algebra of generic elements $\mathcal{F}(A)$ is generated by $y_1, \ldots, y_k, \ldots$. It is convenient to assume that there is a one-to-one correspondence between the set of generic elements and the set of variables X.

LEMMA 1.1. *Let $\pi : \mathrm{A}\langle X\rangle \to \mathcal{F}(A)$ be the epimorphism defined by the mapping $x_i \to y_i$, then* $\mathrm{Ker}(\pi) = T(A)$.

Proof. Obviously, $T(A) \subseteq \mathrm{Ker}(\pi)$; conversely, if $f = f(x_1, \ldots, x_n)$ is not an identity of A, then $f(a_1, \ldots, a_n) \neq 0$ for suitable $a_1, \ldots, a_n \in A$, hence $f(y_1, \ldots, y_n)$ is a nonzero function, .i.e, $\pi(f) \neq 0$. □

So, $\mathcal{F}(A) \simeq \mathrm{A}\langle X\rangle / T(A)$ and theorem 1.1 can be restated in the following form.

COROLLARY 1.1. *Let A be a finitely generated associative PI algebra. Then there exists a finite dimensional algebra B such that*

$$\mathcal{F}(A) \simeq \mathcal{F}(B) \; .$$

The underlying idea of the proof can be described briefly as follows. It is known that the Jacobson radical of a finitely generated PI algebra is nilpotent [**32, 18, 8**], hence, by [**24**], it satisfies all identities of a suitable finite dimensional algebra. Let $\Gamma = T(A)$, then there exists a f.d. algebra B such that

$$T(B) \subseteq \Gamma \; ; \tag{1.1}$$

the idea is to approximate gradually Γ by ideals of identities of finite dimensional algebras. To that end, we estimate the difference $\Gamma \backslash T(B)$ and, changing B, reduce it to the empty set.

2. Linearization

In this section we give some miscellaneous facts mostly related to linearization of polynomials, cf. [**34, 37, 5**].

Let V be a vector space; like in the previous section the projector of $\vec{V}$ onto the i-th component is called a generic element; let X denote the set of them. Next, let $\mathrm{Map}(\vec{V}, W)$ be the vector space of all mappings from $\vec{V}$ to a vector space W defined at all points of $\vec{V}$. For example, $X \subseteq \mathrm{Map}(\vec{V}, V)$; pick a generic element x_i of V and let $x_i^{-1} : V \to \vec{V}$ denote the mapping of V onto the i-th component, so the composition $x_i^{-1} \circ x_i$ is the identity mapping of V.

Take $f \in \mathrm{Map}(\vec{V}, W)$, $g \in \mathrm{Map}(\vec{V}, V)$, then the composition of g, x_i^{-1} and f means the substitution of g in place of x_i and denoted by $f|_{x_i = g}$. In particular, if $g = a$ is a constant, then the substitution $x_i \to a$ means that we fix the value of x_i.

Take a generic element $x \in X$; a mapping $f \in \mathrm{Map}(\vec{V}, W)$ is said to be *homogeneous with respect to* x if there is $k \in \mathbb{N}_0$ such that $\forall \lambda \in F$ we have $f|_{x=\lambda x} = \lambda^k f$. Obviously, for a nonzero f such a k is unique; we call it *the degree of* f *with respect to* x and denote by $\deg_x(f)$. Clearly, if f does not depend on x, then $\deg_x(f) = 0$; besides, $\deg_x(fg) = \deg_x(f) + \deg_x(g)$ for all f, g.

Suppose $f \neq 0$ is homogeneous with respect to all generic elements; since f belongs to $\mathrm{Map}(kV, W)$ for sufficiently big k, the number of x with $\deg_x(f) \neq 0$ is finite, so we may put

$$\deg(f) = \sum_x \deg_x(f) \ .$$

Let U be a subspace of $\mathrm{Map}(\vec{V}, W)$, for each $x \in X$ fix $k_x \in \mathbb{N}_0$ and consider the set of all homogeneous polynomials of degree k_x with respect to $x \in X$. Obviously, it is closed under vector space operations, we call it a homogeneous component of V (of degree k_x with respect to $x \in X$).

A subspace U is said to be homogeneous ,if it is spanned by homogeneous mappings; using Vandermonde determinant argument, e.g., [**37**, p.], one can show that U is the direct sum of its homogeneous components. Moreover, in the same way one can check the following statement.

Lemma 1.2. *Let* U *be a subspace of a homogeneous space* $U' \subseteq \mathrm{Map}(\vec{V}, W)$ *such that for all* $f \in U$ *and* $\lambda \in F$ *we have* $f_{x=\lambda_x} \in U$. *Then* U *is homogeneous too.*

Let $f \neq 0$ be a homogeneous mapping in a homogeneous space $U \subseteq \mathrm{Map}(\vec{V}, W)$, denote by $\mathrm{varb}(f)$ the set of all x such that $\deg_x(f) \neq 0$; take an arbitrary nonzero $f \in U$ and let $f = \sum_{i=1}^k f_i$ be the decomposition into the sum of homogeneous components, then $\mathrm{varb}(f)$ is defined as $\bigcup_{i=1}^k \mathrm{varb}(f_i)$.

Next, let U be a S-subspace of $\mathrm{Map}(\vec{V}, W)$, i.e., a homogeneous subspace which is invariant under substitution of linear combination of generic elements. We will define the opeartor of linearization $L_x(g)$ for $x \in X$ and $g \in \mathrm{Map}(\vec{V}, V)$, cf. [**37**, p.9].

Pick f and take $y \in X$ such that $\deg_y(f) = 0$ and consider the decomposition of $f|_{x=x+y}$ into the sum of homogeneous components with respect to y,

$$f|_{x=x+y} = \sum_{i=0}^{\infty} f_i \ , \tag{1.2}$$

where f_i is homogeneous in y of degree i. Set $f \cdot L_x(g) = f_1|_{y=g}$; since U is an S-subspace, this element belongs to U thanks to the lemma above; note that the definition does not depend on y. If $\deg_x(f) = k \neq 0$, then the degree of $f \cdot L_x(y)$, where $y \in X$ and $\deg_y(f) = 0$, is equal to $\deg_x(f) = k - 1$. Moreover,

$$f \cdot L_x(y)|_{y=x} = k \cdot f \ . \tag{1.3}$$

Indeed, for all $\lambda \in F$

$$(1+\lambda)^k f = f|_{x=(1+\lambda)x} = (f|_{x=x+y})|_{y=\lambda x} = \sum_{i=0}^{\infty} \lambda^i f_i \ ,$$

matching the coefficients at powers of λ in the both sides, we get $C_k^i f = f_i$.

Now, suppose that U' is generated as an S-subspace by multilinear mappings (i.e., mappings which are linear with respect to every $x \in X$), then an S-subspace U is homogeneous and also generated by multilinear mappings. Indeed, U is homogeneous by the lemma above and, therefore, it is invariant under operators of partial linearization. Moreover, if $\deg_x(f) = 1$, then f is linear with respect to x; taking into account (1.3), we can replace every homogeneous generator of U by its linearizations.

For example, let V, W be an algebra A and let U' be the algebra of generic elements $\mathcal{F}(A)$; clearly, it is a homogeneous S-subspace generated by multilinear mappings, hence, every S-subspace (e.g., a T-ideal) is generated by multilinear elements; thanks to lemma 1.1 the same fact is valid in relatively free algebras. Another example is the vector space of polynomial mappings $\mathrm{Pol}(\vec{V}, W) \subseteq \mathrm{Map}(\vec{V}, W)$; clearly, it is a S-subspace generated by multilinear mappings.

The symmetric group $\mathrm{Sym}(X)$ acts naturally on $\mathrm{Map}(\vec{V}, W)$ by permutation of variables. by definition, every S-subspace is invariant under $\mathrm{Sym}(X)$. Using the technique of Young diagrams, e.g. [**9**], one can prove the following fact, cf. [**33, 5, 13**].

PROPOSITION 1.1. *Let U be a homogeneous S-subspace of $\mathrm{Map}(\vec{V}, W)$ generated by multilinear mappings. Fix $n \in \mathbb{N}$ and suppose that every $f \in U$, which is linear and skew-symmetric with respect to n variables, is equal to 0. Then U is generated as an S-subspace by $U \cap \mathrm{Map}(nV, W)$.*

In particular, U must be equal to 0, if all mappings in U depending only on $x_1, \ldots, x_n$ are equal identically to 0.

For example, let A be an algebra satisfying the Capelli identity of rank $n + 1$

$$\sum_{\sigma \in \S_{n+1}} \mathrm{sgn}(\sigma) x_{\sigma(1)} y_1 \ldots, y_n x_{\sigma(n+1)} \ , \tag{1.4}$$

where $S_{n+1} = \mathrm{Sym}(\{1, \ldots, n+1\})$, cf.[**5, 34**]. Then $T(A) = T(A')$, where A' is an algebra generated by n elements.

Indeed, let Cap_n denote T-ideal generated by (1.4); it contains every polynomial in $\mathrm{A}\langle X \rangle$ which is linear and skew-symmetric with respect to n variables. By hypothesis, $\mathrm{Cap}_{n+1} \subseteq T(A)$, hence, from the proposition above it follows that modulo Cap_{n+1} the S-subspace $T(A)$ is generated by $T(A) \cap \mathrm{A}\langle X_n \rangle$, where $X_n = \{x_1, \ldots, x_n\}$. Put $A' = \mathrm{A}\langle X_n \rangle / T(A) \cap \mathrm{A}\langle X_n \rangle$, we claim that $T(A') = T(A)$. Indeed, A' is embedded into $\mathrm{A}\langle X \rangle / T(A)$, hence $T(A) \subseteq T(A')$. In particluar, $\mathrm{Cap}_{n+1} \subseteq T(A')$, hence $T(A')$ is generated by $T(A') \cap \mathrm{A}\langle X_n \rangle$ by the proposition above. Take $f \in T(A') \cap \mathrm{A}\langle X_n \rangle$, it vanishes under the substitution of $x_i / T(A) \cap \mathrm{A}\langle X_n \rangle$ in place of x_i, which means that $f \in T(A) \cap \mathrm{A}\langle X_n \rangle$; hence, $T(A') = T(A)$.

3. Specializations of noncommutative polynomials

We begin to prove theorem 1.1.

First of all, we can assume that the ground field F is algebraically closed. Indeed, let $\bar{F}$ be the algebraic closure of F and let $\bar{A} = \bar{F} \otimes_F A$, the scalar extension of A by $\bar{F}$. Besides, $\bar{F} \otimes_F \mathrm{A}\langle X \rangle$ is the free algebra over $\bar{F}$; since F is infinite, every T-ideal is homogeneous by lemma 1.2, therefore the algebras A and $\bar{A}$ have the same identities over F, cf. [**37**, p.8] Hence, the ideal of identities of $\bar{A}$ over $\bar{F}$ is equal to $\bar{F} \otimes_F T(A)$.

By hypothesis, $T(\bar{A}) = T(C)$ where C is a finite dimensional algebra over $\bar{F}$; take a basis $c_1, \dots, c_n$ of C and let γ_{ij}^k be the structure constants of C,

$$c_i c_j = \sum_{k=1}^{n} \gamma_{ij}^k c_k \ .$$

Let K be the field extension of F by γ_{ij}^k, since $\bar{F}$ is algebraic over F, it is a finite extension. Hence, the span B of $c_1, \dots, c_n$ over K is a subalgebra of C over F of finite dimension. Cleraly, in $\mathrm{A}\langle X\rangle$ we have $T(B) = T(C) = T(\bar{A}) = T(A)$, as required.

Let A be a finite dimensional algebra over F. We fix a decomposition of A into the sum of the radical $\mathsf{R} = \mathsf{R}(A)$ and a semisimple subalgebra $\mathsf{P} \simeq \mathsf{P}(A) = A/\mathsf{R}(A)$ [**35**, p.192]. The algebra $\mathsf{P}(A)$ is the direct sum of simple algebras; let $\mathrm{pr}(\mathsf{P})$ denote the set of projectors onto simple summands; if $\alpha \in \mathrm{pr}(\mathsf{P})$, then P_α means the corresponding simple subalgebra. Put

$$\mathsf{Spl}(A) = \mathsf{R} \cup \bigcup_{\alpha \in \mathrm{pr}(\mathsf{P})} \mathsf{P}_\alpha \ . \tag{1.5}$$

Pick a multilinear polynomial $f(\mathbf{x}) = f(x_1, \dots, x_n) \in \mathrm{A}\langle X\rangle$ of degree n, where $\mathbf{x}$ means the n-tuple of variables $(x_1, \dots, x_n)$ (here the subindices are not related to any fixed enumeration of elements of X).

Consider a substitution $x \to \bar{x}$ of elements A in place $x \in \mathrm{varb}(f)$, where

$$\bar{x} \in \mathsf{Spl}(A) \ ; \tag{1.6}$$

the n-tuple $\bar{\mathbf{x}}$ is said to be *a specialization of* $\mathbf{x}$ and $\bar{f} = f(\bar{\mathbf{x}}) = f(\bar{x}_1, \dots, \bar{x}_n)$ is *a specialization of* f in A. Put

$$\mathsf{V}(f, A) = \{\bar{\mathbf{x}} |\ f(\bar{\mathbf{x}}) \neq 0\} \tag{1.7}$$

where $\bar{\mathbf{x}}$ runs over all specializations of $\mathbf{x}$ in A.

Take a T-ideal $\Gamma \nsubseteq T(A)$; to estimate the difference $\Gamma \backslash T(A)$ we consider the set

$$\mathsf{V}(\Gamma, A) = \bigcup_f \mathsf{V}(f, A) \ ,$$

where f runs over the set of all multilinear polynomials in Γ. Since Γ is generated by multilinear polynomials, $\mathsf{V}(\Gamma, A)$ is empty iff $\Gamma \subseteq T(A)$.

Type $t(\Gamma, A)$**.**

For $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathsf{V}(f, A)$ put

$$\pi(\bar{\mathbf{x}}) = \{\alpha \in \pi(A) |\ \exists i\ \bar{x}_i \in \mathsf{P}_\alpha\} \tag{1.8}$$

and

$$\mathrm{a}(\bar{\mathbf{x}}) = \sum_{\alpha \in \pi(\bar{x})} \dim(\mathsf{P}_\alpha) \ . \tag{1.9}$$

Next, let $\mathrm{b}(\bar{\mathbf{x}})$ denote the number of x_i such that $\bar{x}_i \in \mathsf{R}$,

$$\mathrm{b}(\bar{\mathbf{x}}) = |\{x_i |\ \bar{x}_i \in \mathsf{R}\}| \ ; \tag{1.10}$$

finally, put $\mathrm{t}(\bar{\mathbf{x}})$ to be the pair $(\mathrm{a}(\bar{\mathbf{x}}), \mathrm{b}(\bar{\mathbf{x}}))$ in the well-ordered set $\mathbb{N}_0 \times \mathbb{N}_0$ with the lexicographical order: $(n, m) \geq (n', m')$ if $n > n'$ or $n = n'$ and $m \geq m'$.

For a multilinear $f \in \Gamma\backslash T(A)$ put

$$\mathrm{t}(f, A) = \min\{\mathrm{t}(\bar{x})|\ \bar{x} \in \mathsf{V}(f, A)\}\ ; \tag{1.11}$$

if $\Gamma \not\subseteq T(A)$ we can define the type $\mathrm{t}(\Gamma, A)$ as the maximum of $\mathrm{t}(f, A)$, where f runs over the subspace of all multilinear polynomials in $\Gamma\backslash T(A)$. By definition, $\mathrm{t}(\Gamma, A)$ is a pair $(a, b) \in \mathbb{N}_0 \times \mathbb{N}_0$; put $\mathrm{a}(\Gamma, A) = a$ and $\mathrm{b}(\Gamma, A) = b$,

It turns out that the type $\mathrm{t}(\Gamma, A)$ can be described in terms of identities.

T-ideal $\mathcal{S}_{a,b,r}(\Gamma)$.

Take $a, b \in \mathbb{N}_0$, $r \in \mathbb{N}$ and a multilinear polynomial $f \in \Gamma$, which contains disjoint subsets $\Lambda_1, \ldots, \Lambda_b$ and $\Theta_1, \ldots, \Theta_r$ in $\mathrm{varb}(f)$ such that

$$|\Lambda_i| = a + 1\ , \tag{1.12}$$

$$|\Theta_j| = a \tag{1.13}$$

for all $i \in \{1, \ldots, b\}$ and $j \in \{1, \ldots, r\}$.

Denote by h the polynomial obtained from f by alternating variables in each Λ_i and Ω_j. More precisely, if M' is a subset of M, then $\mathrm{Sym}(M')$ is embedded naturally in $\mathrm{Sym}(M)$ (all permutations of M' extend trivially to M). Let G be the product of subgroups in $\mathrm{Sym}(X)$:

$$G = \prod_{j=1}^{r} \mathrm{Sym}(\Theta_j) \times \prod_{i=1}^{b} \mathrm{Sym}(\Lambda_i)\ ; \tag{1.14}$$

it is the direct product, because the corresponding subsets are disjoint. It acts on $\mathrm{A}\langle X\rangle$ by permutation of variables and $h = \mathcal{S}_G \cdot f$, where

$$\mathcal{S}_G = \sum_{\sigma\in G} \mathrm{sgn}(\sigma)\sigma. \tag{1.15}$$

So, $\mathcal{S}_{a,b,r}(\Gamma)$ denotes the T-ideal generated by all h defined in this way.

PROPOSITION 1.2. *Let A be a f.d. algebra and let Γ be a T-ideal, $\Gamma \not\subseteq T(A)$. Let $S(\Gamma, A)$ denote the set of all $(a, b) \in \mathbb{N}_0 \times \mathbb{N}_0$ such that*

$$\mathcal{S}_{a,b,r}(\Gamma) \not\subseteq T(A) \tag{1.16}$$

for all $r \in \mathbb{N}$. Then

$$S(\Gamma, A) = \{(a, b) \in \mathbb{N}_0 \times \mathbb{N}_0|\ (a, b) \leq \mathrm{t}(\Gamma, A)\}\ . \tag{1.17}$$

Proof. In particular, the proposition says that $\mathrm{t}(\Gamma, A)$ is the greatest element of $S(\Gamma, A)$; we claim that this statement is equivalent to (1.17). Indeed, suppose $(a, b) = \mathrm{t}(\Gamma, A) \in S(\Gamma, A)$ and take any $(a', b') < (a, b)$; by definition, for all $a' < a$ and $b' \in \mathbb{N}_0$ we have

$$\mathcal{S}_{a,b,r+b'}(\Gamma) \subseteq \mathcal{S}_{a',b',r}(\Gamma)$$

hence, in this case $(a', b') \in S(\Gamma, A)$. Next, if $b' < b$, then

$$\mathcal{S}_{a,b,r}(\Gamma) \subseteq \mathcal{S}_{a,b',r}(\Gamma)$$

hence, the same holds if $a = a'$.

Now, suppose that $(a, b) \in S(\Gamma, A)$, let us show that $(a, b) \leq \mathrm{t}(\Gamma, A)$. Let $r \in \mathbb{N}$ such that $\mathsf{R}^r = 0$, by hypothesis there is a multilinear $h \in \mathcal{S}_{a,b,r}(\Gamma)$ which is not an identity of

A; we may assume that it is equal to $\mathcal{S}_{a,b,r}(f)$ for suitable $f \in \Gamma$. So, the set $\mathsf{V}(h, A)$ is not empty, take $\bar{\mathbf{x}}$ such that $\mathrm{t}(h, A) = (\mathrm{a}(\bar{\mathbf{x}}), b(\bar{\mathbf{x}}))$.

First, suppose $a > \mathrm{a}(\bar{\mathbf{x}})$; since h is skew-symmetric with respect to variables in Θ_j and $|\Theta_j| = a$, there is $x \in \Theta_j$ such that $\bar{x} \in \mathsf{R}$. Hence, $f(\bar{x}) \in \mathsf{R}^r = 0$ which contradicts to the hypothesis. So, $a \leq \mathrm{a}(\bar{\mathbf{x}})$; if the inequality is strict, then $(a, b) < \mathrm{t}(h, A) \leq \mathrm{t}(\Gamma, A)$, as required. Thus, we assume $a = a(\bar{\mathbf{x}})$; hence, for all $i \in \{1, \dots, b\}$ there is $x \in \Lambda_i$ such that $\bar{x} \in \mathsf{R}$, hence $\mathrm{b}(\bar{\mathbf{x}}) \leq b$ and $(a, b) \leq \mathrm{t}(h, A) \leq \mathrm{t}(\Gamma, A)$.

Now we need only to prove that $\mathrm{t}(\Gamma, A) \in S(\Gamma, A)$; let $a = \mathrm{a}(\Gamma, A)$, $b = \mathrm{b}(\Gamma, A)$, we need only to show that (1.16) holds for all r.

LEMMA 1.3. *Let P be a simple algebra and let $f = f(x_1, \dots, x_n) \notin T(P)$. For all $a \in P$ there exists $\bar{x}_i, b_j, c_j \in P$ such that*

$$a = \sum_{j=1}^{m} b_j f(\bar{x}_1, \dots, \bar{x}_n) c_j \ . \tag{1.18}$$

Proof. Take $\bar{x}_i \in P$ such that $\bar{f} = f(\bar{x}_1, \dots, \bar{x}_n) \neq 0$, then the ideal generated by $\bar{f}$ is equal to P. On the other hand, it is equal to $P\bar{f}P$. $\square$

It is well-known [**34**, p.23] that Cap_{t^2} is not contained in the ideal of identities of the algebra of $t \times t$ matrices $M_t(F)$, hence any $a \in M_t(F)$ is represented in the form (1.18) where f is skew-symmetric with respect to t^2 variables. Moreover, for all $r \in \mathbb{N}$ the T-ideal $\mathrm{Cap}_{t^2}^r \not\subseteq T(M_t(F))$, hence, we can find f with disjoint family of subsets $\lambda_1, \dots, \lambda_r \subseteq \mathrm{varb}(f)$ such that f is skew-symmetric with respect to each λ_j.

By the previous argument, for each $\alpha \in \mathrm{pr}(\mathsf{P})$ there exists a polynomial h_α such that

(*) a specialization of h_α in P_α is equal to the identity element e_α of P_α;

(**) there are disjoint subsets of variables $\lambda_1^\alpha, \dots, \lambda_{r+b}^\alpha$ such that h_α is multilinear and skew-symmetric with respect to each λ_α^j and $|\lambda_j^\alpha| = \dim(\mathsf{P}_\alpha)$ for all j.

Take a multilinear $f = f(\mathbf{x}) \in \Gamma$ such that $\mathrm{t}(\Gamma, A) = \mathrm{t}(f, A)$ and let $\bar{\mathbf{x}} \in \mathsf{V}(f, A)$ such that $\mathrm{t}(\bar{\mathbf{x}}) = \mathrm{t}(f, A)$; recall it is the least one among $\mathrm{t}(\tilde{\mathbf{x}})$, $\tilde{\mathbf{x}} \in \mathsf{V}(f, \Gamma)$. Hence, for any specialization $x \to \tilde{x}$ of variables of f with $\mathrm{t}(\tilde{\mathbf{x}}) < \mathrm{t}(\bar{\mathbf{x}})$ we have $f(\tilde{\mathbf{x}}) = 0$. Denote by $Z(f, \bar{\mathbf{x}})$ the set $\{z_1, \dots, z_b\}$ of all variables in $\mathrm{varb}(f)$ such that $\bar{z}_i \in \mathsf{R}$.

First, conisder the case when $\pi(\bar{\mathbf{x}})$ contains a single projector α and $\bar{f} \cdot e_\alpha$ (or $e_\alpha \cdot \bar{f}$) is not zero. Since F is algebraically closed, P_α is isomorphic to a matrix algebra $M_t(F)$, where $a = t^2$. By the lemma above, for all $r \in \mathbb{N}$ there is a polynomial $h = h_\alpha$ satisfying (*), (**); put $f_0 = f \cdot h$. We can assume $\mathrm{varb}(h) \cap \mathrm{varb}(f)$, so this substitution extends $\bar{\mathbf{x}}$; denote it by $x \to \bar{x}$, thus $\bar{f}_0 = \bar{f} \cdot \bar{h} = \bar{f} \neq 0$.

For all $i \in \{1, \dots, b\}$ and $j \in \{1, \dots, r\}$ put $\Lambda_i = \{z_i\} \cup \lambda_i^\alpha$, $\Theta_j = \lambda_{j+b}^\alpha$ and let

$$g = \mathcal{S}_G(f_0) \ ,$$

where G is the group (1.14). Clearly, $g \in \mathcal{S}_{a,b,r}(\Gamma)$, so it suffices to show that g does not vanish under $x \to \bar{x}$.

Consider $\overline{\sigma \cdot f_0}$, where $\sigma \in G$; suppose $\sigma(z_j) \neq z_j$ for some j, then it is equal to $\widetilde{f} \cdot \widetilde{h}$, where $\mathrm{a}(\tilde{\mathbf{x}}) = \mathrm{a}(\bar{\mathbf{x}})$ and $|Z(f, \tilde{\mathbf{x}})| < b$. Hence, $\mathrm{t}(\tilde{\mathbf{x}}) < \mathrm{t}(\bar{\mathbf{x}})$ and therefore $\overline{\sigma \cdot f_0} = 0$, hence, $\bar{g} = |G_1| \cdot \bar{f}\bar{h} \neq 0$, where

$$G_1 = \prod_{i=1}^{r+b} \mathrm{Sym}(\lambda_i^\alpha) \leq G \ .$$

Now consider the general case; recall that e_α denotes the identity element of P_α; put

$$e = \sum_{\alpha \in \pi(\bar{\mathbf{x}})} e_\alpha$$

and $e_0 = 1 - e \in A^\# = F \dotplus A$. It defines the Peirce decomposition of R with respect to the orthogonal system of idempotents e_0, e_α, e.g. [**35**, p.35],

$$\mathsf{R} = \sum_{\alpha,\beta \in \pi(\bar{\mathbf{x}})^\#} \mathsf{R}_{\alpha,\beta} \ ,$$

where $\mathsf{R}_{\alpha,\beta} = e_\alpha \mathsf{R} e_\beta$ and $\pi(\bar{\mathbf{x}})^\# = \pi(\bar{\mathbf{x}}) \cup \{0\}$. Obviously, we can assume that all $\bar{x}$ from R lie in the Pierce components of R; besides, by the previous case, we can assume that $|\pi(\bar{\mathbf{x}})| > 1$ or $e_0 \bar{f} e_0 = (1-e)f(1-e) = \bar{f} \neq 0$.

By definition, for every $\alpha \in \pi(\bar{\mathbf{x}})$ there exists $y \in \mathrm{varb}(f)$ such that $\bar{y} \in \mathsf{P}_\alpha$; hence there exists $z \in Z(f, \bar{\mathbf{x}})$ such that

$$\bar{z} \in \mathsf{R}_{\alpha,\beta} \cup \mathsf{R}_{\beta,\alpha} \ , \tag{1.19}$$

where $\beta \neq \alpha$; fix such $z = z_\alpha \in Z(f, \bar{\mathbf{x}})$, in this case we say that z is marked for α. Denote by $\mathbf{m}(z)$ the set of $\alpha \in \pi(\bar{\mathbf{x}})$ such that z is marked for α; obviously, $0 \leq |\mathbf{m}(z)| \leq 2$.

By lemma 1.3, for all α there is a polynomial h_α satisfying (*), (**); put

$$h'_\alpha = x_\alpha h_\alpha y_\alpha$$

with $\bar{x}_\alpha = e_\alpha$ and $\bar{y}_\beta = e_\beta$ and to every $z \in Z(h, \bar{\mathbf{x}})$ define a polynomial ϕ_z:

(1) if $\mathbf{m}(z) = \emptyset$, then $\phi_z = z$;
(2) let $\mathbf{m}(z) = \{\alpha\}$; then $\bar{z}_\alpha \in \mathsf{R}_{\beta,\alpha}$ (or $\bar{z}_\alpha \in \mathsf{R}_{\alpha,\beta}$) for $\beta \neq \alpha$, put $\phi_z = z' z h'_\alpha$ ($\phi_\alpha = h'_\alpha z z'$, respectively) with $\bar{z}' = e_\beta$;
(3) if $\mathbf{m}(z) = \{\alpha, \beta\}$ and $\bar{z} \in \mathsf{R}_{\alpha,\beta}$, then $\phi_z = h'_\alpha \cdot z \cdot h'_\beta$.

Next, put

$$f_0 = f|_{\forall z \ z = \phi_z} \ ,$$

where z runs over $Z(f, \bar{\mathbf{x}})$; we may assume that the variables envolved in every ϕ_z and $\mathrm{varb}(f) \backslash Z(f, \bar{\mathbf{x}})$ form disjoint sets, so the substitution $x \to \bar{x}$ extends to that for all ϕ_z and gives $\bar{\psi}_z = \bar{z}$, therefore $\bar{f}_0 = \bar{f} \neq 0$. Join variables letting

$$\Lambda_i = \{z_i\} \cup \bigcup_{\alpha \in \pi(\bar{\mathbf{x}})} \lambda_i^\alpha \ ,$$

$$\Theta_j = \bigcup_{\alpha \in \pi(\bar{\mathbf{x}})} \lambda_{b+j}^\alpha \ ,$$

for all $i \in \{1, \dots, b\}$ and $j \in \{1, \dots, r\}$; by definition $|\Lambda_i| = \mathrm{a}(\bar{\mathbf{x}}) + 1 = a + 1$ and $|\Theta_j| = a + 1$. Put $g = \mathcal{S}_G \cdot f_0$, where G is the group (1.14), then $g \in \mathcal{S}_{a,b,r}(\Gamma)$, so it suffices to check that $\bar{g} \neq 0$.

Take $\sigma \in G$ and suppose that $\overline{\sigma \cdot f_0} \neq 0$, we claim that $\sigma(z) = z$ for all $z = z_i \in Z(f, \bar{\mathbf{x}})$ and $\sigma(\lambda_j^\alpha) = \lambda_j^\alpha$ for all j, α. First, consider z with $\mathbf{m}_z \neq \emptyset$ and suppose $\sigma(z) \neq z$, then $\sigma(z) \in \lambda_i^\alpha$ and $\overline{\sigma(z)} \in \mathsf{P}_\alpha$. Recall that the variable z in ϕ_z is surrounded by two variables being fixed under σ and taking value in different simple components of P under $x \to \bar{x}$. Hence, $\overline{\sigma \cdot \phi_z} = 0$ and therefore $\overline{\sigma \cdot f_0} = 0$. Next, take z with $\mathbf{m}(z) = \emptyset$ and suppose $\sigma(z) \neq z$, then $\sigma^{-1}(z) = x \in \lambda_i^\alpha$ for some i, α. There exists $z' \in Z(f, \bar{\mathbf{x}})$ such that

$\alpha \in \mathbf{m}(z')$; by the previous, $\sigma(z') = z'$, hence $\overline{\sigma \cdot f_0} = f(\tilde{\mathbf{x}})$, where $\tilde{z}' \in \mathsf{R}^2$. Hence, $\mathrm{b}(\tilde{\mathbf{x}}) < \mathrm{b}(\bar{\mathbf{x}})$; besides, $\mathrm{a}(\tilde{\mathbf{x}}) = \mathrm{a}(\bar{\mathbf{x}})$, hence $\mathrm{t}(f, \tilde{\mathbf{x}}) < \mathrm{t}(f, \bar{\mathbf{x}})$ and $f(\tilde{\mathbf{x}}) = 0$.

So, $\sigma(z) = z$ for all $z \in Z(f, \bar{\mathbf{x}})$; now, suppose that $\sigma(x) \in \lambda_i^\alpha$ for some $x \in \lambda_i^\beta$, $\beta \neq \alpha$. Observe that

$$\sigma \cdot h'_\alpha = x_\alpha(\sigma \cdot h_\alpha) y_\alpha$$

and all variables involved in $\sigma \cdot h_\alpha$ take value in P, hence $\overline{\sigma \cdot h'_\alpha} = 0$. So $\sigma(\lambda_i^\alpha) = \lambda_i^\alpha$ for all α, i; hence $\bar{g} = q \cdot \bar{f}_0$ for some $q \in \mathbb{N}$. The proposition is proven. □

4. Decomposition of ideals of identities

We will need a general construction of finite dimensional algebra based on free products of algebras, e.g., [**5**]. Recall that an algebra $A * B$ is called a free product of algebras A, B if

(i) A, B are subalgebras of $A * B$;
(ii) $A * B$ is generated by A and B;
(iii) if $\phi : A \to D$, $\psi : B \to D$ are homomorphisms of algebras, then there is a homomorphism $\pi : A * B \to D$ such that $\pi|_A = \phi$ and $\pi|_B = \psi$.

Now, let P be a finite dimensional algebra and let Y be a finite set of variables, $|Y| = d$. Denote by $P\langle Y\rangle$ the free product of P and the free algebra $\mathrm{A}\langle Y\rangle$. By (iii), any mapping $\pi : Y \to P\langle Y\rangle$ can be extended to an endomorphism $\tilde{\pi}$ of $P\langle Y\rangle$ such that $\tilde{\pi}|_P = \mathrm{id}_P$. In terms of variables it means that we may replace variables from Y by elements in $P\langle Y\rangle$. Recall that an element $f \in P$ is homogeneous with respect to $y \in Y$ of degree m, if for every $\lambda \in F$ we have $f|_{y=\lambda y} = \lambda^m f$. Denote by $P\langle Y\rangle^{(i)}$ the subspace of homogeneous elements in $P\langle Y\rangle$ of total degree i; in particular, $P\langle Y\rangle^{(0)} = P$.

Clearly, algebra $P\langle Y\rangle$ has the following properties:

(i) $P\langle Y\rangle$ is the direct sum of its homogeneous components,

$$P\langle Y\rangle = \bigoplus_{i=0}^{\infty} P\langle Y\rangle^{(i)} ;$$

(ii) for all i the subspace $P\langle Y\rangle^{(i)}$ is finite dimensional.

Let Γ be a T-ideal of $\mathrm{A}\langle X\rangle$; denote by I the verbal ideal of Γ in $P\langle Y\rangle$,

$$I = \Gamma[P\langle Y\rangle] = \{f(a_1, \dots , a_n)|\ f \in \Gamma;\ a_i \in P\langle Y\rangle\} ;$$

also, let J be the ideal generated by Y; take $b \in \mathbb{N}_0$ and put

$$\mathcal{K}(P, Y, \Gamma, b) = P\langle Y\rangle/V \tag{1.20}$$

where $V = I + J^{b+1}$.

LEMMA 1.4. *The algebra $\mathcal{K}(P, Y, \Gamma, b)$ is finite dimensional and satisfies all identities in Γ. Besides, if $\Gamma \subseteq T(P)$, then P is isomorphic to $P + V/V \subseteq \mathcal{K}(P, Y, \Gamma, b)$.*

Proof. The first conjecture follows from (i) above and the definition. Next, since Γ is homogeneous, I and J^{b+1} are homogeneous with respect to every variable from Y, hence $P \cap V \subseteq P \cap I$; moreover, $I \cap P = \Gamma[P] = 0$ because $\Gamma \subseteq T(P)$, hence $P \simeq P + V/V \subseteq \mathcal{K}(P, Y, \Gamma, b)$. □

Let A be a finite dimensional algebra; denote by $\mathrm{a}(A)$ the dimension of $\mathsf{P}(A)$, also let $\mathrm{b}(A) \in \mathbb{N}_0$ such that $\mathrm{b}(A)+1$ is the nilpotency index of $\mathsf{R}(A)$; finally $\mathrm{t}(A) = (\mathrm{a}(A), \mathrm{b}(A))$.

PROPOSITION 1.3. *Let A be a f.d. algebra. For all $a, b \in \mathbb{N}_0$ there exists a finite family of algebras B^ξ ($\xi \in \Xi$) such that*

(a) *$T(A) \subseteq T(B^\xi)$ for all $\xi \in \Xi$;*
(b) *$\mathrm{t}(B^\xi) \le (a, b)$ for all $\xi \in \Xi$;*
(c) *for all multilinear $f \in \mathrm{A}\langle X\rangle$, if $f \notin T(A)$ and $\mathrm{t}(f, A) \le (a, b)$, then*

$$f \notin \bigcap_{\xi \in \Xi} T(B^\xi) \ .$$

Proof. Let Ξ denote the set of all subsets $\xi \subseteq \mathrm{pr}(\mathsf{P})$, where $\mathsf{P} = \mathsf{P}(A)$, such that $\sum_{\alpha\in\xi} \dim(\mathsf{P}_\alpha) \le a$. Put $\mathsf{P}^\xi = \sum_{\alpha\in\xi} \mathsf{P}_\alpha$ and $B^\xi = \mathcal{K}(\mathsf{P}^\xi, Y, T(A), c)$, where $|Y| = \dim(\mathsf{R})$ and

$$c = \begin{cases} b & , \dim(\mathsf{P}^\xi) = a \\ b(A) & , \text{otherwise.} \end{cases}$$

By lemma 1.4 the algebra B^ξ is finite dimensional and $T(A) \subseteq T(B^\xi)$; also, $T(A) \subseteq T(\mathsf{P}^\xi)$, hence P^ξ is a subalgebra of B^ξ. The radical $\mathsf{R}(B^\xi)$ is generated by Y/V (in notations of lemma 1.4), hence P^ξ is a semisimple part of B^ξ. Thus, $\mathrm{t}(B^\xi) \le (a, b)$.

Let us prove (c); take $\bar{\mathbf{x}} \in \mathsf{V}(f, \mathcal{A})$ such that $\mathrm{t}(\bar{\mathbf{x}}) = \mathrm{t}(f, \mathcal{A})$; then $|\pi(\bar{\mathbf{x}})| = \mathrm{a}(\bar{\mathbf{x}}) \le a$ and $\xi = \pi(\bar{\mathbf{x}})$ belongs to J. Define a mapping from Y onto a basis of R; it extends to a homomorphism ψ from $K^\xi = \mathsf{P}^\xi\langle Y\rangle$ to A, where ψ restricts to the identity mapping on P^ξ. By definition, for all $x \in \mathrm{varb}(f)$ the specialization $\bar{x}$ lies in the image of ψ. Denote by $\tilde{x} \in K^\xi$ a pre-image of $\bar{x}$, where $\tilde{x} \in \mathrm{vect}(Y)$ provided $\bar{x} \in \mathsf{R}$ and $\tilde{x} = \bar{x}$ if $\bar{x} \in \mathsf{P}^\xi$. Then the specialization $\tilde{f} = f(\tilde{x}_1, \ldots, \tilde{x}_m)$ is not equal to 0 and, since $\mathrm{b}(\bar{\mathbf{x}}) \le b$, its degree with respect to Y is $\le c$. Besides, the verbal ideal $\Gamma[K^\xi]$ lies in the kernel of ψ, hence $\tilde{f} \notin \Gamma[K^\xi]$ and therefore $\tilde{f} \notin \Gamma[K^\xi] + J^{c+1}$ and $f \notin T(\mathcal{B}^\xi)$. □

An algebra A is called *representable* if there exists an extension K of the ground field F and a finite dimensional algebra A' over K such that A is embedded in A' as an algebra over F. Let $B \subseteq A'$ denote the span of A over K, then $T(B) = T(A)$. Indeed, since $A \subseteq B$, we have $T(B) \subseteq T(A)$; on the other hand, $T(A)$ is generated by multilinear polynomials, hence $T(A) \subseteq T(B)$.

LEMMA 1.5. *Let B an algebra over F that is finite dimensional over an extension K of F. Then there exists a finite dimensional algebra C over F such that $T(C) = T(B)$.*

Proof. Recall that B is decomposed into the sum of the radical $\mathsf{R}(B)$ and a semisimple subalgebra $\mathsf{P}(B)$; besides, we may assume that H is algebraically closed, hence $P(B)$ is the direct sum of matrix algebras, and therefore $\mathsf{P}(B) \simeq K \otimes_F P$, where P is a direct sum of matrix algebras over F.

Next, put

$$C = \mathcal{K}(P, Y, T(B), \mathrm{b}(B))$$

where $|Y| = \dim_K(\mathsf{R}(B))$. By lemma 1.4, C is a finite dimensional algebra over F and $T(B) \subseteq T(C)$. Now suppose that the reciprocal inclusion is untrue, then there is a multilinear polynomial $f(x_1, \ldots, x_n) \in T(C)$ which is not an identity of B. Let $\bar{f} \ne 0$ be a specialization of f in B, moreover , we may assume that $\bar{x}_i$ are elements of a basis of B. Hence, we can define a homomorphism π from $P\langle Y\rangle$ to B which takes elements from Y to the basis elements of the radical $\mathsf{R}(B)$ and restricts to the identity mapping of $P \subseteq \mathsf{P}(B)$. Hence, there is $\tilde{f} \in \pi^{-1}(\bar{f})$ being homogeneous with respect to Y and degree $\leq b$, hence, $\tilde{f} \notin V$ and $\hat{f} = \tilde{f}/V$ is a nonzero specialization of f in C which is impossible. $\square$

So, to prove theorem 1.1 it suffices to show that $\mathrm{A}\langle X\rangle/T(A)$ is representable.

5. Specht's conjecture

A positive solution of Specht's problem was given by A.Kemer in [**19**]; it says that any associative algebra over a field of characteristic 0 has a finite basis of identities. In fact, the proof consists of two parts; the first one shows that the variety of any PI algebra is generated by the Grassmann envelope of a finitely generated PI superalgebra, and the second part states d.c.c. conditions for subvarieties. In this section we gives the argument of the second part adopted to the case of algebras which gives Specht's conjecture for finitely generated algebras.

COROLLARY 1.2. *The ideal of identities of a finitely generated algebra A over F is a finitely generated T-ideal.*

Suppose it is untrue; we assume that $T(A) \neq 0$, hence A is a finitely generated PI algebra. By [**18**], $\mathrm{Cap}_n \subseteq T(A)$ for some n; besides, by proposition 1.1 Cap_n is the ideal of identities of an algebra generated by $n-1$ elements; besides, by [**32, 18, 8**], the Jacobson radical of of a finitely generated PI algebra is nilpotent, hence, thanks to [**24**], Cap_n contains the ideal of identities of a finite dimensional algebra B. Since Cap_n is a finitely generated T-ideal, it suffices to show that $T(A)$ is finitely generated modulo $T(B)$.

PROPOSITION 1.4. *Let A be a finite dimensional algebra, then modulo $T(A)$ any T-ideal is generated by a finite number of polynomials.*

Proof. Suppose it does not hold; since every T-ideal is generated by multilinear polynomials, there exists a sequence of multilinear polynomials f_n, $n = 1, \dots, \infty$, such that $deg(f_n) < deg(f_{n+1})$ for all n and $f_{n+1} \notin T(A) + (f_1, \dots, f_n)_{\mathsf{T}}$.

Put

$$(f_n)^{+}_{\mathsf{T}} = (f_n x,\ x f_n,\ f_n|_{x_i = x_i x},\ f_n|_{x_i = x x_i} \mid \forall i,\ f_n = f_n(x_1, \dots, x_i, \dots, x_n),\ \forall x \in X)_{\mathsf{T}}\ .$$

In other words, it is the T-ideal of all "proper" consequences of the identity f_n; let $\Gamma = T(A) + \sum_{n=1}^{\infty} (f_n)^{+}_{\mathsf{T}}$.

From homogeneity of T-ideals it follows that $f_n \notin \Gamma$ for all n, but $(f_n)^{+}_{\mathsf{T}} \subseteq \Gamma$. We call such a sequence of polynomials $\{f_n \mid n \in \mathbb{N}\}$ *appropriate.* A T-ideal which has an appropriate sequence is called *critical.*

PROPOSITION 1.5. *Let Γ be a critical T-ideal, then $\Gamma \neq T(C)$, where C is a finite-dimensional algebra.*

Proof. On the contrary, suppose that $\Gamma = T(C)$, where C is a finite dimensional algebra; recall that C is the sum of the radical $\mathsf{R}(C)$ and a semisimple subalgebra $\mathsf{P}(C)$. Take an element f_n of the appropriate sequence such that $\deg(f_n) \geq \mathrm{b}(C)$; since $f_n = f_n(x_1, \dots, x_k) \notin \Gamma = T(C)$, there are $a_1, \dots, a_k \in C$ such that $\bar{f}_n = f_n(a_1, \dots, a_k) \neq 0$.

Recall that f_n is multilinear, therefore, we may assume that $a_i \in \mathsf{R}(C) \cup \mathsf{P}(C)$ for all i. Suppose that $a_j \in \mathsf{P}(C)$ for some j; by the definition, $f'_n(x_1, \dots, x_{k+1}) = f_n|_{x_j = x_j x_{k+1}} \in \Gamma = T(C)$, hence, $\bar{f}_n$ must be zero. On the other hand, let e be the identity element of $\mathsf{P}(C)$, then $f'_n(a_1, \dots, a_k, e) = \bar{f}_n$. So, $a_i \in \mathsf{R}(C)$ for all i, but in this case $\bar{f}_n$ also vanishes, because $\deg(f_n) \geq c$. □

From theorem 1.1 it follows that critical T-ideals do not exist, hence, it proves Specht's conjecture. However, we give a shorter proof of this fact.

"Counter-example".

Call a pair of T-ideals

$$T(A) \subset \Gamma$$

a "counter-example", if A is a finite-dimensional algebra and Γ is a critical T-ideal.

Now suppose that a "counter-example" exists, then there exists a "counter-example" $T(A) \subset \Gamma$ of the smallest type $\mathrm{t}(\Gamma, A)$.

Lemma 1.6. *Let Γ, Γ' be T-ideals, where Γ is critical. Then $\Gamma + \Gamma'$ or $\Gamma \cap \Gamma'$ is critical.*

Proof. Let $\{f_i |\ i = 1, 2, \dots\}$ be the appropriate sequence for Γ. Suppose that every subsequence is not appropriate for $\Gamma + \Gamma'$, then there exists n such that for all $m \geq n$ we have $f_m = h_m + g_m$, where $h_m \in \Gamma$, $g_m \in \Gamma'$ are multilineal polynomials of degree $\deg(f_m)$. Since $f_m \notin \Gamma$ and $h_m \in \Gamma$, we get $g_m \notin \Gamma$. Besides, $(g_m)^{+}_{\mathsf{T}} \subseteq (f_m)^{+}_{\mathsf{T}} + (h_m)^{+}_{\mathsf{T}} \subseteq \Gamma$ and therefore

$$(g_m)^{+}_{\mathsf{T}} \subseteq \Gamma \cap \Gamma' \ .$$

Hence, $\{g_n |\ n \geq m\}$ is an appropriate sequence for $\Gamma \cap \Gamma'$. □

Lemma 1.7. *Let $\Gamma \subseteq T(C)$ be a "counter-example". Suppose there is a family of finite dimensional algebras C_i, $i = \{1, \dots, k\}$ such that*

$$T(C) = \Gamma \cap \bigcap_{i=1}^{k} T(C_i) \ .$$

Then for some i there exists a "counter-example" $T(C_i) \subset \Gamma'$, where $\Gamma' \subseteq \Gamma + T(C_i)$.

Proof. Put $\Upsilon_0 = \Gamma$ and

$$\Upsilon_m = \Gamma \cap \bigcap_{i=1}^{m} T(C_i) \ ;$$

take the maximal $m \in \{0, \dots, k\}$ such that Υ_m is critical, then $m < k$ and by the previous lemma,

$$\Gamma' = \Upsilon + T(C_{m+1})$$

is critical. □

From proposition 1.3 we get immediately

Corollary 1.3. *There exists a counter example $T(B) \subset \Gamma'$, where $\mathrm{t}(B) \leq \mathrm{t}(\Gamma, A)$.*

Since $\mathrm{t}(\Gamma, A)$ is minimal and $\mathrm{t}(\Gamma', B) \leq \mathrm{t}(B)$, we have $\mathrm{t}(\Gamma', B) = \mathrm{t}(B)$. In the next section we will exhibit a "counter example" of lesser type.

6. Finite-dimensional embedding

Let A be a finite dimensional algebra and let $\mathrm{tr}(x)$ be a linear mapping to the centroid $\mathcal{C}(A)$ of A,

$$\mathcal{C}(A) = \{\phi \in \mathrm{End}(A) | \ \forall a, b \in A \ \phi(ab) = \phi(a)b = a\phi(b)\} \ .$$

Denote by Ψ the subalgebra of $\mathrm{Pol}(\vec{A}, \mathcal{C}(A))$ generated by 1 and $\mathrm{tr}(u)$, where u runs over $\mathcal{F}(A)$ and let $\Psi \cdot \mathcal{F}(A)$ denote the Ψ-module generated by Ψ in $\mathrm{Pol}(\vec{A}, A)$. Clearly, $\Psi \cdot \mathcal{F}(A)$ is a fully characteristic ideal, i.e., it is invariant under substitution of elements from $\mathcal{F}(A)$ in place of variables (generic elements).

We say $\mathcal{F}(A)$ is integral over Ψ if for all $f \in \mathcal{F}(A)$ there are $\phi_1, \dots, \phi_n \in \Psi$ such that

$$f^{n+1} = \sum_{i=1}^{n} \phi_i f^i \ . \tag{1.21}$$

Lemma 1.8. *Suppose that $\mathcal{F}(A)$ is integral over Ψ. Let I be a fully characteristic ideal of $\Psi \cdot \mathcal{F}(A)$, then the full pre-image of $I \cap \mathcal{F}(A)$ with respect to the canonical homomorphism fron $\mathrm{A}\langle X \rangle$ onto $\mathcal{F}(A)$, see proposition 1.1, is equal to the ideal of identities of a finite dimensional algebra C,*

$$I \cap \mathcal{F}(A) = T(C)/T(A) \ .$$

Proof. Let Υ be the full pre-image of $I \cap \mathcal{F}(A)$, i.e., $\Upsilon/T(A) = I \cap \mathcal{F}(A)$; since I is fully characteristic, Υ is a T-ideal. Let $k = \dim(A)$, then $\mathrm{Cap}_{k+1} \subseteq T(A) \subseteq \Upsilon$; by lemma 1.1 Υ is the ideal of identities of $\mathrm{A}\langle X_k \rangle/\Upsilon_k$, where $\Upsilon_k = \Upsilon \cap \mathrm{A}\langle X_k \rangle$. Thanks to lemma 1.5 it suffices to show that $\mathrm{A}\langle X_k \rangle/\Upsilon_k$ is representable.

Recall that $\mathcal{F}_k(A)$ is the subalgebra of $\mathcal{F}(A)$ generated by $x_1, \dots, x_k$; in fact, it is the span of all elements in $\mathcal{F}(A)$ of zero degree with respect to all $x \notin \{x_1, \dots, x_k\}$. Let Ψ_k denote the span of such elements in in Φ; clearly, it is generated by $\mathrm{tr}(u)$, where $u \in \mathcal{F}_k(A)$. Put $B_k = \Psi_k \cdot \mathcal{F}_k(A)$; it is the corresponding homogeneous component of $\Psi \cdot \mathcal{F}(A)$; besides,

$$\Upsilon_k + T(A)/T(A) = I \cap \mathcal{F}_k(A) = I_k \cap \mathcal{F}_k(A)$$

where $I_k = I \cap B_k$. Hence, $\mathrm{A}\langle X_k \rangle/\Upsilon_k$ is embedded into B_k/I_k and we need only to show that the latter algebra is representable.

By (1.21) every element in $\mathcal{F}_k(A)$ is integral over Ψ; thanks to homogeniety, we can assume that all ϕ_j in (1.21) are in Ψ_k, so $\mathcal{F}_k(A)$ is integral over Ψ_k, Besides, B_k inherits all identities of A, in particular, it is PI; finally, it is generated by $x_1, \dots, x_k$ over Φ_k, hence, by Shirshov's theorem on hieght [**34, 37**], B_k is finitely generated Ψ_k-module, hence so is B_k/I_k. Let $u_1, \dots, u_m$ be the generators, then Ψ_k as an algebra over F is generated by 1 and $\mathrm{tr}(u_1), \dots, \mathrm{tr}(u_m)$. From [**7, 3**] it follows that B_k/I_k is representable. □

Now we continue the proof of theorem 1.1. Let $\mathrm{t}(\Gamma, A) = (a, b)$, by proposition 1.3 there is a finte family of finite dimensional algebras B^ξ, $\xi \in \Xi$, such that $\mathrm{t}(B) \leq \mathrm{t}(\Gamma, A)$.

Proposition 1.6. *There exists $r_0 \in \mathbb{N}$ such that for all $r \geq r_0$ there exists a f.d. algebra B such that*

$$T(B) = \mathcal{S}_{a,b,r}(\Gamma) + \bigcap_{xi \in \Xi} T(B^\xi) \ . \tag{1.22}$$

Proof. Take $\xi \in \Xi$ such that $\mathrm{t}(B^\xi) < \mathrm{t}(\Gamma, A)$, then $\mathrm{t}(\Gamma, B^\xi) < (a,b)$ by proposition 1.2. Hence, there exists $r \in \mathbb{N}$ such that

$$\mathcal{S}_{a,b,r}(\Gamma) \subset T(B^\xi) \ ; \tag{1.23}$$

since the set Ξ is finite, we can assume that it holds for all $\xi \in \Xi$ with $\mathrm{t}(B^\xi) < (a,b)$.

Consider the direct sum of algebras $C = \bigoplus_{\xi\in\Xi} B^\xi$ and define a linear mapping tr : $C \to \bigoplus_{\xi\in\Xi} F^\xi$, where $F^\xi = F \cdot \mathrm{id}_{B^\xi} \subseteq \mathrm{End}(B^\xi)$, letting

$$\mathrm{tr}(\oplus_\xi a^\xi) = \oplus_\xi \mathrm{tr}_\xi(a^\xi) \cdot \mathrm{id}_{B^\xi} \ ,$$

where $\mathrm{tr}_\xi(x)$ is the trace of the operator $R_{p_\xi(x)} \in \mathrm{End}(\mathsf{P}(B^\xi))$ and $p_\xi(x)$ is the projection of $x \in C$ onto $\mathsf{P}(B^\xi)$. Observe that $\bigoplus_{\xi\in\Xi} F^\xi$ lies in the centroid of C; consider the subalgebra Ψ with respect to this $\mathrm{tr}(x)$.

Lemma 1.9. *The algebra $\mathcal{F}(C)$ is integral over Ψ.*

Proof. Let $u \in \mathcal{F}(C)$; by Cayley-Hamilton's theorem in Linear Algebra, $E_\xi = R_{p_\xi(u)}$ satisfies the equation

$$E_\xi^{m+1} - \mathrm{tr}_\xi(E_\xi)E_\xi^m + \ldots + (-1)^m \det(E_\xi)E_\xi = 0 \ ,$$

where $m = \dim(\mathsf{P}(B^\xi))$ and all coeficients are polynomials in $\mathrm{tr}_\xi(E_\xi^j)$ [**34**, p.18]. If we replace each $\mathrm{tr}_\xi(E_\xi^j)$ with $\mathrm{tr}(E^j)$, where $E = \bigoplus_\xi E_\xi$, we get a similar equality for E_ξ, thus, E_ξ is integral over Ψ; hence, E is also integral over Ψ.

Let $f(t) = t^n + \sum_{j=1}^{n-1} t^j\phi_j$ be a polynomial over Ψ such that $f(E) = 0$; put $E' = R_{p(u)}$, where R_x means the right multiplication operator in C and $p(x) = \bigoplus_\xi p_\xi(x)$. Then E' restricts to E on $\mathsf{P}(C) = \bigoplus_\xi \mathsf{P}(B^\xi)$, hence $f(E')$ restricts to 0 on $\mathsf{P}(C)$ and therefore $f(E') = 0$ on C. Notice that $u = p(u) + d(u)$, where $d(x)$ is the projection of x onto the radical of C, hence $f(R_u) = f(R_{p(u)} + R_{d(u)}) = R_v$, where v is a mapping in $\Psi \cdot \mathcal{F}(C)$ which takes value in the radical $\mathsf{R}(C)$ under any specialization of variables in A. Hence, $u \cdot f^c(R_u) = 0$ for all $c > \mathrm{b}(C)$, therefore u is integral over Ψ. □

Put $\Upsilon = \mathcal{S}_{a,b,r}(\Gamma) + T(C)$, it is precisely the right hand side of (1.22). Next, let $I = \Upsilon/T(C) \subseteq \mathcal{F}(A)$; to use proposition 1.8 it suffices to show that I is invariant under multiplication by elements in Ψ.

Lemma 1.10. *For all $h \in \Upsilon$ and $u \in \mathcal{F}(C)$ there exist $g \in \Upsilon$ such that modulo $T(C)$*

$$\mathrm{tr}(u) \cdot h = g \tag{1.24}$$

Proof. We need to prove equality (1.24) under any specialization of variables in C. Since I is a fully characteristic ideal, we may assume that $h = \mathcal{S}_{a,b,r}(f)$, where f is a multilinear polynomial in Γ and u is a variable, $u \notin \mathrm{varb}(f)$. To find g we will need the following important observation [**29**].

Lemma 1.11. *Let V, W be vector spaces over a field F, where $\dim(V) = n$. Let $f = f(x_1, \ldots, x_n)$ be a multilinear skew symmetric mapping from V to W. Then for all $E \in \mathrm{End}(V)$*

$$\mathrm{tr}(E) \cdot f = \sum_{i=1}^n f|_{x_i = x_iE} \ .$$

Proof. It suffices to check it for all $x_1, \ldots, x_n \in V$; since f is skew symmetric, we can assume that they form a basis of V. Let (a_{ij}) be the matrix of E in this base. Then

$$f|_{x_i=x_iE} = f|_{x_i=\sum_j a_{ij}x_j} = f|_{x_i=a_{ii}x_i} = f \cdot a_{ii} \ .$$

Taking the sum , we get the required statement. □

Put

$$g = f * \mathrm{tr}(u) = \sum_{y\in\Theta_1} f|_{y=yu} \ ,$$

clearly, it belongs to $\mathcal{S}_{a,b,r}(\Gamma)$. Since the both sides of (1.24) are multilinear, it suffices to check the equality under specialization $x \to \bar{x} \in \mathsf{Spl}(C)$. Moreover, since C is the direct sum of algebras B^ξ and $\mathrm{tr}(x)$, respectively, is the direct sum of $\mathrm{tr}_\xi(x)$, we need only to prove that

$$f * \mathrm{tr}(u) = \mathrm{tr}_\xi(u) \cdot f$$

under any substitution $x \to \bar{x} \in \mathsf{Spl}(B^\xi)$. Moreover, thanks to (1.23) we can assume that $\mathrm{t}(B^\xi) = (a, b)$.

Consider the case when the right hand side does not vanish under $x \to \bar{x}$ (the proof similar under the same hypothesis for the other side). First, observe that $\bar{u} \in \mathsf{P}(B^\xi)$; next, $\mathrm{a}(B^\xi) = a$, hence for every $i \in \{1, \ldots, b\}$ there exists $x \in \Lambda_i$ with $\bar{x} \in \mathsf{R}(B^\xi)$. Moreover, $\mathrm{b}(B^\xi) = b$, hence for all $x \notin \bigcup_{i=1}^b \Lambda_i$ the value $\bar{x}$ belongs to $\mathsf{P}(B^\xi)$. In particular, $\bar{x} \in \mathsf{P}(B^\xi)$ for all $x \in \Theta_1$; besides, $|\Theta_1| = a$ is equal to $\mathrm{a}(B^\xi)$, the dimension of $\mathsf{P}(B^\xi)$, hence the required statement follows from lemma 1.11. □

Thus, proposition 1.6 is proven. Consider the "counter-example" from corollary 1.3; the family consisting of a single algebra $B^\xi = A$ satisfies the hypothesis of proposition 1.6, hence $T(C) = \mathcal{S}_{a,b,r}(\Gamma) + T(A)$ for some finite dimensional algebra C, so $T(C) \subset \Gamma$ is a "counter-example". From proposition 1.2 it follows that $\mathrm{t}(\Gamma, C) < \mathrm{t}(\Gamma, A)$ which contradicts to the minimality of $\mathrm{t}(\Gamma, A)$, hence, corollary 1.2 is proven.

7. Representability of relatively free algebras

In this section we prove theorem 1.1; suppose that it is untrue, then there exists a T-ideal Γ and a finite dimensional algebra A such that $T(A) \subset \Gamma$ and $\mathrm{t}(\Gamma, A)$ is the smallest one for such pairs. Put $\Gamma' = \Gamma + T(B)$, where B is the algebra from proposition 1.6, then there exists a finite dimensional algebra C such that $T(C) = \Gamma'$. Indeed,

$$\mathcal{S}_{a,b,r}(\Gamma') \subseteq \mathcal{S}_{a,b,r}(\Gamma) \subseteq T(B) \ ,$$

hence, by proposition 1.2 we have $\mathrm{t}(\Gamma, B) < (a, b) = \mathrm{t}(\Gamma, A)$, hence $\mathrm{t}(\Gamma', B) < \mathrm{t}(\Gamma, A)$ and, by the induction hypothesis, $T(C) = \Gamma'$ for some finite dimensional algebra C.

Since $\mathrm{t}(\Gamma, A) < (a, b+1)$, there exists $l \in \mathbb{N}$ such that $\mathcal{S}_{a,b+1,l}(\Gamma) \subseteq T(A)$. Put $d = (a+1)(b+1) + a \cdot l$ and let $Q = \mathcal{K}(A, Y, T(A), d)$; by lemma 1.4, $T(A) \subseteq T(Q)$ and $A \subseteq Q$, hence $T(A) = T(Q)$. Consider the direct sum of algebras $P = Q \oplus C$; notice $\mathsf{P}(Q) \oplus \mathsf{P}(C)$ is a maximal semisimple subalgebra, so we denote it by $\mathsf{P}(P)$. Let $\mathrm{tr}_Q(x)$ $(\mathrm{tr}_C(x))$ denote the trace of the linear operator $R_{p(x)} \in \mathrm{End}(\mathsf{P}(Q))$ $(\mathrm{End}(\mathsf{P}(C))$, respectively), where $p(x)$ is the projection onto the semisimple part; define a linear mapping $\mathrm{tr} : P \to \mathcal{C}(P)$ letting

$$\mathrm{tr}(x \oplus y) = \mathrm{tr}_Q(x) \oplus \mathrm{tr}_C(y) \in F \cdot \mathrm{id}_P \oplus F \cdot \mathrm{id}_C \ .$$

Consider the algebra $\Psi \subseteq \mathrm{Pol}(\vec{P}, \mathcal{C}(P))$ generated by $\mathrm{tr}(u)$, where $u \in \mathcal{F}(P)$, then, like in lemma 1.9, one can show that $\mathcal{F}(P)$ is integral over Ψ. Let I denote the Ψ-submodule generated by $\bar{\Gamma} = \Gamma/T(P) \subseteq \mathcal{F}(P)$ in $\Psi \cdot \mathcal{F}(P)$.

LEMMA 1.12. $I \cap \mathcal{F}(P) = \bar{\Gamma}$.

Proof. Obviously, $\bar{\Gamma} \subseteq I \cap \mathcal{F}(P)$. Conversely, take a multilinear polynomial $f \in I \cap \mathcal{F}(P)$, it belongs to the Ψ-submodule I' generated by $\bar{\Gamma}' = \Gamma'/T(P) \subseteq \mathcal{F}(P)$. Observe that $I' \cap \mathcal{F}(P) = \bar{\Gamma}'$; indeed, $\bar{\Gamma}' \subseteq I' \cap \mathcal{F}(P)$ and, conversely, $I' \cap \mathcal{F}(P)$ consists of functions which vanish under substitution of elements in C in place of variables, hence $I' \cap \mathcal{F}(P) \subseteq T(C)/T(P) = \bar{\Gamma}'$. So, $f = g + h$, where $g \in \Gamma$ and

$$h \in T(B) = \bigcap_{\xi \in \Xi} T(B^{\xi}) \ . \tag{1.25}$$

On the other hand, $f = \sum_{p=1}^{m} f_i \phi_i$, where $f_i \in \bar{\Gamma}$ and $\phi_i \in \Psi$. Hence,

$$h = -g + \sum_{i=1}^{m} f_i \phi_i \ . \tag{1.26}$$

Since Γ and Ψ are homogeneous, we may assume that all terms are multilinear,

$$\mathrm{varb}(h) = \mathrm{varb}(g) = \mathrm{varb}(f_i) \cup \mathrm{varb}(\phi_i) \tag{1.27}$$

and $\mathrm{varb}(f_i) \cap \mathrm{varb}(\phi_i) = \emptyset$ for all $i \in \{1, \ldots, m\}$. We claim that that $h \in T(A)$.

On the contrary, suppose that $\mathsf{V}(h, A)$ is not empty; if $\mathrm{t}(h, A) \leq t(\Gamma, A)$, then, by proposition 1.3 $h \notin \bigcap_{\xi \in \Xi} T(B^{\xi})$ which contradicts (1.25). So, $\mathrm{t}(h, A) > \mathrm{t}(\Gamma, A) = (a, b)$, hence, $\mathrm{t}(h, A) \geq (a, b+1)$. By proposition 1.2, $\mathcal{S}_{a,b+1,l}(\Upsilon) \nsubseteq T(A)$, where $\Upsilon = (h)_{\mathrm{T}}$. Take

$$h' = \mathcal{S}_{a,b+1,l}(f')$$

for some multilinear $f' \in \Upsilon$ and let $\bar{\mathbf{x}} \in \mathsf{V}(h', A)$.

The order of the set $(\bigcup_{i=1}^{b+1} \Lambda_i) \cup (\bigcup_{j=1}^{r} \Theta_j)$ is equal to $(b+1)(a+1) + a \cdot l = |Y|$, so we can identify it with Y. Consider a homomorphsim $\psi : F\langle Y \rangle \to A$ which takes every $x \in Y$ to $\bar{x}$; it extends to a homomorphism π from $A * F\langle Y \rangle$ to A with $\pi|_A = \mathrm{id}_A$. For each $x \in \mathrm{varb}(g)$ we define $\widetilde{x} \in A * F\langle Y \rangle$ in the following way

$$\widetilde{x} = \begin{cases} x & , \text{if } x \in Y, \\ \bar{x} & , \text{otherwise} \ . \end{cases}$$

Observe that $\pi(h'(\tilde{x})) = h'(\pi(\tilde{x})) = h'(\bar{x}) \neq 0$; hence, $\widetilde{h}' = h'(\widetilde{x}) \neq 0$.

Also, since h' is multilinear, $\widetilde{h}'$ is homogeneous with respect to Y of total degree d, hence $\widetilde{h}' \notin V$ (in notation of lemma 1.4) and therefore $h'(\widehat{\mathbf{x}}) \neq 0$, where $\widehat{x} = \widetilde{x}/V \in Q$. On the other hand, in the algebra $\Psi \cdot \mathcal{F}(P)$ we have

$$h' = -g' + \sum_{n=1}^{m} f'_n \phi'_n$$

where $g', f'_n \in \bar{\Gamma}$, $\phi'_n \in \Psi$; also, we may assume that g', f'_n, ϕ'_n are multilinear and

$$\mathrm{varb}(h') = \mathrm{varb}(g') = \mathrm{varb}(f'_n \phi'_n)$$

for each n. Moreover, since h' is skew-symmetric with respect to all Λ_i and Ω_j, so is g' and $f'_n \phi'_n$, for just apply the operator $\frac{1}{|G|}\mathcal{S}_G$ (see (1.15)) to both sides.

Next, we claim that $g'(\widehat{x}) = (f'_n h'_n)(\widehat{x}) = 0$ for all n. Indeed, $g' \in \mathcal{S}_{a,b+1,r}(\Gamma) \subseteq T(A) = T(Q)$; the same holds for f'_n, if $Y \subseteq \mathrm{varb}(f_n)$. So, there is $x \in Y \cap \mathrm{varb}(\phi'_n)$, by definition $\widehat{x} \in \mathsf{R}(Q)$, hence, $\widehat{\phi}'_i = 0$. Thus, $h'(\widehat{x}) = 0$ which is impossible, hence $h \in T(A)/T(R) \subseteq \bar{\Gamma}$ and therefore $f = g + h \in \bar{\Gamma}$. Lemma 1.12 is proven. $\square$

By lemma 1.8, $\bar{\Gamma} = I \cap \mathcal{F}(Q)$ is equal to $T(D)/T(Q)$, where D is a finite dimensional algebra. Thus, $\Gamma = T(D)$ and theorem 1.1 is proven. $\square$

CHAPTER 2

Identities of Lie algebra representations

The rest of these notes is devoted to the problem of finite basis of identities in finite dimensional Lie algebras and their representations over a field of characteristic zero. In this chapter we formulate the theorem and obtain some preliminary results. In particular, we introduce the class of generalized Lie modules, where the theorem corresponds to a more general fact; then we give the notion of "counter-example" and define the type of a "counter-example.

1. Theorem

Let L be a Lie algebra over a field F of characteristic 0 and let $\rho : L \to \mathrm{End}(V)$ be a Lie representation. An associative polynomial $f = f(x_1, \ldots, x_n) \in \mathrm{A}\langle X\rangle$ is said to be an identity of the representation ρ, if $f(\rho(a_1), \ldots, \rho(a_n)) = 0$ for all $a_i \in L$. Obviously, we may assume that ρ is faithful, hence, the definition can be rewritten in terms of Lie pairs as follows (cf. [**33**, chap.1]).

Let A be an envelope of a Lie algebra L, then $\mathcal{P} = (A, L)$ is called *a Lie pair*; the class of Lie pairs over F is a variety of 2-base algebraic systems. By Poincaré-Birkhoff-Witt's theorem, $\mathrm{A}\langle X\rangle^{\#}$ is the universal envelope of the free Lie algebra $\mathrm{L}\langle X\rangle$, hence $(\mathrm{A}\langle X\rangle^{\#}, \mathrm{L}\langle X\rangle)$ is a free Lie pair $\mathcal{P}\langle X\rangle$. An identity of $\mathcal{P}$ is a polynomial $f(x_1, x_2, ..., x_n) \in \mathrm{A}\langle X\rangle$ such that $f(a_1, a_2, ..., a_n) = 0$ for all $a_i \in L$, where the products are taken in A. The set of all identities of $\mathcal{A}$ is an ideal $T(\mathcal{P})$ of $\mathrm{A}\langle X\rangle^{\#}$ closed under replacing variables with Lie polynomials. In other words, $T(\mathcal{P})$ is a fully characteristic ideal of $\mathcal{P}\langle X\rangle$; following Rasmyslov's terminology [**33**], we call it a *weak T-ideal.*

The rest is devoted to the proof of the following theorem.

THEOREM 2.1. *Let $\mathcal{P} = (A, L)$ be a Lie pair, where L is a finitely generated Lie algebra and A is an envelope of L satisfying a polynomial identity (i.e., A is a PI algebra). Then, $T(\mathcal{P})$ is generated as a weak T-ideal by a finite number of polynomials.*

In terms of Lie representations the theorem says that a PI representation of a finitely generated Lie algebra has a finite basis of identities, cf. [**4**, p.182].

Recall that the adjoint representation $\mathrm{ad} : L \to \mathrm{End}(L)$ takes an element a to the operator of right multiplication $\mathrm{ad}(a) : x \to [x, a]$ (we use this notation to distinguish it from the operator of right multiplication $R_a : x \to xa$ in its envelope).

COROLLARY 2.1. *Let L be a finitely generated Lie algebra such that the envelope of* $\mathrm{ad}(L)$ *in* $\mathrm{End}(L)$ *is PI, then $T(L)$ is a finitely generated T-ideal in the free Lie algebra* $\mathrm{L}\langle X\rangle$.

Proof. Consider a Lie pair $\mathcal{P} = (A, \mathrm{ad}(L))$ where A is the envelope of $\mathrm{ad}(L)$ in $\mathrm{End}(L)$. Obviously, it satisfies the hypothesis of the theorem, hence, the weak T-ideal $T(\mathcal{P})$ has a finite set of generators $f_1, f_2, ..., f_n$. Let $y \in X$ such that $f_1, f_2, ..., f_n$ do not depend on y, then the set of Lie polynomials

$$S = \{ \ y f_i(\mathrm{ad}(x_1), \mathrm{ad}(x_2), ..., \mathrm{ad}(x_m)) \mid f_i = f_i(x_1, x_2, ..., x_m), i = 1, ..., n \ \}$$

generates the T-ideal $T(L)$. Indeed, $T(L)$ is generated by multilinear polynomials and we may assume that they depend on y; pick such a polynomial, it can be written in the form, cf. [**5**, p.71],

$$y \cdot h(\mathrm{ad}(x_1), \mathrm{ad}(x_2), ..., \mathrm{ad}(x_k)) \ ,$$

where h is an associative polynomial. Obviously, h is an identity of $\mathcal{P}$, hence we can replace it by $E_1 f_i(\mathrm{ad}(v_1), \mathrm{ad}(v_2), ..., \mathrm{ad}(v_m))E_2$, where v_i are Lie polynomials and E_1, E_2 are products of adjoint operators. Obviously,

$$y \cdot E_1 f_i(\mathrm{ad}(v_1), \mathrm{ad}(v_2), ..., \mathrm{ad}(v_m))E_2$$

belongs to the T-ideal generated by $y \cdot f_i(\mathrm{ad}(x_1), \mathrm{ad}(x_2), ..., \mathrm{ad}(x_m))$. $\square$

Since any finite dimensional algebra is PI (it satisfies a Capelli identity), we get immediately an analog of Kemer's theorem about finite basis of identities in finite dimensional Lie algebras.

COROLLARY 2.2. *The ideal of identities of a finite dimensional Lie algebra is a finitely generated T-ideal.*

Since F is infinite, every weak T-ideal is homogeneous by lemma 1.2, hence in the hypothesis of theorem 2.1 we can assume that F is algebraically closed. Indeed, let H be is a field extension of F, then the free associative algebra over H is isomorphic to $H \otimes_F \mathrm{A}\langle X \rangle$. Consider the scalar extension $\mathcal{P}_H = (A_H, L_H)$, where $A_H = H \otimes_F A$ and $L_H = H \otimes_F L$. Then $T(\mathcal{P}_H) = H \otimes_F T(\mathcal{P})$ and $T(\mathcal{P}_H)$ is a finitely generated weak T-ideal iff so is $T(\mathcal{P})$.

2. Capelli polynomials

Recall that the T-ideal of Capelli identities of rank n, Cap_n, is generated by multilinear polynomials that are skew-symmetric with respect to n variables, see (1.4).

PROPOSITION 2.1. Cap_n *is a finitely generated weak T-ideal.*

Proof. Denote by $\mathrm{Cap}_{n,k}$ the weak T-ideal generated by polynomials

$$g \ = \ f(u_1, u_2, ..., u_n, ...), \tag{2.1}$$

where $f(x_1, x_2, ..., x_n, ...)$ is multilinear and skew-symmetric with respect to $\{x_1, x_2, ..., x_n\}$ and $u_1, u_2, ..., u_n$ are associative polynomials, $\deg u_i \le k$ for all $i \in \{1, \dots, n\}$. Then $\mathrm{Cap}_n = \bigcup_{k \ge 1} \mathrm{Cap}_{n,k}$.

First, let us show that $\mathrm{Cap}_{n,k}$ is a finitely generated weak T-ideal for all k; in fact, we follow Rasmyslov's proof, see [**5**, p.225]. Obviously, as a weak T-ideal, $\mathrm{Cap}_{n,k}$ is generated by polynomials

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(u_{\sigma(1)}E_1)\dots(u_{\sigma(n-1)}E_{n-1})(u_{\sigma(n)}E_n) \ , \tag{2.2}$$

where u_i are associative monomials, $\deg(u_i) \le k$, and E_j are products of adjoint operators (E_j may be equal to 1). We claim that all polynomials of this form, where $\deg(E_j) \le n$, generate $\mathrm{Cap}_{n,k}$ as a weak T-ideal. Notice that their degree is bounded, hence it will yield the required statement.

Let I denote the weak T-ideal generated by polynomials (2.2); clearly, $I \subseteq \mathrm{Cap}_{n,k}$. To prove the reciprocal inclusion consider a tensor product of vector spaces over F

$$V = \mathrm{A}\langle X\rangle^{\otimes n} \otimes B^{\otimes n}$$

where B denotes the envelope of $\mathrm{ad}\,(\mathrm{L}\langle X\rangle)$ in the algebra of endomorphisms of the vector space $\mathrm{A}\langle X\rangle$. Also, define a linear mapping $g : V \to \mathrm{A}\langle X\rangle^{\#}$ letting

$$g(u_1 \otimes \ldots \otimes u_n \otimes E_1 \otimes \ldots \otimes E_n)$$

be equal to element (2.2), where u_i, E_j are elements from the corresponding tensor factors of V; obviously, the image $g(V)$ generates $\mathrm{Cap}_{n,k}$ as a weak T-ideal.

Denote by W_m the span of elements

$$w = u_1 \otimes \ldots \otimes u_n \otimes E_1 \otimes \ldots \otimes E_n \ , \tag{2.3}$$

where u_i is an associative product of $\leq k$ Lie polynomials and the operator length of E_j is $\leq m$. By definition, the ideal I is generated by $g(W_n)$, and we need to show that $g(W_m) \subseteq I$ for all $m > n$. We shall carry out the induction on m; the initial step, $m = n$, is trivial, suppose that it holds for $m = s$, let us prove it for $m = s + 1$.

Let S denote the algebra of commutative polynomials over F in variables $t_1, t_2, ..., t_n$. Fix a free generator $y \in X$ and define a representation of S in $\mathrm{End}(V)$ setting

$$(u_1 \otimes \ldots \otimes u_n \otimes E_1 \otimes \ldots \otimes E_n) * t_j =$$
$$= (u_1 \otimes \ldots \otimes u_n \otimes E_1 \otimes \ldots \otimes E_n)|_{E_j = \mathrm{ad}(y)E_j} \ .$$

Lemma 2.1. *Suppose that ϕ is a symmetric polynomial in S, then $g(W_m * \phi) \subseteq g(W_m)$ for all $m \in \mathbb{N}$.*

Proof. Take a generator $w \in W_m$ of form (2.3), then

$$w * t_1^{q_1} t_2^{q_2} ... t_n^{q_n} = w|_{\forall i \ E_i = \mathrm{ad}(y)^{q_i} E_i} =$$
$$= (u_1 \otimes \ldots \otimes u_n \otimes \mathrm{ad}(y)^{q_1} E_1 \otimes \ldots \otimes \mathrm{ad}(y)^{q_n} E_n) \ .$$

Hence,

$$g(w * \sum_{\tau \in S_n} t_{\tau(1)}^{q_1} t_{\tau(2)}^{q_2} ... t_{\tau(n)}^{q_n}) = g(w * \sum_{\tau \in S_n} t_1^{q_{\tau(1)}} t_2^{q_{\tau(2)}} ... t_n^{q_{\tau(n)}}) =$$
$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \sum_{\tau \in S_n} (u_{\sigma(1)} \mathrm{ad}(y)^{q_{\tau\sigma^{-1}\sigma(1)}} E_1) \ldots (u_{\sigma(n)} \mathrm{ad}(y)^{q_{\tau\sigma^{-1}\sigma(n)}} E_n) =$$
$$= \sum_{\tau \in S_n} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) (u_{\sigma(1)} \mathrm{ad}(y)^{q_{\tau\sigma(1)}} E_1) \ldots (u_{\sigma(n)} \mathrm{ad}(y)^{q_{\tau\sigma(n)}} E_n) = \sum_{\tau \in \S_n} f_\tau,$$

where

$$f_\tau = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) (v_{\sigma(1)}^\tau E_1) \ldots (v_{\sigma(n)}^\tau E_n)$$

with $v_j^\tau = u_j \mathrm{ad}(y)^{\tau(j)}$ for all j. Since v_j is a linear combination of associative products of $\leq k$ Lie polynomials, we have $f_\tau \in g(W_m)$ and therefore $g(w * \phi) \in g(W_m)$. □

Now take $w \in W_{s+1}$ of form (2.3); notice that a transposition of operators in E_j reduces the operator length of E_j, hence, we may assume that $E_j = \mathrm{ad}(y)^n E_j'$ for some

$y \in X$. Hence $w = w' * t_j^n$, where w' is of form (2.3) with E_j replaced by E'_j, in particular, $w' \in W_{s-n+1}$.

Next, t_j is the root of the polynomial

$$t^n + t^{n-1}\phi_1 + \ldots + \phi_{n-1}t + \phi_n = \prod_{i=1}^{n}(t - t_i)$$

where ϕ_r are symmetric polynomials. Hence, by the induction hypothesis and the lemma above for all $r \geq 1$

$$g(w' * (t_j^{n-r}\phi_r)) = g((w' * t_j^{n-r}) * \phi_r) \in g(W_{s+1-r} * \phi_r) \subseteq g(W_{s+1-r}) \subseteq g(W_s) \subseteq I$$

and

$$g(w) = g(w' * t_j^n) = -\sum_{r=1}^{n} g(w' * (t_j^{n-r}\phi_r)) \in I$$

as required.

In a similar fashion one can show that $\mathrm{Cap}_n = \mathrm{Cap}_{n,n}$; actually, Cap_n is generated by

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) v_0 u_{\sigma(1)} v_1 \ldots u_{\sigma(n-1)} v_{n-1} u_{\sigma(n)} v_n \ ,$$

where u_j, v_j are associative polynomials; we need to prove that Cap_n is generated by such polynomials where $\deg(u_j) \leq n$. Suppose that $\deg(u_j) > n$, then we can assume that $u_j = u'_j y^n$. Distributing "uniformly" powers of y on $v_1, \ldots, v_n$, we reduce the power of y in u_j. $\square$

LEMMA 2.2. *Theorem 2.1 follows from a.c.c. for weak T-ideals containing the ideal of identities of a finite dimensional Lie pair.*

Proof. By [**18**] there is a natural n such that $\mathrm{Cap}_n \subseteq T(A) \subseteq T(\mathcal{A})$. Also, from [**24**] and nilpotency of the radical of a f.g. PI algebra [**33, 18, 8**] it follows that there exists a f.d. algebra B such that the ideal of identities $T(B)$ is contained in Cap_n. Obviously, $T(B)$ is equal to the ideal of identities of the Lie pair $\mathcal{B} = (B, B^{(-)})$. By hypothesis, there are $f_1, \ldots, f_m \in T(\mathcal{A})$ which generate $T(\mathcal{A})$ modulo $T(\mathcal{B})$ as a weak T-ideal. Also, let $g_1, \ldots, g_k$ be generators of the weak T-ideal Cap_n, then $f_1, \ldots, f_n, g_1, \ldots, g_k$ generate $T(\mathcal{A})$. $\square$

3. Generalized Lie modules

Generalized Lie representation. The notion of representation of an algebraic system (ring, algebra, group) is related to some action of this system. Thus, a representation μ of a Lie algebra L means that an element a from L acts as a linear operator $\mu(a) = \mu_a$ on a vector space V, and μ is a homomorphism from L to the Lie algebra $\mathrm{End}(V)^{(-)}$. It induces the dual representation ρ^* of L in the endomorphism algebra of the space of linear forms V^*, where for $f = f(v) \in V^*$ and $a \in L$ we put $(f \cdot \mu_a^*)(v) = -f(v \cdot \mu_a)$. Moreover, we can replace V^* with the space of linear mappings $\mathrm{Hom}(V, W)$ from V to another vector space W.

Now suppose that μ is the adjoint representation $\mathrm{ad}(x)$, then the induced representation $\rho = \mu^*$ in $\mathrm{Hom}(L, W)$ has the form

$$f \cdot \rho(a) = f|_{x=[a,x]}$$

In this situation there is some additional type of action of L. Namely, fixing the value of the variable, we get an element from W which can be regarded as a constant function on W,

$$f \cdot \delta(a) = f|_{x=a} \ .$$

The resulting element is not in $\mathrm{Hom}_F(L, W)$ which forces us to consider the sum $V = \mathrm{Hom}(L, W) + W$; clearly, it is direct. We can extend trivially $\rho(a)$, $\delta(a)$ to V and get finally the following definition:

$$f \cdot \rho(a) = \begin{cases} f|_{x=[a,x]} & \text{,if } f \in \mathrm{Hom}_F(L, W), \\ 0 & \text{,if } f \in W \ ; \end{cases} \tag{2.4}$$

$$f \cdot \delta(a) = \begin{cases} f|_{x=a} & \text{,if } f \in \mathrm{Hom}_F(L, W), \\ 0 & \text{if } f \in W. \end{cases} \tag{2.5}$$

One can easily check the following relations: for all $a, b \in L$

$$\rho(a)\delta(b) = \delta([ab]) \ , \tag{2.6}$$

$$\delta(a)\rho(b) = \delta(a)\delta(b) = 0 \ . \tag{2.7}$$

Consider a more general situation; take the direct sum $kL = L \oplus \ldots \oplus L$ and let V be the subspace of $\mathrm{Pol}(kL, W)$ spanned by all homogeneous polynomials of degree 0 or 1 with respect to every generic element x_i of L. Fix x_i and for all $a \in L$ define $\rho_i(a), \delta_i(a) \in \mathrm{End}(V)$ setting for all $f \in V$

$$f \cdot \rho_i(a) = \begin{cases} f|_{x_i=[a,x_i]} & \text{,if } \deg_{x_i} f = 1, \\ 0 & \text{,if } \deg_{x_i} f = 0, \end{cases} \tag{2.8}$$

$$f \cdot \delta_i(a) = \begin{cases} f|_{x_i=a} & \text{,if } \deg_{x_i} f = 1, \\ 0 & \text{,if } \deg_{x_i} f = 0. \end{cases} \tag{2.9}$$

We can extend this construction to a relatively free Lie algebra $\mathrm{L}_{\mathfrak{M}}\langle X\rangle$ in the following way. First, enlarge the set of free variables setting $X' = X \cup \{x_1, \ldots, x_k\}$ and let V be the vector space of Lie polynomials from $\mathrm{L}_{\mathfrak{M}}\langle X'\rangle$ of degree ≤ 1 with respect to each x_i; in particular, V includes $\mathrm{L}_{\mathfrak{M}}\langle X\rangle$.

Every element $f = f(x_1, \ldots, x_k)$ from V can be regarded as a mapping from $k\mathrm{L}_{\mathfrak{M}}\langle X\rangle$ to $\mathrm{L}_{\mathfrak{M}}\langle X\rangle$:

$$(a_1, \ldots, a_k) \to f(a_1, \ldots, a_k) \ ,$$

hence, (2.8), (2.9) define a similar representation of $\mathrm{L}_{\mathfrak{M}}\langle X\rangle$ in $\mathrm{End}(V)$.

We have the following relations between ρ_i, δ_j. For all $a, b \in L$

$$[\rho_i(a), \rho_i(b)] = \rho_i([a, b]) \ , \tag{2.10}$$

i.e. , ρ_i is an ordinary Lie algebra representation,

$$\rho_i(a)\delta_i(b) = \delta_i([a, b]) \ \ , \ \ \delta_i(b)\rho_i(a) = 0 \ \ , \ \ \delta_i(a)\delta_i(b) = 0 \ ; \tag{2.11}$$

also, if $i \neq j$, then

$$[\rho_i(a), \rho_j(b)] = 0 \ \ , \ \ [\rho_i(a), \delta_j(b)] = 0 \ \ , \ \ [\delta_i(a), \delta_j(b)] = 0. \tag{2.12}$$

A *generalized Lie (g.Lie) representation of rank* n of a Lie algebra L in an associative algebra A ($1 \in A$) is an n-tuple $\nu = (\nu_1, \dots, \nu_n)$, where ν_i is a pair (ρ_i, δ_i) of linear mappings from L to A and relations (2.10)-(2.12) hold for all $a, b \in L$.

We have $\mathrm{Ker}(\rho_j) \triangleleft L$, because ρ_j is a representation of L; also from (2.11) it follows that $\mathrm{Ker}(\delta_j) \triangleleft L$, hence, we have the following statement.

LEMMA 2.3. *The kernel of a g.Lie representation* $\nu : L \to A$,

$$\mathrm{Ker}(\nu) = \bigcap_{j=1}^{s} (\mathrm{Ker}(\rho_j) \cap \mathrm{Ker}(\delta_j) \ ,$$

is an ideal of L.

Put $\nu_i(L) = \rho_i(L) + \delta_i(L)$, then by (2.10), (2.11) it is a Lie subalgebra of $A^{(-)}$, where $\delta_i(L)$ is a trivial ideal of $\nu_i(L)$ (its square is equal to 0). Moreover, $\nu_i(L)$, $\nu_j(L)$ commute for different i, j, hence

$$\nu(L) = \sum_{j=1}^{s} \nu_j(L) \tag{2.13}$$

is a Lie algebra. The associative subalgebra of A generated by $\nu(L)$ and $1 \in A$ is called a *generalized (g.) envelope* and denoted by $\mathsf{En}(L, \nu)$.

Fix $n \in \mathbb{N}$ and a Lie algebra L; a g.Lie representation $\nu : L \to A$, $\nu = (\nu_1, \dots, \nu_n)$ is said to be *universal* one, if for all g.Lie representation $\nu' : L \to A'$ there exists an algebra homomorphism $\psi : A \to A'$ such that ν' is equal to the composition ν and ψ. We may assume that A is generated $\nu(L)$ and $1 \in A$; one can easily check that a universal representation exists and unique up to an isomorphism; the algebra A is called the universal generalized (g.) envelope and denoted by $\mathsf{U}(L, n)$.

Generalized Lie modules. It would be natural to give this name to a vector space V with a g.Lie representation of L in $\mathrm{End}(V)$. However, in the proof of the theorem we will use it for a slightly different object; namely, we fix several vectors in V and consider them as additional operations. Therefore, a g.Lie module is defined as an ordered set

$$\mathcal{A} = (L; V; \nu; \mathrm{v}_1, \dots, \mathrm{v}_r) \tag{2.14}$$

where L is a Lie algebra, V is a vector space, $\nu = (\nu_1, \dots, \nu_s)$ is a g. Lie representation of L in $\mathrm{End}(V)$ and $\mathrm{v}_1, \dots, \mathrm{v}_r \in V$ are fixed vectors in V; we will use often a shorter notation, e.g. $\mathcal{A} = (L; V)$.

Let $\mathfrak{M}$ denote the class of g.Lie modules (2.14) over F, where the class of the g.Lie representation s and the number of the fixed vectors r are fixed; a g. Lie module $\mathcal{A} = (L; V) \in \mathfrak{M}$ can be regarded as a 2-base algebraic system, where the operations are

1) the operations of the Lie algebra L;
2) $\sum_{i=1}^{r} \mathrm{v}_i E_i$, where E_i are linear combinations of products of $\rho_j(a)$, $\delta_l(b)$, where a, b are Lie polinomials from $\mathrm{L}\langle X \rangle$,

satisfying all identities following from the linearity of ρ_j, δ_l and identities (2.10)-(2.12). Obviously, we may assume that V is generated by $\mathrm{v}_1, \dots, \mathrm{v}_r$ as a module over $\mathsf{En}(L, \nu) \subseteq \mathrm{End}(V)$; also, put $\mathsf{En}(\mathcal{A}) = \mathsf{En}(L, \nu)$.

A homomorphism from $\mathcal{A} = (L; V)$ to $\mathcal{A}' = (L', V')$ is a pair (ϕ, ψ), where $\phi : L \to L'$ is a Lie algebra homomorphism and ψ is a linear mapping from V to V' taking v_j to v'_j and

$$\psi(v \cdot \rho_j(a)) = \psi(v) \cdot \rho_j(\phi(a)) \ ; \quad \psi(v \cdot \delta_j(a)) = \psi(v) \cdot \delta_j(\phi(a))$$

for all $v \in V$, $a \in L$ and $j \in \{1, \dots , s\}$.

A submodule $\mathcal{B} = (M; W)$ of $\mathcal{A}$ consists of a Lie subalgebra $M \subseteq L$ and a subspace $W \subseteq V$ generated as a module over $\mathsf{En}(M, \nu) \subseteq \mathsf{En}(\mathcal{A})$ by $\mathrm{v}_1, \dots , \mathrm{v}_n$. Let $\mathcal{A}_i = (L_i; V)$, $i \in I$, be a family of g.Lie modules, then the direct product $\prod_{i\in I} \mathcal{A}_i$ is equal to $(\prod_{i\in I} L_i; \prod_{i\in I} V_i)$, where the j-th fixed vector is the product of the corresponding vectors from V_i. Let X be a set of variables; the free g. Lie module is $\mathcal{M}\langle X\rangle = (\mathrm{L}\langle X\rangle; V\langle X\rangle)$, where $\mathrm{L}\langle X\rangle$ is a free Lie algebra and $V\langle X\rangle$ is a free module over $\mathsf{U}(\mathrm{L}\langle X\rangle, s)$ generated by $\mathrm{v}_1, \dots , \mathrm{v}_r$.

In fact, we need to modify the notion of g.Lie module: we have to replace an ordinary Lie algebra with k-sorted one.

Generalized Lie modules over k-sorted Lie algebras.

Let L be a Lie algebra which is a sum of fixed ideals $L_1, \dots , L_k \lhd L$, then L is called a *k-sorted Lie algebra*. Formally speaking, it is a $k + 1$-tuple $\mathbf{L} = (L; L_1, \dots , L_k)$; we consider it as $k + 1$-base algebraic system, cf. [**33**], where all operations are defined on $L_{i_1} \times \ldots \times L_{i_r}$, where $i_1, \dots , i_r \in \{1, \dots , k\}$ and take value either in some L_i or in L. Operations of the first type include all Lie algebra operations on L_i and action of L_j on L_i by derivations; operations of the second type are defined by Lie polynomials in many variables. One can write down identities which enable us to identify each L_i with an ideal of L; so the class of k-sorted Lie algebras over F form a variety of algebraic systems. In case $k = 1$ we get ordinary Lie algebras.

The expression $x \in \mathbf{L}$ will mean that x belongs to one of L_i.

Let $\mathbf{L}, \mathbf{L}'$ be k-sorted Lie algebras; a homomorphism from $\mathbf{L}$ to $\mathbf{L}'$ is Lie algebra homomorphism $\pi : L \to L'$ taking every L_i to L'_i. A k-sorted subalgebra of $\mathbf{L}$ is a subalgebra M of L which is the sum of ideals $M_i \subseteq M \cap L_i$, $i \in \{1, \dots , k\}$. Take a k-sorted subset $S = \cup_{i=1}^k S_i$, $S_i \subseteq L_i$; then the k-sorted subalgebra generated by S is equal to $(M; M_1, \dots , M_k)$, where M is a subalgebra of L generated by S and M_i is the ideal of M generated by S_i.

Let X be a disjoint union of infinite sets of variables, $X = \bigcup_{i=1}^k X_i$, and let $\mathrm{L}_i\langle X\rangle$ denote the ideal of $\mathrm{L}\langle X\rangle$ generated by X_i. Then it defines a k-sorting on $\mathrm{L}\langle X\rangle$; moreover, every mapping $\pi : X \to L$, which takes X_i to L_i, extends to a homomorphism of k-sorted Lie algebras. Thus, $(\mathrm{L}\langle X\rangle; \mathrm{L}_1\langle X\rangle, \dots , \mathrm{L}_k\langle X\rangle)$ is the free k-sorted Lie algebra $\mathbf{L}\langle X\rangle$.

A generalized Lie representation of rank s of a k-sorted Lie algebra $\mathbf{L} = (L; L_1, \dots , L_k)$ in an associative algebra A $(1 \in A)$ is that of L; put $\mathsf{En}(\mathbf{L}, \nu) = \mathsf{En}(L, \nu)$. Finally, a generalized Lie module $\mathcal{A}$ is

$$(\mathbf{L}; V) = (\mathbf{L}; V; \nu; \mathrm{v}_1, \dots , \mathrm{v}_r)$$

where $\mathbf{L}$ is a k-sorted Lie algebra, V is a vector space, ν is a g.Lie representation of $\mathbf{L}$ in $\mathrm{End}(V)$ and $\mathrm{v}_1, \dots , \mathrm{v}_r$ are fixed vectors from V.

Cleraly, we can replace the ground field F with an arbitrary associative commutative ring K and get g.Lie modules over K.

Like in the case of ordinary Lie algebras (i.e., 1-sorted Lie algebras) we consider a g.Lie module as an algebraic system; so we assume that V is generated by $v_1, \dots, v_r$ over $\mathsf{En}(\mathcal{A}) = \mathsf{En}(L, \nu) \subseteq \mathrm{End}(V)$.

Let $\mathfrak{M}_{k,s,r}$ denote the class of g.Lie modules over F with k, s, r fixed; clearly, it is a variety and all basic algebraic constructions extend automatically to this case. In partucular, the free g.Lie module is

$$\mathcal{M}\langle X\rangle = (\mathbf{L}\langle X\rangle; V\langle X\rangle) . \tag{2.15}$$

where $V\langle X\rangle$ is a free module over $\mathsf{U}(\mathbf{L}\langle X\rangle, s) = \mathsf{U}(\mathrm{L}\langle X\rangle, s)$ generated by $\mathrm{v}_1, \dots, \mathrm{v}_r$.

Clearly, $\mathsf{U}(\mathbf{L}\langle X\rangle; s)$ and, therefore, $V\langle X\rangle$ are homogeneous with respect to every $x \in X$, so $\deg_x(f)$, $\deg_x(E)$ are defined for all $f \in v\langle X\rangle$, $E \in \mathsf{U}(\mathbf{L}\langle X\rangle; s)$ which are homogeneous with respect to x.

Let $\mathcal{A}$ be a g.Lie module from $\mathfrak{M}_{k,s,r}$; a polynomial $f = f(x_1, \dots, x_n) \in V\langle X\rangle$ is said to be *an identity* of $\mathcal{A}$, if it vanishes under any homomorphism from $\mathcal{M}\langle X\rangle$ to $\mathcal{A}$. In other words, f is equal to $0 \in V$ under any substitutions of elements from L in place of variables (note that every variable from X_i must be replaced with an element from L_i).

Let $T(\mathcal{A})$ denote the set of all identities of $\mathcal{A}$; it is a submodule of $V\langle X\rangle$ over $\mathsf{En}(\mathcal{M}\langle X\rangle)$ closed under substitutions of Lie polynomials $g \in \mathrm{L}_i\langle X\rangle$ in place of $x \in X_i$. In other words, $T(\mathcal{A})$ is a fully characteristic submodule of $V\langle X\rangle$, i.e., it is invariant under any g.Lie module endomorphism of $\mathcal{M}\langle X\rangle$. We prefer to call such a submodule *a T-ideal* like in the case of algebras.

Since F is infinite, every T-ideal Γ is homogeneous with respect to every variable $x \in X$; it means that Γ is decomposed into the direct sum of its homogeneous components. It follows from lemma 1.2; recall that the proof is based on the Vandermonde determinant argument employing substitutions of the form $x \to \lambda x$, where $\lambda \in F$, hence, the statement can be rewritten as follows. Let $f \in V\langle X\rangle$ and let g be its homogeneous component with respect to some variables, then there exist endomorphisms $(\phi_1, \psi_1), \dots, (\phi_l, \psi_q)$ of $\mathcal{M}\langle X\rangle$ such that $g = \psi(f)$, where ψ is a linear combination of $\psi_1, \dots, \psi_q$. In other words, the projector onto a homogeneous component of $V\langle X\rangle$ acts on a given element like a linear combination of endomorphisms of $\mathcal{M}\langle X\rangle$, however, this representation is local, it depends on the degree of the element.

Let $\mathrm{E}(X) \subseteq \mathrm{End}(V\langle X\rangle)$ denote the subalgebra generated by the following elements:

(1) endomorphisms of $\mathcal{M}\langle X\rangle$;
(2) projectors onto homogeneous components of $V\langle X\rangle$ with respect to $Y \subseteq X$.

So, a submodule of $V\langle X\rangle$ is a T-ideal iff it is invariant under $E(X)$.

Recall that $V\langle X\rangle$ is the free module over $\mathsf{En}(\mathcal{M}\langle X\rangle) = \mathsf{U}(\mathbf{L}\langle X\rangle, s)$, hence, every endomorphism $\phi : \mathbf{L}\langle X\rangle \to \mathbf{L}\langle X\rangle$ extends to an endomorphism ψ' of $\mathsf{U}(\mathbf{L}\langle X\rangle, s)$ and therefore defines an endomorphism (ϕ, ψ) of $\mathcal{M}\langle X\rangle$: $\psi(\sum_{j=1}^r \mathrm{v}_j \cdot E_j) = \sum_{j=1}^r \mathrm{v}_j \cdot \psi'(E_j)$. Clearly, the correspondence between ψ and ψ' is one-to-one, so it does not make any confusion, if we denote ψ' by ψ. So, if $f = \sum_{j=1}^r \mathrm{v}_j \cdot E_j$ is an arbitrary element in $V\langle X\rangle$, then the action of ψ can be rewritten as

$$\psi(f) = \sum_{j=1}^{r} \mathrm{v}_j \cdot \psi(E_j) . \tag{2.16}$$

Also, the same holds for projectors onto homogeneous components; moreover, it extends to the linear combinations and, in particular, to all $\psi \in E(X)$.

If I is a T-ideal, then it is equal to the ideal of identities $T(\mathcal{A})$, where $\mathcal{A}$ is the quotient $(\mathbf{L}\langle X\rangle, V\langle X\rangle/I)$. Let $\mathcal{A} = (\mathbf{L}; V)$ be a g.Lie module, then elements of $\mathcal{F}(\mathbf{L}; V) = V\langle X\rangle/T(\mathcal{A})$ can be regarded as functions from $L_1 \times \ldots \times L_k$ to V in many variables.

Next, consider the algebra of all mappings from $L_1 \times \ldots \times L_k$ to L in many variables; a mapping from L_i (for some i) to L that takes $a \in L_i$ to $a \in L$ is called a generic element of $\mathbf{L}$ (of type i). Let X_i denote an infinite set of generic elements of type i; also, put $X = \bigcup_{i=1}^{k} X_i$ and let $\mathcal{F}(\mathbf{L})$ denote the subalgebra generated by X. It is equal to the sum of ideals $\mathcal{F}(\mathbf{L})_i$ generated by X_i, so

$$\mathbf{F}(\mathbf{L}) = (\mathcal{F}(\mathbf{L}); \mathcal{F}(\mathbf{L})_1, \ldots, \mathcal{F}(\mathbf{L})_k) \tag{2.17}$$

is a k-sorted Lie algebra called *the algebra of generic elements* of $\mathbf{L}$.

The g.Lie representation of $\mathbf{L}$ in $\mathrm{End}(V)$ induces that of $\mathbf{F}(\mathbf{L})$ in $\mathrm{End}(\mathcal{F}(V, \mathbf{L}))$; it defines a g.Lie module

$$\mathcal{F}(\mathcal{A}) = (\mathbf{F}(\mathbf{L}); \mathcal{F}(\mathbf{L}; V)) \tag{2.18}$$

called *the g.Lie module of generic elements of* $\mathcal{A}$.

Now we are going to state a theorem about T-ideals in $\mathfrak{M}_{k,s,r}$ which yields theorem 2.1. In fact, we consider a subclass of g.Lie modules satisfying some identities allowing us to rearrange elements in $\nu(L_i)$ and $\nu(L_j)$ for $i \neq j$.

Let J be the ideal of $\mathsf{En}(\mathcal{A})$ generated by Lie commutators $[\nu(L_i), \nu(L_j)]$, $\forall i, j \in \{1, \ldots, k\}$, $i \neq j$, then

$$V \cdot J^m = 0 \ . \tag{2.19}$$

for some $m \in \mathbb{N}$; clearly, it is equivalent to a set of identities. By (2.10)-(2.12), $[L_i, L_j]^m \subseteq \mathrm{Ker}(\nu)$; so, if ν is faithful, then

$$[L_i, L_j]^m = 0 \ . \tag{2.20}$$

A g.Lie module $\mathcal{A} = (\mathbf{L}; V)$ is said to be finite dimensional (f.d) if the dimension of $L_1, \ldots, L_k$ and V is finite. Suppose $\mathbf{L}$ satisfies (2.20), then $[L_i, L_j]$ is contained in the solvable radical of $[L, L]$ denoted by $\mathsf{N}(L)$; from (2.10)-(2.12) it follows that the image of every solvable ideal is a solvable ideal of the Lie algebra $\nu(L)$, hence, $\nu(\mathsf{N}(L)) \subseteq \mathsf{N}(\nu(L))$. Thanks to ([**17**, p.50,56]) $\mathsf{N}(\nu(L)$ generates a nilpotent ideal in $\mathsf{En}(\mathcal{A})$, so (2.19) follows from (2.20) in this case. Hereafter every f.d. k-sorted Lie algebra is supposed to satisfy (2.20) for all $i \neq j$.

Let $\mathcal{A} = (\mathbf{L}, V)$ be a g.Lie module; denote by $\mathcal{Q}(\mathcal{A})$ the T-ideal generated by all multilinear $f \in V\langle X\rangle$ satisfying the following condition: for all $i, j \in \{1, \ldots, k\}$, $i \neq j$, and $x \in X_i$, $y \in X_j$, where $\deg_x(f) \neq 0$,

$$f|_{x=[x,y]} \in T(\mathcal{A}) \ . \tag{2.21}$$

In other words, f vanishes under substitution any elements in $\mathbf{L}$ in place of variables, where x is replaced by an element from $\sum_{j\neq i}[L_i, L_j]$. Note that in case $k = 1$ this condition becomes trivial, so we put $\mathcal{Q}(\mathcal{A}) = V\langle X\rangle$.

THEOREM 2.2. *Let $\mathcal{A}$ be a finite dimensional g.Lie module from $\mathfrak{M}_{k,s,r}$ satisfying (2.19). Then any ascending chain of T-ideals*

$$T(\mathcal{A}) \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_q \ \subseteq \ldots \subset \mathcal{Q}(\mathcal{A}) \tag{2.22}$$

becomes stationary.

Let us show that it yields theorem 2.1. Actually, by lemma 2.2 it suffices to show that any ascending chain of weak T-ideals

$$T(\mathcal{B}) \subseteq I_1 \subseteq \ldots \subseteq I_q \subseteq \ldots \tag{2.23}$$

where $\mathcal{B} = (B, M)$ is a finite dimensional Lie pair, becomes stationary.

Define a covariant functor F from the category of Lie pairs to the category of g.Lie modules $\mathfrak{M}_{1,1,1}$. Given Lie pair $\mathcal{A} = (A, L)$ put $\mathsf{F}(\mathcal{A}) = (L; A; (\rho_1, \delta_1); \mathrm{v}_1)$, where v_1 is the identity element of A , $\rho_1(x)$ is the operator of right multiplication $R_x \in \mathrm{End}(A)$ and δ_1 is equal to 0 identically. One can check that any homomorphism $f = (\phi, \psi)$ between two Lie pairs $\mathcal{A}$ and $\mathcal{B}$ induces a homomorphism $\mathsf{F}(f)$ and this mapping preserves the composition. Moreover, since v_1 is the identity element, every homomorphism from $\mathsf{F}(\mathcal{A})$ to $\mathsf{F}(\mathcal{B})$ is induced by a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ and this correspondence is one-to-one.

Let (id, ψ) be the homomorphism of the free g.Lie module $\mathcal{M}\langle X\rangle = (\mathrm{L}\langle X\rangle, V\langle X\rangle)$ to $\mathsf{F}(\mathcal{P}\langle X\rangle)$ induced by the identity mapping of X. Every endomorphism (π, ϕ) of $\mathcal{M}\langle X\rangle$ induces an endomorphism ϕ' of $\mathcal{P}\langle X\rangle$ such that $\psi(\phi(f)) = \phi'(\psi(f))$ for all $f \in V\langle X\rangle$; hence, the full pre-image of (2.23),

$$\psi^{-1}(T(\mathcal{B})) \subseteq \psi^{-1}(I_1) \subseteq \ldots \subseteq \psi^{-1}(I_q) \subseteq , \ldots \tag{2.24}$$

consists of T-ideals of $\mathcal{M}\langle X\rangle$.

Moreover, $\psi^{-1}(T(\mathcal{B}))$ is equal to the ideal of identities of $\mathsf{F}(\mathcal{B})$. Actually, let (π, μ) be a homomorphism from $\mathcal{M}\langle X\rangle$ to $\mathsf{F}(\mathcal{B})$, then the algebra homomorphism $\pi : \mathrm{L}\langle X\rangle \to M$ defines a homomorphism of Lie pairs $(\pi, \tau) : \mathcal{P}\langle X\rangle \to \mathcal{B}$ and, therefore, a homomorphism of g.Lie modules $(\pi, \nu) : \mathsf{F}(\mathcal{P}\langle X\rangle) \to \mathsf{F}(\mathcal{B})$, where μ is equal to the composition of ψ and ν. Hence, $\psi^{-1}(T(\mathcal{B})) \subseteq T(\mathsf{F}(\mathcal{B}))$. Conversely, the composition of (id, ψ) with any homomorphism from $\mathsf{F}(\mathcal{P}\langle X\rangle)$ to $\mathsf{F}(\mathcal{B})$ must take elements from $T(\mathsf{F}(\mathcal{B}))$ to 0, hence $\psi^{-1}(T(\mathcal{B})) = T(\mathsf{F}(\mathcal{B}))$.

By theorem 2.2 the chain (2.24) becomes stationary; since ψ is epic, the original chain (2.23) has the same property. □

4. Canonical decomposition

The Kemer's theorem is based on the splitting theorem for a finite dimensional associative algebra A. Roughly speaking, two numbers, the dimension of the semisimple part $\mathsf{P}(A)$ and the nilpotency index of the radical $\mathsf{R}(A)$, characterizes a "counter-example" $T(A) \subset \Gamma$.

A similar situation should be expected in other classes of algebras, where a suitable decomposition exists. Actually, Kushkulei-Rasmyslov' theorem [**23**] says that identities distinguish non-isomorphic simple algebras (algebraic systems with multilinear operations) over an algebraically closed field of characteristic 0. Thus, we need to decompose properly a finite dimensional Lie algebra into "simple" components.

Levi's theorem provides a decomposition of a finite dimensional algebra L into the direct sum a semisimple algebra D and the solvable radical $\mathsf{S}(L)$ [**17**, p.91]. However, $\mathsf{S}(L)$ cannot play the same role as the nilpotent radical in associative algebras, because its action on finite dimensional modules might be not nilpotent. We can replace $\mathsf{S}(L)$ with the radical of the commutator subalgebra, $\mathsf{N}(L) = [\mathsf{S}(L), L]$, therefore we need to find a "suitable" complement to $\mathsf{S}(L)$.

Besides, in the associative case a finite dimensional irreducible modules over a semisimple algebra are embedded into the regular one, so the structure of the radical as a module over the semisimple part is not essential for the proof. In the case of Lie algebras the situation is absolutely different, a simple algebra has a wide range of non-isomorphic finite dimensional irreducible modules.

Let $\mathbf{L} = (L, L_1, \dots, L_k)$ be a k-sorted finite dimensional Lie algebra over F. Put $\mathsf{S}_i = \mathsf{S}(L_i) = \mathsf{S}(L) \cap L_i$, hence $\mathsf{S}(L) = \sum_{i=1}^k \mathsf{S}_i$. A maximal semisimple subalgebra of L is defined up to an automorphism, let $\mathsf{D} = \mathsf{D}(L)$ denote one of them. Also, denote by $\mathsf{D}_i = \mathsf{D}_i(\mathbf{L})$ a semisimple part of L_i, $\mathsf{D}(L_i)$; by [**17**, p.92] there exists an inner automorphism ϕ_i of L taking D_i to D; since L_i is invariant under ϕ_i, we may assume that $\mathsf{D}_i \subseteq \mathsf{D}$. Hence, $\mathsf{D} = \mathsf{D}(L)$ is the direct sum of D_i,

$$\mathsf{D} = \bigoplus_{i=1}^{k} \mathsf{D}_i \ . \tag{2.25}$$

Also, put $\mathsf{N} = \mathsf{N}(L)$; if $\rho : L \to \operatorname{End}(V)$ is a finite dimensional representation, then the image $\rho(\mathsf{N})$ generates a nilpotent ideal of the envelope of $\rho(L)$ [**17**, p.50,56]; in particular, N is a nilpotent ideal.

Next, let N_i denote

$$\mathsf{N}_i(\mathbf{L}) = \mathsf{N}(L_i) + \mathsf{I}_i \ , \tag{2.26}$$

where

$$\mathsf{I}_i = \mathsf{I}_i(\mathbf{L}) = \sum_{j \neq i} [L_i, L_j] \ , \tag{2.27}$$

then, $\mathsf{N} = \sum_{i=1}^k \mathsf{N}_i$. Indeed, $\mathsf{N}_i \subseteq \mathsf{N}$ for all i by (2.20) and conversely,

$$\mathsf{N} = [L, \mathsf{S}] \subseteq \sum_{i=1}^{k} [L_i, \mathsf{S}] \subseteq \sum_{i=1}^{k} [L_i, \mathsf{S}(L_i)] + \mathsf{I}_i = \sum_{i=1}^{k} \mathsf{N}_i \ .$$

Since D is semisimple, we can find a D-submodule C_i completing N_i to S_i [**17**, p.79], hence $[C_i, \mathsf{D}] = 0$.

Lemma 2.4. *For all i there is a complement C_i such that $[C_i, \mathsf{D}] = 0$ and the Lie subalgebra $\langle C \rangle$, where $C = \sum_{i=1}^k C_i$, is nilpotent.*

Proof. Let $c_1, \dots, c_n$ be the union of bases of C_i; hence they generate the vector space C. We are going to repalce $c_1, \dots, c_n$ by suitable elements $c'_1, \dots, c'_n \in \langle C \rangle$ such that $c_j - c'_j \in (c_j, C)$ where $(c_j, C) = \sum_{t \geq 1} c_j \cdot \operatorname{ad}(C)^t$. Using the induction on p, let us prove the following statement.

There exist $c_1^{(p)}, \dots, c_n^{(p)} \in \langle \mathsf{C} \rangle$ and $k_p \in \mathbb{N}$ such that

(*) for all $j \in \{1, \dots, n\}$ the element $c_j - c_j^{(p)}$ belongs to (c_j, C);

(**) let M_p denote the subalgebra $\langle c_1^{(p)}, \dots, c_n^{(p)}\rangle$, then for all $j \le p$ and $c \in [M_p, M_p]$

$$M_p \cdot \mathrm{ad}(c_j^{(p)} + c)^{k_p} = 0 \ .$$

In fact, we can start with $p = 0$ assuming that $M_0 = \langle C\rangle$ and the statement is trivial in this case. Next, suppose it holds for $p \ge 0$, let us prove it for $p + 1$.

Since M_p is finite dimensional, there exists a polynomial $f(x) = g(x) \cdot x^k + x^k$, where $k > 0$ and $g(x)$ is a polynomial without the absolute term, such that $M_p \cdot f(\mathrm{ad}(c_{p+1}^{(p)})) = 0$. For all $j \in \{1, 2, \dots, n\}$ put $b_j = c_j^{(p)} \cdot g(\mathrm{ad}(c_{p+1}^{(p)}))$; notice that

$$b_j \in (c_j^{(p)}, C) \subseteq (c_j, C)$$

for all j and $b_{p+1} = 0$. Put $c_j^{(p+1)} = c_j^{(p)} + b_j$, hence $c_{p+1}^{(p+1)} = c_{p+1}^{(p)}$ and

$$c_j^{(p+1)} \cdot \mathrm{ad}(c_{p+1}^{(p+1)})^k = (c_j^{(p)} + b_j) \cdot \mathrm{ad}(c_{p+1}^{(p+1)})^k = c_j^{(p)} \cdot f(\mathrm{ad}(c_{p+1}^{(p)})) = 0 \ .$$

In addition, by (**) for all $j \in \{1, 2, \dots, p\}$ we have

$$M_p \cdot \mathrm{ad}(c_j^{(p+1)})^{k_p} = M_p \cdot \mathrm{ad}(c_j^{(p)} + b_j)^{k_p} = 0.$$

The subalgebra $M_{p+1} = \langle c_1^{(p+1)}, \dots, c_n^{(p+1)}\rangle$ is finite dimensional, hence it is spanned by a finite number of commutators in the generators (including the generators). Since $\mathrm{ad}(x)$ is a derivation, there exists $l \in N$ such that $M_{p+1} \cdot \mathrm{ad}(c_{p+1}^{(p+1)})^l = 0$. Let $m \in \mathbb{N}$ be the nilpotency index of the commutator subalgebra $[M_{p+1}, M_{p+1}]$. Then one can check that for all $c \in [M_{p+1}, M_{p+1}]$

$$M_{p+1} \cdot \mathrm{ad}(c_{p+1}^{(p+1)} + c)^{l \cdot m+1} = 0 \ .$$

Put k_{p+1} to be the maximum of k_p and $l \cdot m + 1$, therefore the statement holds for $p + 1$.

So, if $p = n$, then there exist $c'_1 = c_1^{(n)}, \dots, c'_n = c_n^{(n)}$ such that $c'_j - c_j \in (c_j, C) \subseteq \mathsf{N}(L)$ and for every c from the commutator subalgebra of $M_n = \langle c'_1, \dots, c'_n\rangle$ the operator $\mathrm{ad}(c'_i + c)$ acts nilpotently on M_n. Hence, the same holds if we replace c'_i with a linear combination of $c'_1, \dots, c'_n$, hence M_n is a nil algebra and by Engel's theorem [**17**, p.3] it is nilpotent. Clearly, the span of $\{c'_1, \dots, c'_n\} \cap L_i$ completes N_i to S_i. $\square$

Let

$$\mathsf{C}_i = \mathsf{C}_i(\mathbf{L}) \tag{2.28}$$

denote a complement of N_i to S_i such that $[\mathsf{C}_i, \mathsf{D}] = 0$ and the subalgebra $\langle \mathsf{C}\rangle$, where $\mathsf{C} = \sum_{i=1}^k \mathsf{C}_i$, is nilpotent. Then $L = \mathsf{D} + \mathsf{C} + \mathsf{N}$ and $\langle \mathsf{C}\rangle$ commutes with D.

Now we are going to split the ideal N_i into a sum of "simple" components. First of all, let $\bar{\mathsf{N}}_i$ be a complement of $\mathsf{N}_i^2 + \mathsf{I}_i$ to N_i; we can regard it as the L-module N/N^2; it turns out that all necessary information about l-module structure of N_i is contained in $\bar{\mathsf{N}}_i$. By complete reducibility of modules over D, we may assume that $\bar{\mathsf{N}}$ is a D-submodule of N_i. Also, modulo $\mathsf{N}_i^2 + \mathsf{I}_i$ the ideal N acts trivially on $\bar{\mathsf{N}}_i$, so we need only to describe $\bar{\mathsf{N}}_i$ with respect to the adjoint action of $\langle \mathsf{C}\rangle$.

The $\langle \mathsf{C}\rangle$-module N_i is the direct sum of weight subspaces $\bigoplus_{\lambda\in\Lambda_i} \mathsf{N}_i(\lambda)$ [**17**, p.41]. For a given subspace $M \subseteq \mathsf{N}_i(\lambda)$ put

$$M^{\{0\}} = M \quad ;$$

$$M^{\{l\}} = \sum_{c_1,\dots,c_l \in C} M \cdot (\mathrm{ad}(c_1) - \lambda(c_1)) \dots (\mathrm{ad}(c_l) - \lambda(c_l)) \ ;$$

$$M^{[k]} = \sum_{l \geq k} M^{\{l\}} \ .$$

Recall that D commutes with elements in C, hence, if M is a D-module, then $M^{\{k\}}$ is a D-submodule too. In addition, if M is an irreducible D-module and

$$M_1 = M \cdot (\mathrm{ad}(c_1) - \lambda(c_1)) \dots (\mathrm{ad}(c_l) - \lambda(c_l)) \tag{2.29}$$

is not equal to 0, then M_1 and M are isomorphic as D-modules.

Let M be a $\langle \mathsf{C} \rangle$-submodule of N_i, then M is the direct sum of weight subspaces $M(\lambda) = M \cap \mathsf{N}_i(\lambda)$ and

$$\mathsf{N}/M = \bigoplus_{\lambda \in \Lambda_i} (\mathsf{N}_i(\lambda) + M)/M \simeq \bigoplus_{\lambda \in \Lambda} \mathsf{N}_i(\lambda)/M(\lambda) \ . \tag{2.30}$$

Also, put

$$M^{\{m\}} = \sum_{\lambda \in \Lambda_i} M(\lambda)^{\{m\}} \quad , \quad M^{[m]} = \sum_{l \geq m} M^{\{l\}}$$

for all $m \geq 0$.

Now let

$$\mathsf{M}_i = \mathsf{M}_i(\mathbf{L}) = \mathsf{N}_i^2 + \mathsf{N}_i^{[1]} + \langle \mathsf{C}_i \rangle^2 + \mathsf{I}_i \ ; \tag{2.31}$$

clearly, it is an ideal of L. By (2.30), $\mathsf{N}_i/\mathsf{M}_i$ is the direct sum of $\mathsf{N}_i(\lambda)/\mathsf{M}_i(\lambda)$; observe that for all $c \in \mathsf{C}$ the operator $\mathrm{ad}(c)$ acts as the scalar $\lambda(c)$, moreover $\lambda(\mathsf{C}_j) = 0$ for all $j \neq i$, because $[\mathsf{N}_i, \mathsf{C}_j] \subseteq \mathsf{I}_i \subseteq \mathsf{M}_i$.

Choose a D-module $\mathsf{W}_i(\lambda)$ completing $\mathsf{M}_i(\lambda)$ to $\mathsf{N}_i(\lambda)$ and put

$$\mathsf{W}_i = \mathsf{W}_i(\mathbf{L}) = \sum_{\lambda \in \Lambda_i} \mathsf{W}_i(\lambda) \ , \tag{2.32}$$

then the sum is direct and W_i completes M_i to N_i. Besides,

$$[\mathsf{D}_j, \mathsf{W}_i] = 0 \tag{2.33}$$

for all $j \neq i$. Fix a decomposition of $\mathsf{W}_i(\lambda)$, $\lambda \in \Lambda_i$, into a direct sum of irreducible D-submodules; then $\mathsf{W}_i(\lambda) \simeq \mathsf{W}_i(\lambda) + \mathsf{M}_i/\mathsf{M}_i$, it equips $\mathsf{W}_i(\lambda)$ with structure of an L-module. Put

$$\mathrm{pr}(\mathsf{W}_i) = \bigcup_{\lambda \in \Lambda_i} \mathrm{pr}\,(\mathsf{W}_i(\lambda)) \tag{2.34}$$

so W_i is equal to the direct sum of all $\mathsf{W}_{i,\alpha}$, $\alpha \in \mathrm{pr}(\mathsf{W}_i)$.

LEMMA 2.5. *The L_i-module W_i has no trivial submodules (i.e., nonzero submodules with the trivial action of L_i).*

Proof. Suppose there is $a \in \mathsf{N}_i \backslash \mathsf{M}_i$ such that $[a, L_i] \subseteq \mathsf{M}_i$, then $[a, L] \subseteq \mathsf{M}_i$. We may assume that $a \in \mathsf{W}_i(0)$, hence the subspace $Fa = \mathrm{vect}_F\{a\}$ is a trivial D-module. Let U denote a D-submodule of N_i which contains M_i and completes Fa to N_i; since $a \in \mathsf{W}_i(0)$, it contains all $\mathsf{W}_i(\lambda)$, $\lambda \neq 0$, hence, $[U, \mathsf{C}] \subseteq U$. Thus, $L_i = \mathsf{D}_i + \mathsf{C}_i + Fa + U$ and

$$[L_i, L] = [\mathsf{D}_i + \mathsf{C}_i + Fa + U, \mathsf{D}_i + \mathsf{C}_i + Fa + U] + \mathsf{I}_i \subseteq$$

$$\subseteq [\mathsf{D}_i, \mathsf{D}_i] + [\mathsf{D}_i, U] + [\mathsf{C}_i, \mathsf{C}_i] + [\mathsf{C}_i, U] + [a, U] + [U, U] + \mathsf{I}_i \subseteq \mathsf{D}_i + U \ .$$

Hence, $a \notin [L_i, L] \supseteq \mathsf{N}_i$, which contradicts to the choice of a. □

COROLLARY 2.3. *The sum* $\mathsf{W} = \sum_{i=1}^k \mathsf{W}_i$ *is direct.*

Proof. Pick i and let $W = \mathsf{W}_i \cap \sum_{j \neq i} \mathsf{W}_j$, by definition, L_i acts trivially on W_j, hence, W is a trivial L_i-module and by the lemma above, $W = 0$. □

Put $\mathrm{pr}(\mathsf{W}) = \bigcup_{i=1}^k \mathrm{pr}(\mathsf{W}_i)$, then W is the direct sum of L-modules W_β, $\beta \in \mathrm{pr}(\mathsf{W})$; also, set $\mathsf{W}^{[l]} = \sum_{i=1}^k \mathsf{W}_i^{[l]}$ for all $l \geq 0$.

Next, let $\mathsf{Gn}_i(\mathbf{L}) = \mathsf{D}_i + \mathsf{C}_i + \mathsf{W}_i$, it is a D-module of L_i; we also regard it as an L-module L_i/M_i. It turns out, $\bigcup_{i=1}^k \mathsf{Gn}_i(\mathbf{L})$ contains all necessary information about L-module structure of N.

LEMMA 2.6. *For all* $i \in \{1, \dots, k\}$ *the Lie algebra* L_i *is generated modulo* I_i *by* $\mathsf{Gn}_i(\mathbf{L})$. *Moreover,*

$$\mathsf{N}_i = \langle \mathsf{C}_i \rangle^2 + \sum_{t \geq 1} [\underbrace{\mathsf{W}_i^{[0]}, \mathsf{W}_i^{[0]}, \dots, \mathsf{W}_i^{[0]}}_{t}] + \mathsf{I}_i \ . \tag{2.35}$$

Proof. Obviously, we only need to check (2.35). First, since N_i is nilpotent, modulo I_i it is generated as a Lie subalgebra by a complement $\bar{\mathsf{N}}_i$ of $\mathsf{N}_i^2 + \mathsf{I}_i$ to N_i. Actually, if $\mathsf{N}_i^c = 0$, then modulo I_i

$$\mathsf{N}_i \subseteq \bar{\mathsf{N}}_i + \mathsf{N}^2 \subseteq\subseteq \bar{\mathsf{N}}_i + (\bar{\mathsf{N}}_i + \mathsf{N}_i^2)^2 \subseteq$$

$$\subseteq \bar{\mathsf{N}}_i + \bar{\mathsf{N}}_i^2 + \mathsf{N}_i^3 \subseteq \dots \subseteq \bar{\mathsf{N}}_i + \bar{\mathsf{N}}_i^2 + \bar{\mathsf{N}}_i^3 + \dots \bar{\mathsf{N}}_i^{c-1} \ .$$

Observe that $I = \mathsf{N}_i^2 + \langle \mathsf{C}_i \rangle^2 + \mathsf{I}_i$ is a submodule over $\langle C \rangle$; let us show that modulo I for all λ

$$\mathsf{N}_i(\lambda) \subseteq \mathsf{W}_i(\lambda)^{[0]} \ .$$

Since λ is a weight of C, there exists l such that $\mathsf{N}_i^{[l]}(\lambda) = 0$, hence, by (2.31), modulo I

$$\mathsf{N}_i(\lambda) \subseteq \mathsf{W}_i(\lambda) + \mathsf{N}_i(\lambda)^{[1]} \subseteq \mathsf{W}_i(\lambda) + \mathsf{N}_i(\lambda)^{\{1\}} + \mathsf{N}_i(\lambda)^{[2]} \subseteq$$

$$\subseteq \mathsf{W}_i(\lambda) + (\mathsf{W}_i(\lambda) + \mathsf{N}_i(\lambda)^{[1]})^{\{1\}} + \mathsf{N}_i(\lambda)^{[2]} \subseteq$$

$$\subseteq \mathsf{W}_i(\lambda) + \mathsf{W}_i(\lambda)^{\{1\}} + \mathsf{N}_i(\lambda)^{[2]} \subseteq \dots \subseteq \sum_{m=0}^{l-1} \mathsf{W}_i(\lambda)^{\{m\}} + \mathsf{N}_i^{[l]}(\lambda) = \mathsf{W}_i^{[0]}(\lambda).$$

So, $\bar{\mathsf{N}}_i \subseteq \mathsf{W}_i^{[0]} + \langle \mathsf{C}_i \rangle^2$, hence by the previous statement, the right hand side generates the algebra N_i modulo I_i. □

So, fixing $\mathsf{Gn}_i(\mathbf{L})$ for all i, we define a decomposition of $\mathbf{L}$ called a *canonical* one.

Recall that we study T-ideals lying in $\mathcal{Q}(\mathcal{A})$, where $\mathcal{A} = (\mathbf{L}; V)$ is a finite dimensional g.Lie module satisfying (2.20). Fix a canonical decomposition of $\mathbf{L}$ and consider the k-sorted subalgebra $\mathbf{M} = (M; M_1, \dots, M_k)$ generated by

$$\mathsf{Gn}(\mathbf{L}) = \bigcup_{i=1}^k \mathsf{Gn}_i(\mathbf{L}) \ .$$

By (2.21) the value of any $f \in \mathcal{Q}(\mathcal{A})$ in $\mathcal{A}$ is defined by elements from complements of I_i to L_i. Hence, by the lemma above, we can replace $\mathcal{A}$ with the submodule generated by $\mathbf{M}$,

$$(\mathbf{M}; W)$$

where W is the submodule over $\mathsf{En}(\mathbf{M}, \nu) \subseteq \mathsf{En}(\mathcal{A})$ generated by $\mathrm{v}_1, \dots, \mathrm{v}_r$.

LEMMA 2.7. *For all i the vector space M_i is equal to the direct sum of D_i, C_i, and*

$$(\mathsf{C}_i, \mathsf{C}) + \sum_{t \geq 0} [\mathsf{W}_i^{[0]}, \underbrace{\mathsf{W}^{[0]}, \dots, \mathsf{W}^{[0]}}_{t}] \ , \tag{2.36}$$

and $(\mathsf{C}_i, \mathsf{C})$ denotes the ideal of $\langle \mathsf{C} \rangle$ generated by $[\mathsf{C}_i, \mathsf{C}]$.

Moreover, $\mathbf{M}$ has a canonical decomposition such that $\mathsf{Gn}_i(\mathbf{M}) = \mathsf{Gn}_i(\mathbf{L})$ for all i.

Proof. Let M_i' denote the sum of $\mathsf{D}_i + \mathsf{C}_i$ and the right hand side of (2.36), then $M_i' \subseteq M_i$; moreover, $M_1' + \dots + M_k'$ is a subalgebra of M containing the set of generators $\mathsf{Gn}(\mathbf{L})$, hence $M = M_1 + \dots + M_k$. Next, M_i' is closed under multiplication by elements in $\mathsf{Gn}(\mathbf{L})$, hence it is an ideal of M, also $\mathsf{Gn}_i(\mathbf{L}) \subseteq M_i'$, hence $M_i' = M_i$.

The subalgebra $\mathsf{D}_i \subseteq M_i \subseteq L_i$ is a maximal semisimple one in M_i, hence, we can put $\mathsf{D}_i(\mathbf{M}) = \mathsf{D}_i$. Next, the solvable radical $\mathsf{S}(M_i)$ is equal to the sum of C_i and (2.36); by definition,

$$\mathsf{N}_i(\mathbf{M}) = \mathsf{N}(M_i) + \mathsf{I}_i(\mathbf{M}) \ ,$$

from lemma 2.5 it follows that all $\mathsf{W}_{i,\beta}$ lie in $\mathsf{N}_i(\mathbf{M})$, hence, $\mathsf{N}_i(\mathbf{M})$ coincides with (2.36). On the other hand, $\mathsf{N}_i(\mathbf{M}) \subseteq \mathsf{N}_i$, hence, C_i completes $\mathsf{N}_i(\mathbf{M})$ to $\mathsf{S}(M_i)$, so we can define $\mathsf{C}_i(\mathbf{M})$ as C_i. Finally, by definition

$$\mathsf{M}_i(\mathbf{M}) = \mathsf{N}_i(\mathbf{M})^{[1]} + \langle \mathsf{C}_i(\mathbf{M}) \rangle^2 + \mathsf{N}_i(\mathbf{M})^2 + \sum_{j \neq i} [M_i, M_j] \ ;$$

as $\mathsf{C}_i(\mathbf{M}) = \mathsf{C}_i$, it is equal to

$$\mathsf{W}_i^{[1]} + (\mathsf{C}_i, \mathsf{C}) + \sum_{t \geq 1} [\mathsf{W}_i^{[0]}, \underbrace{\mathsf{W}^{[0]}, \dots, \mathsf{W}^{[0]}}_{t}] \ ,$$

hence, $\mathsf{M}_i(\mathbf{M}) + \mathsf{W}_i = \mathsf{N}_i(\mathbf{M})$. On the other hand, $\mathsf{M}_i(\mathbf{M}) \subseteq \mathsf{M}_i$, hence we can put $\mathsf{W}_i(\mathbf{M}) = \mathsf{W}_i$. □

5. Cayley-Hamilton polynomial

The underlying idea of Kemer's theorem can be described as follows. Let Γ be a T-ideal containing the ideal of identities of a finite dimensional algebra A, we fit gradually a finite dimensional algebra B such that $\Gamma = T(B)$. At some point we build a T-ideal Γ' of a special form related to the structure of A such that the quotient C of the free algebra modulo Γ' admits an extension by a commutative algebra of trace polynomials K. It is important that K has a balanced size. It is not too large so that Γ' is invariant under K modulo identities of A and, therefore, the extension of C by K is possible. On the other hand, it is large enough to provide certain finiteness conditions. This section gives an analog of K in the case of Lie algebras.

Let $\mathbf{L} = (L; L_1, \dots, L_k)$ be a finite dimensional k-sorted Lie algebra over F; if V is a finite dimensional Lie module over L, then the corresponding representation of L in

$\mathrm{End}(V)$ is denoted by ad_V (in fact, we will mostly deal with representations of L induced by the adjoint one).

Recall that $\mathcal{F}(\mathbf{L})$ denotes the Lie algebra of generic elements of $\mathbf{L}$; since $\mathbf{L}$ is finite dimensional, it lies in the algebra of polynomial mappings in many variables from $L_1, \dots, L_k$ to L,

$$\mathrm{Pol}(\vec{\mathbf{L}}, L) \simeq F[\vec{\mathbf{L}}] \otimes_F L \ ,$$

where $F[\vec{\mathbf{L}}] = \prod_{i=1}^k F[\vec{L}_i]$.

Let $\mathcal{UF}(\mathbf{L})$ denote the universal envelope of the Lie algebra $\mathcal{F}(\mathbf{L})$; it consists of associative polynomials in generic elements of $\mathbf{L}$. If V is an L-module, then for all i the representation $\mathrm{ad}_V : L \to \mathrm{End}(V)$ restricts to a representation of L_i and therefore induces a homomorphism π_V from $\mathcal{UF}(\mathbf{L})$ to $\mathrm{Pol}(\vec{\mathbf{L}}, \mathrm{End}(V))$ that takes $f \in \mathcal{F}(\mathbf{L})$ to $\mathrm{ad}_V(f)$; so the image of $E \in \mathcal{UF}(\mathbf{L})$ is obtained from E by replacing the generic elements with the corresponding adjoint operators.

We will consider modules over a fixed algebra, so it makes sence to abbrevate notations and put $\mathrm{Pol}(V) = \mathrm{Pol}(\vec{\mathbf{L}}, \mathrm{End}(V))$; also let $\Pi(V)$ denote the image of $\mathcal{UF}(\mathbf{L})$ in $\mathrm{Pol}(V)$. Let K be a subalgebra of $F[\vec{\mathbf{L}}] \simeq F[\vec{\mathbf{L}}] \cdot \mathrm{id}_V \subseteq \mathrm{Pol}(V)$;

An element $E \in \Pi(V)$ is said to be integral over K, if there exist $s_1, \dots, s_{n-1} \in K$ such that

$$E^n + s_1 \cdot E^{n-1} + \ldots + s_{n-1} \cdot E = 0 \ ;$$

we call $f_E(t) = t^n + s_1 \cdot t^{n-1} + \ldots + s_{n-1} \cdot t \in K[t]$ a *Cayley-Hamilton (C-H) polynomial for E*. If it holds for all $E \in \Pi(V)$, then we say that $\Pi(V)$ is integral over K. For example, let $K = \mathrm{tr}_V[\mathbf{L}]$ be the subalgebra of $F[\vec{L}]$ generated by polynomials $\mathrm{tr}(E)$ where $E \in \Pi(V)$. Then $\Pi(V)$ is integral over K of degree $\dim(V)$, where $f_E(t) = g(t)t$ and $g(t)$ is the standard Cayley-Hamilton polynomial for a linear operator.

Let $K \cdot \Pi(V)$ be the module over K generated by $\Pi(V)$; it is an algebra over K. As V is finite dimensional, it is an associative PI algebra, hence, the integrality of $\Pi(V)$ over a subalgebra $K \subseteq F[\vec{\mathbf{L}}]$ has an equivalent form given by Shirshov's theorem about height [**34, 37**].

LEMMA 2.8. *$\Pi(V)$ is integral over K iff any finitely generated subalgebra of $K \cdot \Pi(V)$ is a finitely generated module over K.*

Now our task is to find a "minimal" V such that $\Pi(L)$ is integral over $K = \mathrm{tr}_V[\mathbf{L}]$. Fix an L-module V and consider the class $Q(V)$ of all finite dimensional L-modules U such that $\Pi(U)$ is integral over $K = \mathrm{tr}_V(\mathbf{L})$.

LEMMA 2.9. *The class Q is closed with respect to*

(i) *submodules,*
(ii) *quotients,*
(iii) *tensor products,*
(iv) *finite direct sums (with the diagonal action of L),*
(v) *extensions: if an L-module V has a chain of submodules*

$$V = V_1 \supseteq V_2 \supseteq \ldots \supseteq V_m = 0 \tag{2.37}$$

where all factors $\bar{V}_i = V_i/V_{i+1}$ belong to Q, then $V \in Q$.

Proof.

(i) Let $U \in Q$ and let V_1 be a submodule; pick $E \in \mathcal{UF}(\mathbf{L})$, by hypothesis there is a C-H polynomial $f(t) \in K[t]$ for $\pi_U(E)$. Hence, $v \cdot f(\pi_U(E)) = 0$ for all $v \in U$; in particular, $V_1 \cdot f(\pi_U(E)) = 0$, hence $f(\pi_{V_1}(E)) = 0$.

(ii) Similar to (i).

(iii) Let $V_1, V_2 \in Q$ and let $U = V_1 \otimes_F V_2$, then for all $a \in L$

$$\mathrm{ad}_U(a) = \mathrm{ad}_{V_1}(a) \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes \mathrm{ad}_{V_2}(a) \ . \tag{2.38}$$

Also, $\mathrm{Pol}(U)$ can be identified with $\mathrm{Pol}(V_1) \otimes_{F[\vec{\mathbf{L}}]} \mathrm{Pol}(V_2)$. Pick $E \in \mathcal{UF}(L)$, it belongs to an associative subalgebra generated by a finite number of generic elements $x_1, \dots, x_q$ of L. Hence, by (2.38) the subalgebra of $\mathrm{Pol}(U)$ over K generated by E is is contained in $A_1^{\#} \otimes A_2^{\#}$, where A_i is the subalgebra of $\mathrm{Pol}(V_i)$ over K generated by $\mathrm{ad}_{V_i}(x_j)$, $j \in \{1, \dots, q\}$. Thanks to lemma 2.8, A_1, A_2 are finitely generated K-modules, hence so is A and (iii) holds.

(iv) Let $U = V' \oplus V''$, where $V', V'' \in Q$; hence U has a chain

$$U = V' \oplus V'' \supseteq V' \supseteq 0$$

where all factors belong to Q; hence, (iv) will follow from (v).

(v) Pick $E \in \mathcal{UF}(\mathbf{L})$ and let $f_i(t)$ be a C-H polynomial for $\pi_{\bar{V}_i}(E)$. Then $f(t) = f_1(t) \dots f_{m-1}(t)$ is a C-H polynomial for $\pi_U(E)$. Indeed, for all i we have $V_i \cdot f_i(\pi_{V_i}(E)) \subseteq V_{i+1}$, hence

$$V \cdot f(\pi_U(E)) = V_1 \cdot f_1(\pi_U(E)) \dots f_{m-1}(\pi_U(E)) \subseteq V_2 \cdot f_2(\pi_{V_2}(E)) \dots f_{m-1}(\pi_{V_2}(E)) \subseteq \dots$$

$$\subseteq V_i \cdot f_i(\pi_{V_i}(E)) \dots f_{m-1}(\pi_{V_i}(E)) \subseteq \dots \subseteq V_m = 0 \ .$$

The lemma is proven. □

Now, let $\mathbf{L}$ be a finite dimensional k-sorted Lie algebra; put

$$\mathsf{V} = \mathsf{V}(\mathbf{L}) = \mathsf{D} + \mathsf{W} \simeq \oplus_{i=1}^{k} \mathsf{D}_i \bigoplus \mathsf{W}_i \ ,$$

so, it is an L-module; as a matter of fact, it is isomorphic to the quotient of L over the ideal $\sum_{i=1}^{k} (\mathsf{M}_i + \mathsf{C}_i)$.

PROPOSITION 2.2. *Let $\mathbf{L} = (L; L_1, \dots, L_k)$ be a finite dimensional k-sorted Lie algebra, where $\mathsf{Gn}(\mathbf{L})$ generates L. Suppose that U is an L-module such that $\mathsf{V} \in Q(U)$, then $\Pi(L)$ is integral over $\mathrm{tr}_U[\mathbf{L}]$.*

Proof. We need to show that the regular L-module belongs to $Q(U)$; notice that L has a chain of submodules

$$L \supseteq \mathsf{N} \supseteq \mathsf{N}^2 \dots \supseteq \mathsf{N}^m = 0 \ ,$$

where m is the nilpotency index of N. By lemma 2.9 (v), we need only to prove that all factors lie in $Q(U)$.

First of all, $\mathsf{V}_i = \mathsf{D}_i + \mathsf{W}_i$ is a submodule of V, hence, it belongs to $Q(U)$. Next, $L_i/\mathsf{N}_i \simeq \mathsf{Gn}_i(\mathbf{L}) \in Q(U)$, because it is an extension of V_i by a trivial module $\mathsf{S}(L_i)/\mathsf{N}_i \simeq \mathsf{C}_i$. Next, $\mathsf{N}_i \subseteq \mathsf{N} \cap L_i$, hence $L_i + \mathsf{N}/\mathsf{N} \in Q(U)$ as quotient of L_i/N_i. Hence $L/\mathsf{N} = \sum_{i=1}^{k} L_i + \mathsf{N}/\mathsf{N} \in Q(U)$, because it is a quotient of the direct sum of $L_i + \mathsf{N}/\mathsf{N}$.

Next, $\mathsf{N}^p/\mathsf{N}^{p+1}$ is a quotient of $\bar{\mathsf{N}}^{\otimes p}$, where $\bar{\mathsf{N}} = \mathsf{N}/\mathsf{N}^2$, so we need only to check that $\bar{\mathsf{N}} \in Q(U)$. By lemma 2.7 it is equal to $\mathsf{W}^{[0]} + \langle \mathsf{C}\rangle^2 + \mathsf{N}^2/\mathsf{N}^2$. Consider a chain of submodules

$$W_1 \supseteq W_2 \supseteq \ldots \supseteq W_n \supseteq 0$$

where $W_j = \mathsf{W}^{[0]} + \langle \mathsf{C}\rangle^{j+1} + \mathsf{N}^2/\mathsf{N}^2$ and $m-1$ is the nilpotency index of C. Clearly, W_j/W_{j+1} is trivial module for all $j < m$, hence, we need only $W_m = \mathsf{W}^{[0]} + \mathsf{N}^2/\mathsf{N}^2 \in Q(U)$. Consider a chain

$$U_0 \supseteq U_1 \supseteq \ldots \supseteq U_l = 0$$

where $U_j = \mathsf{W}^{[j]} + \mathsf{N}^2/\mathsf{N}^2$; by (2.29) for all j the quotient U_j/U_{j+1} is a sum of submodules being isomorphic to irreducible submodules in V, hence it lies in $Q(U)$ which proves the proposition. $\square$

6. Identities of irreducible representations

Take a variable $y \notin X$ and let $\mathrm{L}\langle y; X\rangle$ denote the subspace of all polynomials in $\mathrm{L}\langle X \cup \{y\}\rangle$ that are linear with respect to y. Let V be a module over a Lie algebra L, we denote it by $V : L$; a polynomial $f = f(y; x_1, \ldots, x_n) \in \mathrm{L}\langle y; X\rangle$ is called *an identity* of $V : L$, if $f(v; a_1, \ldots, a_n) = 0$ for all $v \in V$ and $a_1, \ldots, a_n \in L$. Let $T(V : L)$ be the subset of all identities of $V : L$; clearly, it is closed under substitution of Lie polynomials from $\mathrm{L}\langle X\rangle$ in place of variables from X and a polynomial from $\mathrm{L}\langle y; X\rangle$ in place of y. We call such a subspace of $\mathrm{L}\langle y; X\rangle$ a *T-submodule*. Obviously, every T-submodule can be identified with the set of all identities $T(V : L)$ of an appropriate Lie module $V : L$.

In fact, it is equivalent to the language of Lie pairs, cf.[**33**]. Actually, let $\widetilde{L}$ be the image of L in $\mathrm{End}(V)$ and let A denote the envelope of $\widetilde{L}$; then $T(V : L)$ consists of all polynomials of the form $y \cdot f(\mathrm{ad}(x_1), \ldots, \mathrm{ad}(x_n))$, where f ranges over the ideal of identities of the Lie pair $(A, \widetilde{L})$. Moreover, if I is weak T-ideal of $\mathrm{A}\langle X\rangle$, then

$$MI = \{y \cdot f(\mathrm{ad}(x_1), \ldots, \mathrm{ad}(x_n)) |\ f \in I\}$$

is a T-submodule and every T-submodule has such a form; also put

$$LI = \{x \cdot f(\mathrm{ad}(x_1), \ldots, \mathrm{ad}(x_n)) |\ x \in X;\ f \in I\}\ ,$$

it is a T-ideal of $\mathrm{L}\langle X\rangle$. Later we shall need Lie polynomials of the form

$$f(y; E_1, \ldots, E_n, \ldots) = y \cdot g(E_1, \ldots, E_n, \ldots)$$

where $g = g(x_1, \ldots, x_n, \ldots) \in \mathrm{A}\langle X\rangle$ and $E_1, \ldots, E_n$ are polynomials in adjoint operators $\mathrm{ad}(x)$, $x \in X$.

The next fact is an analog of lemma 1.3.

LEMMA 2.10. *Let V be an irreducible finite dimensional module over a Lie algebra L and let $f = f(y; x_1, \ldots, x_n)$ be a multilinear nonidentity of $V : L$, $f \notin T(V : L)$.*

Take a nonzero $v \in V$, then there exist $a_1, \ldots, a_n \in L$ and a polynomial E in adjoint operators $\mathrm{ad}_V(a)$,$a \in L$, such that v is equal to

$$f(v; a_1, \ldots, a_n)E \tag{2.39}$$

Proof. Notice first, that the subset $u \in V$ such that $f(u; L, \dots, L) = 0$ is a submodule of $V : L$; as $f \notin T(V : L)$ and V is irreducible, it must be equal to 0. Hence, $w = f(v; a_1, \dots, a_n) \neq 0$ for suitable $a_1, \dots, a_n \in L$ and the submodule U generated by w coincides with V. Clearly, every element in U has form (2.39) for some E. □

In other words, the statement says that for every nonzero $v \in V$ there is a polynomial of the form

$$g(y; x_1, \dots, x_n, x_{n+1}, \dots, x_m) = f \cdot E \tag{2.40}$$

such that under a suitable substitution of $x_i \to \bar{x}_i \in L$ and $y \to \bar{y} = v$ we get $v = \bar{g}$. Observe that g may be not linear with respect to $x_{n+1}, \dots, x_m$, but we can assume that it is a sum of multilinear polynomials.

Consider some examples.

LEMMA 2.11. *Let V be an irreducible L-module,* $\dim(V) = l$ *and* $\dim(\mathrm{ad}_V(L)) = m$. *Then* $M\mathrm{Cap}_{l^2} \not\subseteq T(V : L)$ *and* $M\mathrm{Cap}_{m,1} \not\subseteq T(V : L)$.

Proof. Since V is irreducible and F is algebraically closed, the envelope A of $\mathrm{ad}_V(L)$ coincides with $\mathrm{End}(V) \simeq M_l(F)$ [**35**, p.152]. Recall (cf. lemma 1.3) that the verbal ideal $\mathrm{Cap}_{l^2}[A]$ is equal to A, in particular, $V \cdot \mathrm{Cap}_{l^2}[A] \neq 0$, hence, $M\mathrm{Cap}_{l^2} \not\subseteq T(V : L)$.

Next, pick

$$f = f(y; E_1, \dots, E_{l^2}, \dots) \in M\mathrm{Cap}_{l^2}$$

which is multilinear and skew-symmetric with respect to $E_1, \dots, E_{l^2}$ and under a suitable substitution $x \to \bar{x}$ does not vanish. Observe that $\bar{E}_1, \dots, \bar{E}_{l^2}$ must be a basis of A; we can replace them with elements of any another basis, hence, we can assume that the first m operators are adjoint ones, $E_i = \mathrm{ad}(x_i)$, where $i \leq m$. □

So, every $v \in V$ can represented as a specialization of polynomial (2.40), where $f = f(y; E_1, \dots, E_{l^2}, \dots)$ is multilinear and skew-symmetric with respect to $E_1, \dots, E_{l^2}$ or $f = f(y; x_1, \dots, x_m, \dots)$ is multilinear and skew-symmetric with respect to $x_1, \dots, x_m$.

Now we go back to the canonical decomposition of a f.d. k-sorted Lie algebra $\mathbf{L} = (L; L_1, \dots, L_k)$. Recall that $\mathsf{Gn}_i(\mathbf{L}) = D_i + \mathsf{C}_i + \mathsf{W}_i$ is a module over L; as N acts trivially, we may consider it as a module over $\bar{L} = L/\mathsf{N}$. Note that $\bar{L}$ is isomorphic to the direct sum of $\mathsf{D} \simeq \mathsf{D} + \mathsf{N}/\mathsf{N}$ and $\bar{\mathsf{C}} = \mathsf{C} + \mathsf{N}/\mathsf{N}$. Hence, the regular $\bar{L}$-module is the direct sum of simple summands D_α, $\alpha \in \mathrm{pr}(\mathsf{D}) = \bigcup_{i=1}^k \mathrm{pr}(\mathsf{D}_i)$, and 1-dimensional trivial submodules of $\bar{\mathsf{C}}$.

Take $\beta \in \mathrm{pr}(\mathsf{W}_i) \subseteq \mathrm{pr}(\mathsf{W})$, the $\bar{L}$-module $\mathsf{W}_{i,\beta}$ is said to be *standard* if it is isomorphic to $\mathsf{D}_\alpha : \bar{L}$ for some $\alpha \in \mathrm{pr}(\mathsf{D})$. In particular, it means that

$$[\mathsf{W}_{i,\beta}, \bar{\mathsf{C}}] = 0 \ , \tag{2.41}$$

$$[\mathsf{W}_{i,\beta}, \mathsf{D}_\gamma] = 0 \tag{2.42}$$

for all $\gamma \neq \alpha$; besides, $\alpha \in \mathrm{pr}(\mathsf{D}_i)$ because $[\mathsf{W}_i, \mathsf{D}_j] = 0$ for $j \neq i$.

For $\beta \in \mathrm{pr}(\mathsf{W}_i)$ put $\mathrm{ad}_\beta = \mathrm{ad}_V$, where $V = \mathsf{W}_{i,\beta}$. Suppose that $\mathsf{W}_{i,\beta}$ is non-standard, then, by Kushkulei-Rasmyslov's theorem [**23**], $T(\mathsf{W}_{i,\beta} : \bar{L}) = T(\mathsf{W}_{i,\beta} : L)$ differs from $T(\mathsf{D}_\alpha : \bar{L}) = T(\mathsf{D}_\alpha : \mathsf{D}_\alpha)$ for all $\alpha \in \mathrm{pr}(\mathsf{D})$.

LEMMA 2.12. *Suppose $W = \mathsf{W}_{i,\beta}$ is a non-standard submodule, where $\dim(\mathrm{ad}_\beta(L)) = m$. Then there exists $\phi = \phi(y; x_1, \ldots, x_m, \ldots) \notin T(W : L)$ which is multilinear and skew-symmetric with respect to $y, x_1, \ldots, x_m$.*

Proof. By lemma 2.11, there is a multilinear $\psi_1(y; x_1, \ldots, x_m, \ldots)$ being skew-symmetric with respect to $x_1, \ldots, x_m$ such that

$$\bar{\psi}_1 = \psi_1(\bar{y}; \bar{x}_1, \ldots, \bar{x}_m, \ldots) \neq 0$$

for some $\bar{y} \in W$ and $\bar{x}_j \in L$.

First, suppose that $\mathrm{ad}_\beta(L)$ is simple, hence, it is isomorphic to some D_α. Consider the case $T(\mathsf{D}_\alpha : \bar{L}) \not\subseteq T(W : L)$; by lemma 2.10, there exists

$$\psi_2 = \psi_2(y; z_1, \ldots, z_n) \in T(\mathsf{D}_\alpha : \bar{L})$$

which a sum of multilinear polynomials and $\bar{y} = \psi_2(\bar{y}; \bar{z}_1, \ldots, \bar{z}_n)$ for appropriate $\bar{z}_1, \ldots, \bar{z}_n \in L$. Let ϕ denote the alternation of

$$\psi_1(\psi_2(y; z_1, \ldots, z_n); x_1, \ldots, x_m, \ldots)$$

with respect to $y, x_1, \ldots, x_m$, then under the same substitution $\bar{\phi} = m! \cdot \bar{\psi}_1$. So, taking a suitable homogeneous component of ϕ, we get the required polynomial.

Thus, thanks to Kushkulei-Rasmyslov's theorem, we can assume that

$$T(\mathsf{D}_\alpha : \mathsf{D}_\alpha) \not\supseteq T(W : L)$$

, hence, there exists

$$\psi_2 = \psi_2(y; z_1, \ldots, z_n) \in T(W : L)$$

such that $\bar{\psi}_2 \neq 0$ under a suitable substitution $x \to \bar{x}$ of elements from D_α in place of $y, z_1, \ldots, z_n$. Since $\dim(\mathsf{D}_\alpha) = \dim(\mathrm{ad}_\beta(\bar{L})) = m$, we can find $\psi_1(y; x_1, \ldots, x_m, \ldots)$ being multilinear and skew-symmetric with respect to $x_1, \ldots, x_m$ such that

$$\bar{y} = \psi_1(\bar{y}; \bar{x}_1, \ldots, \bar{x}_m, \ldots) \; .$$

Take $v \in W$ such that $[v, \bar{\psi}_2] \neq 0$, hence

$$[v, \psi_2(\psi_1(\bar{y}; \bar{x}_1, \ldots, \bar{x}_m, \ldots); \bar{z}_1, \ldots, \bar{z}_n)] \neq 0 \; ;$$

if we swap v and any $\bar{x}_i$, we shall get 0 by the choice of ψ_2. Hence, the alternation of the polynomial

$$[y, \psi_2(\psi_1(x; x_1, \ldots, x_m, \ldots); z_1, \ldots, z_n)]$$

with respect to $y, x_1, \ldots, x_m$ is not an identity of $W : L$; choosing a suitable homogeneous component, we get the required polynomial.

Now, suppose that $\mathrm{ad}_\beta(L)$ is not simple, then it has nonzero ideals L', L'' (maybe equal) such that $[L', L''] = 0$. We can choose $\bar{z}'_j \in L'$, $\bar{z}''_j \in L''$, $j + 1, \ldots, t$ and $\tilde{y} \in W$ such that $\bar{y} = \sum_j [[\tilde{y}, \bar{z}'_j], \bar{z}''_j]$. Then the alternation ϕ of

$$\psi_1([[y, z'_1], z''_1]; x_1, \ldots, x_m, \ldots)$$

with respect to $y, x_1, \ldots, x_m$ is not an identity of $W : L$. $\square$

7. "Counter-example"

Like in the associative case we need the notion of "counter-example". Let $\mathcal{A} = (\mathbf{L}; V)$ be a finite dimensional g.Lie module from $\mathfrak{M}_{k,s,r}$ such that there exists a strictly ascending chain of T-ideals in $\mathcal{Q}(\mathcal{A})$ starting with $T(\mathcal{A})$. Then one can find a sequence of multlinear polynomials $f_1, f_2, \dots, f_n, \dots \in \mathcal{Q}(\mathcal{A})$, $deg(f_i) < deg(f_{i+1})$, such that the chain of T-ideals:

$$T(\mathcal{A}) \subset \Gamma_1 \subset \Gamma_2 \subset \dots \subset \Gamma_n \subset \dots \tag{2.43}$$

where $\Gamma_n = (f_1, f_2, \dots f_n)_{\mathsf{T}} + T(\mathcal{A})$, ascends strictly.

Let Γ_n^+ denote the T-ideal generated by $T(\mathcal{A})$ and all homogeneous polynomials from Γ_n of degree greater than $\deg(f_n)$. Put $\Gamma = \sum_{n=1}^{\infty} \Gamma_n^+ + T(\mathcal{A})$, then for all n the identity f_n does not belong to Γ, but every "proper" consequence of f_n, i.e., a homogeneous polynomial from $(f_n)_{\mathsf{T}}$ of degree $> \deg(f_n)$, lies in Γ. We call such a sequence *appropriate*.

A T-ideal Γ is called *critical* if it has an appropriate sequence; the T-ideal generated by Γ and the appropriate sequence is denoted by $\Gamma^{\#}$. Like in associative algebras, the main property of a critical T-ideal is that it can not be the ideal of identities of a finite dimensional g.Lie module.

LEMMA 2.13. *If Γ is a critical T-ideal, then $\Gamma \neq T(\mathcal{A})$, where $\mathcal{A}$ is a finite dimensional g.Lie module.*

Proof. Suppose that $\Gamma = T(\mathcal{A})$, where $\mathcal{A} = (\mathbf{L}, V)$ is a finite dimensional g.Lie module. Let $f_1, \dots, f_n, \dots$ be the appropriate sequence for Γ. For all n we have $f_n(x_1, x_2, \dots, x_m) \notin T(\mathcal{A})$, hence one can find $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m \in \mathbf{L}$, such that $\bar{f}_n = f_n(\bar{x}_1, \dots, \bar{x}_m) \neq 0$.

By lemma 2.4 we have

$$\mathsf{C} + [L, L] = \sum_{i=1}^{k} \mathsf{C}_i + [L_i, L] = \sum_{i=1}^{k} L_i = L \ ;$$

since f_n is multilinear, we may assume that $\bar{x}_j \in \mathsf{C}_j \cup [L_i, L]$ provided $x_j \in X_i$. Observe that $\bar{x}_j$ cannot be in $[L_i, L]$; actually, the substitution a commutator $[a, b] \in \mathrm{L}_i\langle X\rangle$ in place of x_j means that $\bar{f}_n$ is equal to the value of a "proper" consequence of f_n which belongs to $T(\mathcal{A})$, therefore $\bar{f}_n$ must be equal to 0. So, for all n we can replace the variables of f_n by suitable elements from $\bigcup_{i=1}^{k} \mathsf{C}_i$ and get a nonzero element $\bar{f}_n \in V$. Let $c_1, \dots, c_t$ be the union of bases of C_i, $i \in \{1, \dots, k\}$; since f_n are multilinear we may assume that these elements are among $c_1, \dots, c_t$.

Consider a set of free variables $Y = \{y_1, y_2, \dots, y_t\}$, $Y_j = \{y_i \in Y \mid c_i \in C_j\}$ and the free g.Lie module

$$\mathcal{M}\langle Y\rangle = (\mathbf{L}\langle Y\rangle; V\langle Y\rangle) \ ,$$

and define a homomorphism (ϕ, ψ) from $\mathcal{M}\langle Y\rangle$ to $\mathcal{A}$ which takes y_i to c_i; then the image coincides with the g.Lie submodule $\mathcal{B}$ generated by $c_1, \dots, c_t$.

Furthermore, put $I = \mathrm{L}\langle Y\rangle^q$, where $\langle \mathsf{C}\rangle^q = 0$ (recall that $\langle \mathsf{C}\rangle$ is nilpotent), then $\mathrm{Ker}(\phi)$ contains I; besides, let J denote the ideal of identities of $\mathcal{B}$, then $J \subseteq \mathrm{Ker}(\psi)$. Hence, (ϕ, ψ) induces an epimorphism $(\tilde{\phi}, \tilde{\psi})$ from

$$\widetilde{\mathcal{M}}\langle Y\rangle = (\tilde{\mathbf{L}}; V\langle Y\rangle / J)$$

where $\tilde{\mathbf{L}} = \mathbf{L}\langle Y\rangle/I$, onto $\mathcal{B}$. We claim that $\tilde{V} = V\langle Y\rangle/J$ satisfies a.c.c. for submodules.

LEMMA 2.14. *Let $\mathcal{A} = (\mathbf{L}; V)$ be a g.Lie module over a commutative noetherian ring K. Suppose that L is a finitely generated module over K, then $\mathsf{En}(\mathcal{A})$ is a noetherian algebra (a.c.c. for right ideals) and, therefore, V is a noetherian module over $\mathsf{En}(\mathcal{A})$.*

Proof. Recall that $\nu(L) = \sum_{i=1}^{k} \nu_i(L)$ is a Lie subalgebra of $\mathrm{End}_\Phi(V)^{(-)}$ and $\mathsf{En}(\mathcal{A})$ is its envelope, hence, lemma follows from [**11**, sec.2.3.8]). □

Observe that $\tilde{L}$ is finite dimensional over F, hence the lemma gives the necessary property for $\tilde{V}$.

Recall that f_n does not vanish under replacing all variables x_i with the corresponding basis elements c_{i_j}. Let $g_n \in V\langle Y\rangle$ be the polynomial f_n where x_i is replaced with y_{i_j}. Then it is homogeneous of the same degree as f_n and obviously $\psi(g_n)$ is equal to $\bar{f}_n \neq 0$.

Now let $\tilde{V}_n$ be the submodule of $\tilde{V}$ over $\mathsf{En}(\widetilde{\mathcal{M}}\langle Y\rangle)$ generated by $\tilde{g}_1, \dots, \tilde{g}_n$, where $\tilde{g}_n = \tilde{\psi}(g_n) = g_n/J$. Since $\tilde{V}$ satisfies a.c.c. for submodules, $\tilde{V}_n = \tilde{V}_{n+1}$ for some n, hence $\tilde{g}_{n+1} \in \tilde{V}_n$ and

$$g_{n+1} - g_n E \in J$$

for some $E \in \mathsf{En}(\widetilde{\mathcal{M}}\langle Y\rangle)$. Moreover, since g_{n+1}, g_n are homogeneous elements and J is a homogeneous submodule, we may assume that E is homogeneous of degree $\deg(g_{n+1}) - \deg(g_n) > 0$. Observe that $g_n E$ is obtained from $f_n E$ by the corresponding substitution of variables from Y (since X is infinite, we may assume that $Y \subseteq X$ and no variables from Y occurring in any f_n). By the definition of the appropriate sequence, $f_n E \in \Gamma = T(\mathcal{A})$, hence, $\psi(g_n E) = 0$ and $\psi(g_{n+1})$ must be equal to 0, which is impossible. Thus, the lemma is proven. □

The next statement is an analog of lemma 1.6 and one can prove it in the same way.

PROPOSITION 2.3. *Let Γ, Γ_1 be T-ideals, where Γ is critical. Then $\Gamma + \Gamma_1$ is critical, where the appropriate sequence is a subsequence of that of Γ, or $\Gamma \cap \Gamma_1$ is a critical T-ideal with an appropriate sequence in Γ_1.*

Now we are prepared to define the main object; a pair of T-ideals $T(\mathcal{A}) \subset \Gamma$ is called *a "counter-example"* iff

(i) $\mathcal{A}$ is a finite dimensional generalized Lie module satisfying (2.20);
(ii) Γ is a critical T-ideal, where $\Gamma^{\#} \subseteq \mathcal{Q}(A)$.

The next statement is an analog of lemma 1.7.

LEMMA 2.15. *Let Γ be a critical T-ideal and let $\mathcal{B}_j$, $j = 1, \dots, t$ be finite dimensional g. Lie modules such that $\Gamma^{\#} \subseteq \mathcal{Q}(\mathcal{B}_j)$ for all j and*

$$\Gamma \cap \bigcap_{j=1}^{t} T(\mathcal{B}_j)$$

has no an appropriate sequence in $\Gamma^{\#} \cap \bigcap_{j=1}^{t} T(\mathcal{B}_j)$. Then there is a T-ideal Γ_1 with the appropriate sequence in $\Gamma^{\#}$ such that $T(\mathcal{B}_j) \subset \Gamma_1 \subseteq \Gamma + T(\mathcal{B}_j)$ for some j.

Proof. Put $\Upsilon_0 = \Gamma$ and $\Upsilon_m = \Gamma \cap \bigcap_{j=1}^{m} T(\mathcal{B}_j)$, $\Upsilon_m^{\#} = \Gamma^{\#} \cap \bigcap_{j=1}^{m} T(\mathcal{B}_j)$ for $m \geq 1$. Take the maximal m such that Υ_m has an appropriate sequence in $\Upsilon_m^{\#}$; then $0 \leq m < t$. By the proposition above $\Gamma_1 = \Upsilon_m + T(\mathcal{B}_{m+1})$ is a critical with the appropriate sequence in Υ_m. $\square$

8. Type of a "counter-example"

In the associative case the type of a "counter-example" $T(A) \subset \Gamma$ is defined as a pair of integers ≥ 0 characterizing the difference between $T(A)$ and Γ, and the lexicographical order on $\mathbb{N}_0 \times \mathbb{N}_0$ enables us to carry out the induction on the type. In this section we define the type

$$\operatorname{type}(\mathcal{A}, \Gamma)$$

where $\mathcal{A} = (\mathbf{L}; V)$ is a finite dimensional g.Lie module from $\mathcal{M}_{k,s,r}$ satisfying (2.20), with a fixed canonical decomposition of $\mathbf{L}$, and Γ is a T-ideal in $\mathcal{Q}(\mathcal{A})$ such that $\Gamma \nsubseteq T(\mathcal{A})$.

Clearly, the definition of type should be based on certain numerical characteristics of $\mathcal{A}$; in fact, we will use simple components of the L-modules $\mathsf{Gn}_i(\mathbf{L})$. Since the dimensions of irreducible modules over a finite dimensional Lie algebra are not bounded, finite sequences of an arbitrary length should be employed instead of pairs (or n-tuples); we will use an inductive construction of well-ordered sets with the lexicographical order.

Let $\{M_i\}_{i\in\mathbb{N}}$ be a sequence of well-ordered sets, where 0_i denotes the least element of M_i. Consider the set $\mathsf{Fin}(M_i; i \geq 1)$ of finite sequences $\mathrm{a} = (a_1, \dots, a_i, \dots)$ (i.e., there exists i_0 such that for all $i > i_0$ we have $a_i = 0_i$). Define a linear order on $\mathsf{Fin}(M_i; i \in \mathbb{N})$: for all a, b put $\mathrm{a} > \mathrm{b}$ if there is k such that $a_k > b_k$ and for all $n > k$ we have $a_n = b_n$. Clearly, it is the direct limit of the lexicographically ordered sets $M_1 \times \dots \times M_q$, $q \in \mathbb{N}$; moreover, every ascending chain of elements in $\mathsf{Fin}(M_i;\ i \in \mathbb{N})$ is contained in such subset, hence we get immediately the following statement.

LEMMA 2.16. $\mathsf{Fin}(M_i; i \in \mathbb{N})$ *is a well-ordered set.*

In fact, we shall mostly deal with powers M^∞ of a well-ordered set M, i.e., $M^\infty = \mathsf{Fin}(M_i; i \in \mathbb{N})$, where $M_i = M$ for all i. In particular , a finite power M^k is a well-ordered subset of M^∞ (note that this order is lexicographical from the right to the left).

Like in the associative case, $\operatorname{type}(\Gamma, \mathcal{A})$ should measure the difference $\Gamma \backslash T(\mathcal{A})$. Recall that over a field of characteristic 0 any T-ideal is generated by its multilinear polynomials, hence, we need to estimate the size of the multilinear part of $\Gamma \backslash T(\mathcal{A})$. If $f \in \Gamma$ is a multilinear nonidentity of $\mathcal{A}$, then f does not vanish under some substitution of elements from $\mathcal{A}$ in place of variables. In fact, we do not need to consider all such substitutions.

LEMMA 2.17. *Let $f \in \mathcal{Q}(\mathcal{A})$ be a nonidentity of $\mathcal{A}$, then there exists a multilinear polynomial $g = g(x_1, \dots, x_n) \in (f)_{\mathsf{T}}$ such that g does not vanish under a substitution $x_i \to \bar{x}_i$, where $\bar{x}_i \in \mathsf{Gn}_j(\mathbf{L})$, if $x_i \in X_j$.*

Proof. Suppose $\tilde{f} = f(\tilde{y}_1, \dots, \tilde{y}_m) \neq 0$ for suitable elements $\tilde{y}_i \in L_j$. By lemma 2.6 modulo I_j the Lie algebra L_j is generated by $\mathsf{Gn}(L_j)$, hence $\tilde{y}_i$ has the form of a Lie polynomial $h_i \in \mathrm{L}\langle X_j\rangle \subseteq \mathrm{L}_j\langle X\rangle$ in some elements of $\mathsf{Gn}(L_j)$. Hence, $f_1 = f(h_1, \dots, h_m)$ does not vanish under substitution of the same elements. Taking the linearization of a suitable homogeneous part of f_1, we get the required polynomial g. $\square$

So, we should consider "nonzeros" of multilinear polynomials from Γ consisting of elements in $\bigcup_{i=1}^k \mathsf{Gn}(L_i)$. Moreover, thanks to linearity such elements may be taken only from simple components. Put

$$\mathsf{Spl}(L_i) = \left(\bigcup_{\alpha\in \mathrm{pr}(\mathsf{D}_i)} \mathsf{D}_{i,\alpha}\right) \cup \left(\bigcup_{\beta\in \mathrm{pr}(\mathsf{W}_i)} \mathsf{W}_{i,\beta}\right) \cup \mathsf{C}_i \ ,$$

$$\mathsf{Spl}(\mathbf{L}) = \bigcup_{i=1}^{k} \mathsf{Spl}(L_i) \ .$$

Let $f(\mathbf{x}) = f(x_1, \dots, x_n) \in V\langle X\rangle$, where $\mathbf{x}$ means the n-tuple of variables $(x_1, \dots, x_n)$. A substitution $x \to \bar{x}$, where $\bar{x} \in L_i$ provided $x \in X_i$, defines a homomorphism of the free module to $\mathcal{A}$; the image of f is $\bar{f} = f(\bar{\mathbf{x}}) = f(\bar{x}_1, \dots, \bar{x}_n)$. We will consider mostly substitutions $x \to \bar{x}$ such that

$$\bar{x} \in \mathsf{Spl}(\mathbf{L}) \tag{2.44}$$

for all $x \in X$. Put

$$\mathsf{V}(f, \mathcal{A}) = \{\bar{\mathbf{x}} |\ f(\bar{\mathbf{x}}) \neq 0\} \ , \tag{2.45}$$

where $x \to \bar{x}$ satisfies (2.44) and given a T-ideal Γ of $\mathcal{M}\langle X\rangle$ let

$$\mathsf{V}(\Gamma, \mathcal{A}) = \bigcup_{f} \mathsf{V}(f, \mathcal{A}) \ , \tag{2.46}$$

where f runs over all multilinear polynomials in Γ.

Take $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathsf{V}(f, \mathcal{A})$ and let $\pi^{(m)}(\bar{\mathbf{x}})$ be the set of all projections $\alpha \in \mathrm{pr}(\mathsf{D})$ such that

$$\dim(\mathsf{D}_\alpha) = m \quad \textit{and} \ \ \exists j \quad [\bar{x}_j, \mathsf{D}_\alpha] \neq 0 \ .$$

Put

$$\mathrm{a}^{(m)}(\bar{\mathbf{x}}) = |\pi^{(m)}(\bar{\mathbf{x}})| \tag{2.47}$$

(so, $\mathrm{a}^{(m)}(\bar{\mathbf{x}})$ is the total number of such D_α, not variables x_j with $\bar{x}_j \in \mathsf{D}_\alpha$!),

$$\mathrm{a}(\bar{\mathbf{x}}) = (\mathrm{a}^{(1)}(\bar{\mathbf{x}}), \mathrm{a}^{(2)}(\bar{\mathbf{x}}), \dots, \mathrm{a}^{(m)}(\bar{\mathbf{x}}), \dots). \tag{2.48}$$

As $\dim(\mathsf{D}) < \infty$, it is a finite sequence, therefore, it belongs to the well-ordered set $\mathbb{N}_0^\infty$. Of course, for some natural m simple Lie algebras of dimension m do not exist and, therefore, always $\mathrm{a}^{(m)} = 0$.

At the next stage we count the number of variables x_j with $\bar{x}_j \in \mathsf{N}$, the number of these is bounded by the nilpotency index of the radical of $\mathsf{En}(\mathcal{A})$. Actually, $\mathsf{En}(\mathcal{A})$ is the envelope of $\nu(L)$ and by (2.10)-(2.12) $\nu(\mathsf{N})$ lies in the solvable radical of the commutator subalgebra of $\nu(L)$, which is contained in the radical of $\mathsf{En}(\mathcal{A})$.

Next, let $\mathrm{b}^{(m,0)}(\bar{\mathbf{x}})$ denote the number of $\bar{x}_j \in \mathsf{W}_{i,\beta}$, where $\mathsf{W}_{i,\beta}$ is a standard $\bar{L}$- module with

$$\dim(\mathrm{ad}_\beta(\bar{L})) = m \ . \tag{2.49}$$

Also, let $\mathrm{b}^{(m,l)}(\bar{\mathbf{x}})$ be the number of $\bar{x}_j \in \mathsf{W}_{i,\beta}$ for a nonstandard Lie submodule $\mathsf{W}_{i,\beta}$ of dimension l satisfying (2.49). Put

$$\mathrm{b}^{(m)}(\bar{\mathbf{x}}) = (\mathrm{b}^{(m,0)}(\bar{\mathbf{x}}), \mathrm{b}^{(m,1)}(\bar{\mathbf{x}}), \dots, \mathrm{b}^{(m,l)}(\bar{\mathbf{x}}), \dots) \in \mathbb{N}_0^\infty; \tag{2.50}$$

$$\mathrm{b}(\bar{\mathbf{x}}) = (\mathrm{b}^{(1)}(\bar{\mathbf{x}}), \mathrm{b}^{(2)}(\bar{\mathbf{x}}), \dots, \mathrm{b}^{(m)}(\bar{\mathbf{x}}), \dots), \tag{2.51}$$

it is a element of the well-ordered set $(\mathbb{N}_0^\infty)^\infty$.

For $\mathrm{a} \in \mathbb{N}_0^\infty$,

$$\mathrm{a} = (\mathrm{a}^{(1)}, \mathrm{a}^{(2)}, \dots , \mathrm{a}^{(m)}, \dots) \ , \tag{2.52}$$

put

$$\|\mathrm{a}\| = \sum_{m=1}^{\infty} m \cdot \mathrm{a}^{(m)} \ ; \tag{2.53}$$

also, if

$$\mathrm{b} = (\mathrm{b}^{(1)}, \mathrm{b}^{(2)}, \dots , \mathrm{b}^{(m)}, \dots) \in (\mathbb{N}_0^\infty)^\infty, \tag{2.54}$$

where

$$\mathrm{b}^{(m)} = (\mathrm{b}^{(m,0)}, \mathrm{b}^{(m,1)}, \dots , \mathrm{b}^{(m,l)}, \dots) \ ,$$

then put

$$|\mathrm{b}| = \sum_{m=1}^{\infty} \sum_{l=0}^{\infty} \mathrm{b}^{(m,l)} \ , \tag{2.55}$$

Thus, $|\mathrm{b}(\bar{\mathbf{x}})|$ is the total number of occurences of elements from W (see lemma 2.5), and $\|\mathrm{a}(\bar{\mathbf{x}})\|$ is equal to the total dimension of simple components $\mathsf{D}_{i,\alpha}$ with $[\bar{x}_j, \mathsf{D}_{i,\alpha}] \neq 0$ for some x_j.

Since $\|\mathrm{a}(\bar{\mathbf{x}})\|$ is not greater than the dimension of the semisimple part D, the set $\{\mathrm{a}(\bar{\mathbf{x}}) | \ \bar{\mathbf{x}} \in \mathsf{V}(\mathcal{A}, \Gamma)\}$ is finite; consider the subset of such elements with maximal $\|\mathrm{a}(\bar{\mathbf{x}})\|$ and let $\mathrm{a}(\mathcal{A}, \Gamma)$ denote the maximal element. By definition, we have

$$\|\mathrm{a}(\bar{\mathbf{x}})\| \leq \|\mathrm{a}(\mathcal{A}, \Gamma)\| \leq \dim_F(\mathsf{D}) \ . \tag{2.56}$$

Similarly, $|\mathrm{b}(\bar{\mathbf{x}})|$ does not exceed the nilpotency index of the radical of $\mathsf{En}(\mathcal{A})$. Hence, the set $\{\mathrm{b}(\bar{\mathbf{x}}) | \ \bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})\}$ is finite; we denote by $\mathrm{b}(\Gamma, \mathcal{A})$ the greatest element.

Now, let a, b be elements of form (2.52), (2.54) respectively; also, let $c \in \mathbb{N}$. We are going to define a T-ideal

$$\mathcal{S}_{\mathrm{a},\mathrm{b},c}(\Gamma) \subseteq \Gamma \ . \tag{2.57}$$

In fact, it will be an operator on the set of T-ideals; we will write down it in terms of operators in $E(X)$, see (2.16).

First of all, if $\|\mathrm{a}\| = |\mathrm{b}\| = 0$, then we put (2.57) equal Γ. Next, suppose $\|\mathrm{a}\| \neq 0$ or $|\mathrm{b}| \neq 0$.

Step 1. Take a multilinear polynomial $f \in \Gamma$ with a set $\vartheta \subseteq \mathrm{varb}(f)$. Formally, in terms of $E(X)$, it means that $f \in \tau \cdot \Gamma$, where τ is the projector on a homogeneous component of $V\langle X\rangle$ consistsing of multilinear polynomials of degree 1 with respect to every $x \in \vartheta$.

Step 2. For all $x \in \vartheta$ take multilinear $f_x \in \mathrm{L}_i\langle X\rangle$ such that $\mathrm{varb}(f_x) \cap \mathrm{varb}(f_y) = \emptyset$, where $x \neq y$, and $\bigcup_{x\in\vartheta} \mathrm{varb}(f_x)$ contains a family of disjoint subsets Θ_t, $t \in \{0, \dots , |\mathrm{b}|\}$, where

$$|\Theta_0| = \|\mathrm{a}\|, \tag{2.58}$$

$$|\Theta_t| = \|\mathrm{a}\| + 1 \tag{2.59}$$

for all $t \geq 1$.

Next, for all $x \in \vartheta$ take multilinear

$$g_x \in \sum_{p \geq c} [f_x, \underbrace{\mathrm{L}_i\langle X\rangle, \dots , \mathrm{L}_i\langle X\rangle}_{p}] \tag{2.60}$$

with disjoint sets of variables $\mathrm{varb}(g_x)$, such that the polynomial

$$f_1 = f|_{\forall x = g_x}$$

is multilinear.

Since $\|\mathrm{a}\| \neq 0$ or $|\mathrm{b}| \neq 0$, ϑ must be nonempty, because $\Theta_0 \neq \emptyset$ or $\Theta_1 \neq \emptyset$.

Alternate f with respect to variables of Θ_t for each t, and let h_0 denote the resulting polynomial. More precisely, let G be the product of groups (in the group $\mathrm{Sym}(X)$),

$$G = \prod_{t=0}^{|b|} \mathrm{Sym}(\Theta_t) \ , \tag{2.61}$$

it is the direct product, because the corresponding subsets are disjoint. It acts on f_1 by permutation of variables; observe that $\sigma \cdot f_1 \in \Gamma$ for all $\sigma \in G$ thanks to the specific form of every g_x: since $c \geq 1$, g_x depends on some $y \in X_i$ (where $x \in X_i$) which is fixed under the action of G. So, $h_0 = \mathcal{S}_G \cdot f_1$, where

$$\mathcal{S}_G = \sum_{\sigma \in G} \mathrm{sgn}(\sigma)\sigma \ , \tag{2.62}$$

although $\mathcal{S}_G \notin E(X)$, its composition with the substitution $x \to g_x$ and the projector τ from Step 1 does belong to $E(x)$.

Step 3.

The element $\mathrm{a} \in \mathbb{N}_0^\infty$ defines a partition of Θ_0: we choose a sequence of disjoint subsets, where the m-th term contains $m \cdot \mathrm{a}^{(m)}$ elements. Observe that the total number of elements is $\|\mathrm{a}\| = \sum_{m=1}^\infty m \cdot \mathrm{a}^{(m)}$, therefore it is a partition of Θ_0 indeed. Next, every variable x from the m-th subset is replaced with a multilinear Lie polynomial

$$\phi_x = \psi'_x \cdot \psi_x, \tag{2.63}$$

where ψ_x and ψ'_x are skew-symmetric with respect to m variables, and the total degree of ϕ_x is $\geq c$; clearly, we suppose that $\phi_x \in \mathrm{L}_i\langle X\rangle$ provided $x \in X_i$.

Now we make a similar substitution for elements in Θ_t, $t \geq 1$. First, we fix a variable $y_p \in \Theta_t$; then the order of $\Theta_t \backslash \{y_p\}$ is equal to $\|\mathrm{a}\|$, and we make a substitution of the same form as in the case of Θ_0. So, we need only to define the substitution for y_p.

The element b defines a partition of the set $\{1, \dots , |\mathrm{b}|\}$: we fix disjoint subsets $\eta^{(m,l)}$ such that $|\eta^{(m,l)}| = \mathrm{b}_i^{(m,l)}$ for all m, l. For every $p \in \eta^{(m,0)}$ we replace the fixed variable $y_p \in \Lambda_{i,p}$ with the polynomial ϕ_{y_p} of form (2.63). If $p \in \eta^{(m,l)}$, where $l \geq 1$, then y_p is replaced with a polynomial of the form

$$\phi_{y_p} = \psi'_{y_p}(\psi_{y_p}; E_1, \dots E_{l^2}, \dots) \ , \tag{2.64}$$

where ψ_{y_p} is skew-symmetric with respect to $m+1$ variables, and ψ'_{y_p} is skew-symmetric with respect to $E_1, \dots , E_{l^2}$. Also, the degree of ϕ_x is $\geq c$ and $\phi_{y_p} \in \mathrm{L}_i\langle X\rangle$ if $y_p \in X_i$.

In fact, the substitution $x \to \phi_x$ induces an endomorphism of the free g.Lie module $\mathcal{M}\langle X\rangle$; denote by $\varrho_{\mathrm{a,b},c}$ the corresponding endomorphism of $V\langle X\rangle$, so it belongs to $E(X)$ and

$$h = \varrho_{\mathrm{a,b},c}(h_0)\ ; \tag{2.65}$$

besides, we take ϕ_x depending on different variables, so h is multilinear.

Finally, let $\mathcal{S}_{\mathrm{a,b},c} \in E(X)$ denote the composition of $\tau \in E(X)$ from Step 1, $x \to g_x$, $\mathcal{S}_G$ and $\varrho_{\mathrm{a,b},c}$, then

$$h = \mathcal{S}_{\mathrm{a,b},c}(f)\ . \tag{2.66}$$

Note that for fixed $\mathrm{a}, \mathrm{b}, c$ we have a class of operators of such form; define (2.57) as the T-ideal generated by (2.66), where $\mathcal{S}_{\mathrm{a,b},c}$ ranges over this class. Since Γ is a T-ideal, (2.57) is contained in Γ. Also, an element of (2.66) is called *a typical generator* of (2.57); recall that it is multilinear.

LEMMA 2.18. *Let* $\mathrm{a} = \mathrm{a}(\Gamma, \mathcal{A})$ *and let* S *denote the set of all* $\mathrm{b} \in (\mathbb{N}_0^\infty)^\infty$ *such that for all* $c \in \mathbb{N}$ *the T-ideal (2.57) is not contained in* $T(\mathcal{A})$*. Then* S *is finite.*

Proof. Take all $c \geq 2$; there is a typical generator (2.66) not vanishing under some substitution $x \to \bar{x}$; we can assume that $\bar{\mathbf{x}} \in \mathsf{V}(h, \mathcal{A})$. Indeed, like in lemma 2.17, we can replace each $\bar{x}$ with a Lie polynomial h_x in elements from $\mathsf{Spl}(\mathbf{L})$. We can assume that h_x is the sum of multilinear ones depending on different variables, hence every homogeneous component of

$$h' = h|_{x=h_x\ \forall x}$$

is multilinear and obtained from h by replacing every x with a homogeneous component of h_x. From the definition of (2.66) it follows that all homogeneous components of h' are typical generators of (2.57) and one of these does not vanish under some $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$.

We use notation given above; consider a subset Θ_p, $p \geq 1$; under $x \to \bar{x}$ for all $y \in \Theta_p$ the polynomial ϕ_y takes value in $\mathsf{D} \cup \mathsf{N}$. Indeed, if there is $x \in \mathrm{varb}(\phi_y)$ such that $\bar{x} \in \mathsf{W}$, then $\bar{\phi}_y \in \mathsf{N}$; otherwise $\bar{x} \in \mathsf{D}$ for all $x \in \mathrm{varb}(\phi_y)$, because D commutes with C and the degree of ϕ_y is greater or equal to the nilpotency index of $\langle \mathsf{C}\rangle$.

Collect all $x \in \Theta_p$ such that $\bar{\phi}_x \in \mathsf{D}$, then, by definition of $\mathrm{a}(\bar{\mathbf{x}})$, they all belong to a subalgebra of D of dimension $\leq \|\mathrm{a}(\bar{\mathbf{x}})\| \leq \|\mathrm{a}(\Gamma, \mathcal{A})\| = \|\mathrm{a}\|$ thanks to (2.56). Since Θ_p has $\|\mathrm{a}\| + 1$ elements, there is $y \in \Theta_p$ such that $\bar{\phi}_y \in \mathsf{N}$. Hence, $|\mathrm{b}|$, the total number of the subsets Θ_p, cannot exceed the nilpotency index of the radical of $\mathsf{En}(\mathcal{A})$. Therefore, all coordinates of b are bounded. $\square$.

Put

$$\mathrm{t}(\Gamma, \mathcal{A}) = \max(S)\ , \tag{2.67}$$

where S is the set defined in the lemma above; if S is empty, then we let $\mathrm{t}(\Gamma, \mathcal{A})$ be equal to the least element of $(\mathbb{N}_0^\infty)^\infty$. From the definition we get immediately the following.

LEMMA 2.19. *Let* $\mathcal{B}$ *be a finite dimensional g.Lie module with a fixed canonical decomposition such that* $\Gamma \not\subseteq T(\mathcal{B})$*. Suppose that* $T(\mathcal{A}) \subseteq T(\mathcal{B})$ *and* $\mathrm{a}(\Gamma, \mathcal{B}) = \mathrm{a}(\Gamma, \mathcal{A})$*, then*

$$\mathrm{t}(\Gamma, \mathcal{B}) \leq \mathrm{t}(\Gamma, \mathcal{A})\ .$$

Next, take $\beta \in \mathrm{pr}(\mathsf{W})$, then $\beta \in \mathrm{pr}(\mathsf{W}_i)$ for some i; by definition, $\mathsf{W}_{i,\beta}$ is contained in the weight subspace $\mathsf{N}_i(\lambda)$, hence, there exists $l \in \mathbb{N}$ such that $\mathsf{W}_{i,\beta}^{[l]} = 0$; such a minimal l is called the *height* of $\mathsf{W}_{i,\beta}$ and denoted by $\mathrm{ht}(\mathsf{W}_{i,\beta})$. Also, if $\bar{x}_j \in \mathsf{W}$, then $\bar{x}_j \in \mathsf{W}_{i,\beta}$ (we consider $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$); put $\mathrm{ht}(\bar{x}_j) = \mathrm{ht}(\mathsf{W}_{i,\beta})$. Then $\mathrm{ht}(\bar{\mathbf{x}})$ is defined as the the sum of all $\mathrm{ht}(\bar{x}_j)$, $j \in \{1, \dots, n\}$, with $\bar{x}_j \in \mathsf{W}$. Since $\mathrm{ht}(\bar{\mathbf{x}})$ is not greater than $|\mathrm{b}(\bar{\mathbf{x}})| \cdot m$, where m is the maximum of heights of $\mathsf{W}_{i,\beta}$, $\beta \in \mathrm{pr}(\mathsf{W})$, we can put

$$\mathrm{ht}(\Gamma, \mathcal{A}) = max\{\mathrm{ht}(\bar{\mathbf{x}}) | \ \bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})\} \ . \tag{2.68}$$

Finally, put

$$\begin{aligned} &\mathrm{type}(\Gamma, \mathcal{A}) = \\ &= (\mathrm{ht}(\Gamma, \mathcal{A}) \ , \ \mathrm{b}(\Gamma, \mathcal{A}) \ , \ |\mathrm{b}(\Gamma, \mathcal{A})| \ , \ \mathrm{t}(\Gamma, \mathcal{A}) \ , \ \mathrm{a}(\Gamma, \mathcal{A}) \ , \ \|\mathrm{a}(\Gamma, \mathcal{A})\|) \ , \end{aligned} \tag{2.69}$$

it is an element of the following well-ordered set

$$\mathbb{N}_0 \times (\mathbb{N}_0^\infty)^\infty \times \mathbb{N}_0 \times (\mathbb{N}_0^\infty)^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0 \ . \tag{2.70}$$

By definition, we have

$$\mathrm{type}(\Gamma, \mathcal{A}) = \mathrm{type}(\Gamma + T(\mathcal{A}), \mathcal{A}) \ ; \tag{2.71}$$

besides, let $\Gamma_1 \subseteq \mathcal{Q}(\mathcal{A})$ be a T-ideal containing Γ, then it follows from the definition that

$$\mathrm{type}(\Gamma, \mathcal{A}) \leq \mathrm{type}(\Gamma_1, \mathcal{A}) \ . \tag{2.72}$$

So, the type of a "counter-example" $T(\mathcal{A}) \subset \Gamma$ is defined as $\mathrm{type}(\Gamma, \mathcal{A})$; since the definition depends on the canonical decomposition, we can choose one with respect to which the type takes the smallest value.

CHAPTER 3

Finite basis of identities in finite dimensional Lie algebras

In this chapter, using the induction on the type of "counter-example", we will prove theorem 2.2 and thereby the existence of a finite basis of identities of PI representations of Lie algebras over a field of characteristic zero.

1. Linearization of a "counter-example"

Recall that type$(\Gamma, \mathcal{A})$ is an element of the well-ordered set (2.69); hence, we can compare "counter-examples" from different classes of g.Lie modules (even over different algebraically closed fields). Let $T(\mathcal{A}) \subset \Gamma$ be a "counter-example" of the smallest type in some variety of g.Lie modules $\mathfrak{M}_{k,s,r}$ over an algebraically closed field F and our goal is to construct a "counter-example" of a smaller type in a variety $\mathfrak{M}_{k',s',r'}$ over an algebraically closed field extension of F.

In the first two sections we will replace the given "counter-example" with another one of some canonical form.

Recall that $\mathrm{b}(\mathcal{A}, \Gamma)$ is the maximum of $\mathrm{b}(\bar{\mathbf{x}})$, where $\bar{\mathbf{x}}$ runs over $\mathsf{V}(\mathcal{A}, \Gamma)$; to find $\mathrm{b}(\bar{\mathbf{x}})$ we count $\bar{x}_j$ in irreducible components of W_i, and it would be convenient to have them in different ones; this is the motivation of the following definition.

A finite dimensional g. Lie module is said to be *linearized*, if for all $\beta \in \mathrm{pr}(\mathsf{W})$ the square of the ideal of $\mathsf{En}(\mathcal{A})$ generated by $\nu(\mathsf{W}_\beta)$ is equal to 0.

Take $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$ and put

$$\pi(\bar{\mathbf{x}}) = \bigcup_{m \geq 1} \pi^{(m)}(\bar{\mathbf{x}}) \ , \tag{3.1}$$

it is the set of all projectors $\alpha \in \mathrm{pr}(\mathsf{D})$ such that $[\mathsf{D}_\alpha, \bar{x}] \neq 0$ for some $x \in \mathrm{varb}(f)$. Also, let $\pi_i(\bar{\mathbf{x}}) = \pi(\bar{\mathbf{x}}) \cap \mathrm{pr}(\mathsf{D}_i)$. Next, let $\omega_i(\bar{\mathbf{x}})$ be the collection of all projectors $\beta \in \mathrm{pr}(\mathsf{W}_i)$ such that $\bar{x} \in \mathsf{W}_{i,\beta}$ for some $x \in \mathrm{varb}(f)$. Also, for each $\beta \in \omega_i(\bar{\mathbf{x}})$ put $m_\beta = m_\beta(\bar{\mathbf{x}})$ to be the number of such variables in $\mathrm{varb}_i(f)$; if $\mathcal{A}$ is linearized, then $m_\beta = 1$ for all β. Finally, set $\omega(\bar{\mathbf{x}}) = \bigcup_{i=1}^{k} \omega_i(\bar{\mathbf{x}})$.

PROPOSITION 3.1. *There exists a "counter-example" $T(\mathcal{A}) \subset \Gamma$ of the minimal type such that*

(1) *$\mathcal{A}$ is linearized;*
(2) *there exists $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$ such that*

$$\pi(\bar{\mathbf{x}}) = \mathrm{pr}(\mathsf{D}) \ , \quad \omega(\bar{\mathbf{x}}) = \mathrm{pr}(\mathsf{W}) \ . \tag{3.2}$$

Proof. Take $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$ and put

$$\xi_i(\bar{\mathbf{x}}) = \pi_i(\bar{\mathbf{x}}) \cup \{(\beta, m_\beta) |\ \beta \in \omega_i(\bar{\mathbf{x}})\} \ ,$$

where (β, m_β) means a pair of elements; then $\xi(\bar{\mathbf{x}})$ denotes the k-tuple $(\xi_1(\bar{\mathbf{x}}), \dots, \xi_k(\bar{\mathbf{x}}))$. Obviously, the set

$$J = \{\xi |\ \exists \bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})\ ,\ \xi = \xi(\bar{\mathbf{x}})\}$$

is finite.

LEMMA 3.1. *For each $\xi \in J$ there exists a linearized finite dimensional g.Lie module*

$$\mathcal{A}^\xi = (\mathbf{L}^\xi; V^\xi)$$

such that

$$T(\mathcal{A}) = \Gamma \cap \bigcap_{\xi \in J} T(\mathcal{A}^\xi) \tag{3.3}$$

and for all $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$ if $\xi = \xi(\bar{\mathbf{x}})$, then for all i

$$\mathsf{D}_i(L^\xi) \simeq \bigoplus_{\alpha \in \pi_i(\bar{\mathbf{x}})} \mathsf{D}_{i,\alpha}\ , \tag{3.4}$$

$$\mathsf{W}_i(L^\xi) \simeq \bigoplus_{\beta \in \omega_i(\bar{\mathbf{x}})} m_\beta \mathsf{W}_{i,\beta}\ . \tag{3.5}$$

Proof. Let P_i denote the sum of $\mathsf{D}_{i,\alpha}$, where $\alpha \in \pi_i(\bar{\mathbf{x}})$; also, put $P = \sum_{i=1}^k P_i \subseteq \mathsf{D}$. Take $\beta \in \omega_i(\bar{\mathbf{x}})$ and for every $j \in \{1, \dots, m_\beta\}$ denote by $U_{i,\beta,j}$ the j-th copy of $\mathsf{W}_{i,\beta}$, so

$$U_{i,\beta} = \underbrace{\mathsf{W}_{i,\beta} \bigoplus \dots \bigoplus \mathsf{W}_{i,\beta}}_{m_\beta}$$

is equal to the direct sum of $U_{i,\beta,j}$, $j = 1, \dots, m_\beta$. Finally, let

$$U_i = \bigoplus_{\beta \in \omega_i(\bar{\mathbf{x}})} U_{i,\beta}$$

and $U = \bigoplus_{i=1}^k U_i$.

Consider the k-sorted subalgebra $\mathbf{Q} = (Q; Q_1, \dots, Q_k)$ generated by $\bigcup_{i=1}^k (P_i + \mathsf{C}_i)$, then $Q = P + \langle \mathsf{C} \rangle$ and Q_i is the sum of P_i and the ideal of $\langle C \rangle$ generated by C_i. The Q-submodule generated by $\mathsf{W}_{i,\beta}$ is equal to $\mathsf{W}_{i,\beta}^{[0]}$; denote by $\tilde{U}_{i,\beta,j}$ the j-th copy of this module, it is generated by $U_{i,\beta,j} \subseteq \tilde{U}_{i,\beta,j}$. Furthermore, fix a basis $Z_{i,\beta,j}$ of $U_{i,\beta,j}$; then $\tilde{U}_{i,\beta,j}$ has a basis $\tilde{Z}_{i,\beta,j}$ equal to a disjoint union of $Z_{i,\beta,j}$ and subsets of the form

$$Z_{i,\beta,j}(\mathrm{ad}(c_1) - \lambda(c_1)) \dots \mathrm{ad}(c_l) - \lambda(c_l)) \subseteq \tilde{U}_{i,\beta}^{\{l\}}\ . \tag{3.6}$$

Put

$$\tilde{U}_{i,\beta} = \bigoplus_{j=1}^{m_\beta} \tilde{U}_{i,\beta,j}\ , \quad \tilde{U}_i = \bigoplus_{\beta \in \omega_i(\bar{\mathbf{x}})} \tilde{U}_{i,\beta}\ , \quad \tilde{U} = \bigoplus_{i=1}^{k} \tilde{U}_i\ ;$$

then

$$\tilde{Z}_i = \bigcup_{\beta \in \omega_i(\bar{\mathbf{x}})} \bigcup_{j=1}^{m_\beta} \tilde{Z}_{i,\beta,j}$$

is a basis of $\tilde{U}_i$ and $\tilde{Z} = \bigcup_{i=1}^k \tilde{Z}_i$ is that of $\tilde{U}$.

Consider the free k-sorted Lie algebra $\mathbf{L}\langle \tilde{Z} \rangle$ with $\tilde{Z}$ as the set of free generators; then $\tilde{U} = \mathrm{vect}(\tilde{Z}) \subseteq \mathrm{L}\langle \tilde{Z} \rangle$. We can extend the action of Q from $\tilde{U}$ to $\mathrm{L}\langle \tilde{Z} \rangle$, where every element of Q defines a derivation. Hence, the direct sum of vector spaces $M = Q \oplus \mathrm{L}\langle \tilde{Z} \rangle$ is equipped with a Lie algebra structure, where Q is a subalgebra and $\mathrm{L}\langle \tilde{Z} \rangle$ is a Q-module (therefore, $\mathrm{L}\langle \tilde{Z} \rangle$ is an ideal of M). Also note that U and Q generate M as a Lie algebra.

Let $M(Z_{i,\beta,j}, t)$ denote the subspace of M of all elements homogeneous with respect to $\tilde{Z}_{i,\beta,j}$ of total degree t. In other words, $f \in M(Z_{i,\beta,j}, t)$ iff for all $\gamma \in F$ we have $f = \gamma^t f$ under replacing every basis element $z \in \tilde{Z}_{i,\beta,j}$ with γz. In fact, this mapping is an endomorphism of the Lie algebra M; actually, the simulteneous scaling of all basis vectors from $Z_{i,\beta,j}$ induces an endomorphism of the free Lie algebra $\mathrm{L}\langle \tilde{Z} \rangle$ which agrees with the Q-module structure (i.e., it is a endomorphism over Q). Hence, every homogeneous component is a Q-submodule.

Next, let $q \in \mathbb{N}$, $q \geq 2$ such that $\mathsf{N}^q = 0$, then $\mathrm{L}\langle \tilde{Z} \rangle^q$ is an ideal of M; put

$$\bar{M} = M / \mathrm{L}\langle \tilde{Z} \rangle^q \ .$$

Notice that $\bar{M}$ inherits the homogeneity with respect to each $Z_{i,\beta,j}$. Besides, $q \geq 2$, hence this canonical homomorphism is monic on Q and $\tilde{U}$, so we can identify them with their images in $\bar{M}$. Let $\bar{M}_i$ denote the ideal generated by Q_i and $\tilde{U}_i$, then $\bar{\mathbf{M}} = (\bar{M}; \bar{M}_1, \dots, \bar{M}_k)$ is a k-sorted Lie algebra.

By definition, $P \subseteq Q$ is a maximal semisimple subalgebra of $\bar{M}$ and the solvable radical $\mathsf{S}(\bar{M})$ is generated by $\tilde{U}$ and C; in particular, $\bar{M}$ is finite dimensional. Next, by lemma 2.5, $\mathsf{W}_{i,\beta} \simeq U_{i,\beta,j}$ is a nontrivial L-module; recall that P includes all simple parts acting nontrivially on $\mathsf{W}_{i,\beta}$, hence $\tilde{U}$ is contained in the commutator subalgebra $[\bar{M}, \bar{M}]$. Hence, $\mathsf{N}(\bar{M}) = \mathsf{S}([\bar{M}, \bar{M}])$ is generated as an algebra by U and $\langle \mathsf{C} \rangle^p$, $p \geq 2$; in particular, $\bar{\mathbf{M}}$ satisfies (2.20).

Observe that P_i is a maximal semisimple subalgebra of M_i and the solvable radical of M_i is equal to $\mathrm{L}_i\langle \tilde{Z} \rangle / \mathrm{L}\langle \tilde{Z} \rangle^q$; therefore, we can put

$$\mathsf{D}_i(\bar{M}) = P_i \ .$$

Let N_i' denote the sum of $\langle \mathsf{C}_i \rangle^2$, $\mathsf{I}_i(\bar{\mathbf{M}}) = \sum_{j \neq i} [\bar{M}_i, \bar{M}_j]$ and the subalgebra generated by $\tilde{U}_i$. By lemma 2.5, $\mathsf{N}_i(\bar{\mathbf{M}})$ contains $\tilde{U}_i$, hence, $N_i' \subseteq \mathsf{N}_i(\bar{\mathbf{M}})$ and therefore, $\mathsf{S}(M_i) = \mathsf{N}_i(\bar{\mathbf{M}}) + \mathsf{C}_i = N_i' + \mathsf{C}_i$. On the other hand, $\mathsf{N}_i(\bar{\mathbf{M}}) \cap \mathsf{C}_i = (\mathsf{C}_i, \mathsf{C}) \cap \mathsf{C}_i = 0$, hence, $\mathsf{N}_i(\bar{\mathbf{M}}) = N_i'$ and C_i completes $\mathsf{N}_i(\bar{\mathbf{M}})$ to $\mathsf{S}(\bar{M}_i)$; thus, we can put

$$\mathsf{C}_i(\bar{\mathbf{M}}) = C_i \ .$$

Finally, U_i completes

$$\mathsf{M}_i(\bar{\mathbf{M}}) = \mathsf{N}_i(\bar{\mathbf{M}})^{[1]} + \langle \mathsf{C}_i \rangle^2 + \mathsf{N}_i(\bar{\mathbf{M}})^2 + \mathsf{I}_i(\bar{\mathbf{M}})$$

to $\mathsf{N}_i(\bar{\mathbf{M}})$, and we put

$$\mathsf{W}_i(\bar{\mathbf{M}}) = U_i \ .$$

Thus,

$$\mathsf{Gn}_i(\bar{\mathbf{M}}) = P_i + \mathsf{C}_i + U_i \ . \tag{3.7}$$

Now, let B denote the universal envelope $\mathsf{U}(\bar{\mathbf{M}}, s)$; since every endomorphism of $\bar{M}$ induces that of B, the algebra B is homogeneous with respect to all $\tilde{Z}_{i,\beta,j}$. Let $V(\bar{\mathbf{M}}) =$

$\sum_{p=1}^{r} \mathrm{v}_p \cdot B$ be the free module over B generated by $\mathrm{v}_1, \dots, \mathrm{v}_r$, then

$$(\bar{\mathbf{M}}; V(\bar{\mathbf{M}}); \nu; , \mathrm{v}_1, \dots, \mathrm{v}_r)$$

is a g.Lie module. We can define a homomorphism $\phi : \mathbf{M} \to \mathbf{L}$, which is the identity mapping on Q and its restriction to $\tilde{U}_{i,\beta,j}$ is the isomorphism onto $\mathsf{W}_{i,\beta}^{[0]}$; then ϕ extends to a homomorphism τ from B to the envelope $\mathsf{En}(\mathcal{A})$ and, therefore, induces a homomorphism (ϕ, ψ) from the g.Lie module $(\bar{\mathbf{M}}; V(\bar{\mathbf{M}}))$ to $\mathcal{A}$.

Let V_0 denote the maximal submodule of $V(\bar{\mathbf{M}})$ over B homogeneous with respect to each $Z_{i,\beta,j}$; put

$$\mathcal{B} = (\bar{\mathbf{M}}, \bar{V}), \tag{3.8}$$

where $\bar{V} = V(\bar{\mathbf{M}})/V_0$.

LEMMA 3.2. *$T(\mathcal{B}) \subseteq T(\mathcal{A})$.*

Proof. Pick a multilinear $f = f(x_1, \dots, x_n) \in T(\mathcal{A})$ and suppose $\bar{f} \notin V_0$ under some substitution $x \to \bar{x} \in \bar{\mathbf{M}}$. Since f is multilinear, we may assume that all $\bar{x}_l$ are homogeneous with respect to each $\tilde{Z}_{i,\beta,j}$, hence $\bar{f}$ is homogeneous too. Also,

$$\psi(\bar{f}) = f(\phi(\bar{x}_1), \dots, \phi(\bar{x}_n)) = 0$$

because $f \in T(\mathcal{A})$; hence, $\bar{f}$ must be in V_0. □

LEMMA 3.3. *$\bar{V}$ is finite dimensional.*

Proof. Consider the envelope $\mathsf{En}(\mathcal{B})$; we claim that it is finite dimensional. Indeed, let I_0 be the maximal ideal in the kernel of the homomorphism $\tau : B \to \mathsf{En}(\mathcal{A})$ which is homogeneous with respect to all $\tilde{Z}_{i,\beta,j}$. Obviously, I_0 is generated by all homogeneous elements from $\mathrm{Ker}(\tau)$; also, $\sum_{p+1}^{r} \mathrm{v}_p \cdot I_0 \subseteq V_0$, so it suffices to show that B/I_0 is finite dimensional.

First, $\mathsf{En}(Q, \nu) \subseteq B$ consists of elements of degree 0 with respect to all $\tilde{Z}_{i,\beta,j}$, hence $\mathrm{Ker}(\tau) \cap \mathsf{En}(Q, \nu) \subseteq I_0$ and $\mathsf{En}(Q, \nu)$ is isomorphic to a subalgebra of $\mathsf{En}(\mathcal{A})$, in particular, it is of finite dimension.

Next, B is spanned by elements of the form

$$a_0 b_1 a_1 \dots a_{p-1} b_p a_p \tag{3.9}$$

where $a_0, \dots, a_p$ belong to $\mathsf{En}(Q, \nu)$ and the elements $b_1, \dots, b_p \in \nu(\tilde{U})$ are homogeneous with respect to all $\tilde{Z}_{i,\beta,j}$ and of degree ≥ 1 with respect to one of them. If p is greater then the nilpotency index of the radical of $\mathsf{En}(\mathcal{A})$, then element (3.9) lies in the kernel of τ and therefore it belongs to I_0. Hence modulo I_0 the algebra B is the span of elements (3.9), where p is bounded, hence B is finite dimensional modulo I_0. □

By definition, the space $\bar{V}$ is homogeneous with respect to all $\tilde{Z}_{i,\beta,j}$; let V_1 denote the span of homogeneous elements with respect all $\tilde{Z}_{i,\beta,j}$ and of degree ≥ 2 with respect to one of them. Then V_1 is a submodule; put $V^{\xi} = \bar{V}/V_1$, $\mathbf{L}^{\xi} = \bar{\mathbf{M}}$ and

$$\mathcal{A}^{\xi} = (\mathbf{L}^{\xi}, V^{\xi}) \ .$$

Recall that (3.7) gives a canonical decomposition of $\mathbf{L}^{\xi}$; thanks to factorization modulo V_1, the finite dimensional g.Lie module $\mathcal{A}^{\xi}$ is linearized. Also, by definition we have (3.4), (3.5).

Obviously,

$$T(\mathcal{A}) \subseteq \Gamma \cap \bigcap_{\xi \in J} T(\mathcal{A}^\xi) \ ;$$

let Γ' denote the right hand side and suppose that $\Gamma' \neq T(\mathcal{A})$, then there exists a multilinear $f = f(x_1, \dots, x_n) \in \Gamma' \subseteq \Gamma$ and $\bar{\mathbf{x}} \in \mathsf{V}(f, \mathcal{A})$. We will prove that $f \notin T(\mathcal{A}^\xi)$ where $\xi = \xi(\bar{\mathbf{x}})$ (it will contradict to the choice of f).

To every $\bar{x}_j \in \mathsf{Spl}(\mathbf{L})$ let us define $\tilde{x}_j \in \mathsf{Spl}(\bar{\mathbf{M}})$:

(a) if $\bar{x}_j \in \mathsf{D}_{i,\alpha}$, then $\alpha \in \pi_i(\bar{\mathbf{x}})$ and therefore $\mathsf{D}_{i,\alpha} \subseteq P_i$; hence we can put $\tilde{x}_j = \bar{x}_j$;
(b) similarly, $\tilde{x}_j = \bar{x}_j$ if $\bar{x}_j \in \mathsf{C}_i$;
(c) suppose $\bar{x}_j \in \mathsf{W}_{i,\beta}$ and let $m = m_\beta$, then there are m variables $x_{j_1}, \dots, x_{j_m}$ such that $\bar{x}_{j_t} \in \mathsf{W}_{i,\beta}$; we put $\tilde{x}_{j_t} = \bar{x}_{j_t} \in U_{i,\beta,j}$.

The mapping $\tilde{x}_j \to \bar{x}_j$ extends to a homomorphism (ϕ, ψ) from $\mathcal{B}$ to $\mathcal{A}$, where $\psi(\tilde{f}) = \bar{f} \neq 0$. Also, $\tilde{f}$ is homogeneous of degree ≤ 1 with respect to every $\tilde{Z}_{i,\beta,j}$, hence $\tilde{f} \notin V_1$ and f does not vanish in $V^\xi = \bar{V}/V_1$ under the substitution $x_i \to \tilde{x}_i$. So, (3.3) holds and therefore lemma 3.1 is proven. □

We continue the proof of proposition 3.1. From (3.3) and lemma 2.15 it follows that there is a "counter-example" $T(\mathcal{A}^\xi) \subset \Gamma'$, where $\Gamma' \subseteq \Gamma + T(\mathcal{A}^\xi)$ and

$$(\Gamma')^\# \subseteq \Gamma^\# \subseteq \mathcal{Q}(\mathcal{A}) \subseteq \mathcal{Q}(\mathcal{A}^\xi) \ .$$

The g.Lie module $\mathcal{A}^\xi$ is linearized; let us show that

$$\mathrm{type}(\Gamma, \mathcal{A}^\xi) \leq \mathrm{type}(\Gamma, \mathcal{A}) \ . \tag{3.10}$$

Take $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$ such that $\xi = \xi(\bar{\mathbf{x}})$; by (3.2) we have

$$\|\mathrm{a}(\mathcal{A}^\xi, \Gamma)\| \leq \dim(\mathsf{D}^\xi) = \|\mathrm{a}(\bar{\mathbf{x}})\| \leq \|\mathrm{a}(\mathcal{A}, \Gamma)\|$$

and

$$\mathrm{a}(\mathcal{A}^\xi, \Gamma) \leq \mathrm{a}(\bar{\mathbf{x}}) \leq \mathrm{a}(\mathcal{A}, \Gamma) \ ,$$

without loss of generality we may assume that these are equalities. Similarly, $|\mathrm{b}(\mathcal{A}^\xi, \Gamma)| \leq |\mathrm{b}(\mathcal{A}, \Gamma)|$, $\mathrm{b}(\mathcal{A}^\xi, \Gamma) \leq \mathrm{b}(\mathcal{A}, \Gamma)$ and $\mathrm{ht}(\mathcal{A}^\xi, \Gamma) \leq \mathrm{ht}(\mathcal{A}, \Gamma)$.

Recall that $T(\mathcal{A}) \subseteq T(\mathcal{B}) \subseteq T(\mathcal{A}^\xi)$, hence, by lemma 2.19,

$$\mathrm{t}(\Gamma, \mathcal{A}^\xi) \leq \mathrm{t}(\Gamma, \mathcal{A}) \ ,$$

hence, (3.10) holds.

Finally, let us show that there exists $\bar{\mathbf{y}} \in \mathsf{V}(\Gamma', \mathcal{A}^\xi)$ such that

$$\pi(\bar{\mathbf{y}}) = \mathrm{pr}(\mathsf{D}^\xi) \ , \quad \omega(\bar{\mathbf{y}}) = \mathrm{pr}(\mathsf{W}^\xi) \ .$$

On the contrary, suppose it is untrue for all $\bar{\mathbf{y}} \in \mathsf{V}(\Gamma', \mathcal{A}^\xi)$, then

$$\|\mathrm{a}(\bar{\mathbf{y}})\| < \dim(\mathsf{D}^\xi) = \|\mathrm{a}(\bar{\mathbf{x}})\| \leq |\ a(\Gamma, \mathcal{A})\|$$

or

$$|\mathrm{b}(\bar{\mathbf{y}})| < |\mathrm{pr}(\mathsf{W}^\xi)| = |\mathrm{b}(\bar{\mathbf{x}})| \leq |b(\Gamma, \mathcal{A})| \ .$$

Hence, $\mathrm{type}(\Gamma', \mathcal{A}^\xi)$ must be lesser than that of $T(\mathcal{A}) \subset \Gamma$, which is impossible. So, replacing the given "counter-example" with $T(\mathcal{A}^\xi) \subset \Gamma'$ we get proposition 3.1. □

2. Irreducible "counter-example"

First, we are going to prove that the types of Γ and $\Gamma^{\#}$ with respect to $\mathcal{A}$ are equal; to this end we need the following statement.

LEMMA 3.4. *Let $T(\mathcal{A}) \subset \Gamma$ be a "counter-example" and let $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma^{\#}, \mathcal{A})$ such that $\pi(\bar{\mathbf{x}})$ or $\omega(\bar{\mathbf{x}})$ is not empty. Then there exists $\bar{\mathbf{y}} \in \mathsf{V}(\Gamma, \mathcal{A})$ such that*

$$\pi(\bar{\mathbf{x}}) = \pi(\bar{\mathbf{y}}) \ , \quad \omega(\bar{\mathbf{x}}) = \omega(\bar{\mathbf{y}}) \ . \tag{3.11}$$

Proof. Let $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathsf{V}(f, \mathcal{A})$ for some multilinear $f \in \Gamma^{\#}$; By hypothesis, there is $\bar{x}_j$ lying in a simple component $\mathsf{D}_{i,\alpha}$ or $\mathsf{W}_{i,\beta}$.

Consider the first case; the commutator subalgebra of $\mathsf{D}_{i,\alpha}$ is equal to $\mathsf{D}_{i,\alpha}$, hence $\bar{x}_j \in \mathsf{D}_{i,\alpha}$ can be represented as a Lie polynomial $g = g(x_{n+1}, \dots, x_m)$ from $[\mathrm{L}_i\langle X\rangle, \mathrm{L}\langle X\rangle]$ in some $\bar{x}_{n+1}, \dots, \bar{x}_m \in \mathsf{D}_{i,\alpha}$,

$$\bar{x}_j = g(\bar{x}_{n+1}, \dots, \bar{x}_m) \ .$$

Replacing x_j with g in f, we get $f_1 = f_1 \in \Gamma$ and

$$f_1(\bar{\mathbf{y}}) = f_1(\bar{x}_1, \dots, \bar{x}_{j-1}, \bar{x}_{j+1}, \dots, \bar{x}_n, \bar{x}_{n+1}, \dots, \bar{x}_m) = f(\bar{\mathbf{x}}) \neq 0 \ .$$

We can assume that g is the sum of multilinear polynomials, hence, we can replace g by one of them to make f_1 multilinear and therefore $\bar{\mathbf{y}} \in \mathsf{V}(\Gamma, \mathcal{A})$. Equalities (3.11) hold by definition.

The case $\bar{x}_j \in \mathsf{W}_{i,\alpha}$ is similar. First, if $\mathsf{W}_{i,\alpha}$ is a non-trivial D-module, then $[\mathsf{W}_{i,\alpha}, \mathsf{D}] = \mathsf{W}_{i,\alpha}$ and we make use the same argument as above. Otherwise, by lemma 2.5, $\mathsf{W}_{i,\alpha}$ is contained in a weight subspace $\mathsf{N}_i(\lambda)$ with $\lambda \neq 0$. Hence

$$\bar{x}_j \in \sum_{t \geq 1} [\mathsf{W}_{i,\alpha}, \underbrace{\mathsf{C}, \dots, \mathsf{C}}_{t}]$$

and we can use the same argument. □

COROLLARY 3.1. *Let $T(\mathcal{A}) \subset \Gamma$ be a "counter-example", then*

$$\mathrm{type}(\Gamma, \mathcal{A}) = \mathrm{type}(\Gamma^{\#}, \mathcal{A}) \ .$$

Proof. First, $\mathrm{type}(\Gamma, \mathcal{A}) \leq \mathrm{type}(\Gamma^{\#}, \mathcal{A})$ by (2.72). From the lemma above it folows that $\mathrm{a}(\Gamma^{\#}, \mathcal{A}) = \mathrm{a}(\Gamma, \mathcal{A})$, $\mathrm{b}(\Gamma^{\#}, \mathcal{A}) = \mathrm{b}(\Gamma, \mathcal{A})$ and $\mathrm{ht}(\Gamma^{\#}, \mathcal{A}) = \mathrm{ht}(\Gamma, \mathcal{A})$. So, thanks to lemma 2.19, we need to show only that

$$\mathrm{t}(\Gamma^{\#}, \mathcal{A}) \leq \mathrm{t}(\Gamma, \mathcal{A}) \ . \tag{3.12}$$

Put $\mathrm{a} = \mathrm{a}(\Gamma^{\#}, \mathcal{A})$ and let $\mathrm{b} = \mathrm{t}(\Gamma^{\#}, \mathcal{A})$, we can assume that $|\mathrm{b}| \neq 0$. Then for all $c \in \mathbb{N}$ (see Step 1 of the definition of (2.57))

$$\mathcal{S}_{\mathrm{a},\mathrm{b},c+1}(\Gamma^{\#}) \subseteq \mathcal{S}_{\mathrm{a},\mathrm{b},c}(\Gamma)$$

which yields (3.12). □

PROPOSITION 3.2. *There exists a "counter-example" $T(\mathcal{A}) \subset \Gamma$ of the smallest type such that $\mathcal{A}$ is a linearized g.Lie module and (3.2) holds for all $\bar{\mathbf{x}} \in \mathsf{V}(\mathcal{A}, \Gamma)$.*

Proof. We need g.Lie submodules of $\mathcal{A}$ of a special form.

Take $\alpha \in \mathrm{pr}(\mathsf{D})$; there is $l \in \{1, \dots, k\}$ such that $\alpha \in \mathrm{pr}(\mathsf{D}_l)$; let L_l^α be the ideal of L generated by all $\mathsf{D}_{l,\beta}$, $\beta \neq \alpha$ and $\mathsf{S}(L_l)$. Also, put $L_i^\alpha = L_i$ provided $i \neq l$ and $L^\alpha = \sum_{i=1}^k L_i^\alpha$; then $\mathbf{L}^\alpha = (L^\alpha; L_1^\alpha, \dots, L_k^\alpha)$ is a k-sorted subalgebra of $\mathbf{L}$. Consider the submodule $\mathcal{A}^\alpha = (\mathbf{L}^\alpha; V^\alpha)$ of $\mathcal{A}$ generated by $\mathbf{L}^\alpha$.

LEMMA 3.5. *For all* α *we have* $\mathrm{type}(\Gamma, \mathcal{A}^\alpha) < \mathrm{type}(\Gamma, \mathcal{A})$.

Proof. Recall there exists $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$ such that (3.2) takes place. Hence, we have $\|\mathrm{a}(\Gamma, \mathcal{A})\| = \dim(\mathsf{D})$ and therefore

$$\|\mathrm{a}(\Gamma, \mathcal{A}^\alpha)\| \leq \dim(\mathsf{D}^\alpha) < \dim(\mathsf{D}) = \|\mathrm{a}(\Gamma, \mathcal{A})\| \ .$$

□

Now take $\beta \in \mathrm{pr}(\mathsf{W})$, then $\beta \in \mathrm{pr}(\mathsf{W}_l)$ for some l; put

$$S^\beta = \mathsf{D}_l \cup \bigcup_{\gamma \neq \beta} \mathsf{W}_{l,\gamma} \cup \mathsf{W}_l^{[1]} \cup \mathsf{C}_l \tag{3.13}$$

and let $\mathbf{M}^\beta$ denote the k-sorted subalgebra of $\mathbf{L}$ generated by $S^\alpha \subseteq L_l$ and $\mathsf{Gn}_i(\mathbf{L})$, $i \neq l$. We are going to find a canonical decomposition for $\mathbf{M}^\beta$; for all i put

$$\mathsf{I}_i^\beta = \mathsf{I}_i(\mathbf{M}^\beta) = \sum_{j \neq i} [M_i^\beta, M_j^\beta] \ .$$

Take $i \neq l$ and let M_i be the sum of I_i^β and the subalgebra generated by $\mathsf{Gn}_i(\mathbf{L})$; then M_i is an ideal of M^β, hence $M_i^\beta = M_i$. By lemma 2.6 , $\mathsf{Gn}_i(\mathbf{L})$ generates the algebra L_i modulo I_i, hence we can put $\mathsf{Gn}_i(\mathbf{M}^\beta) = \mathsf{Gn}_i(\mathbf{L})$. Actually, D_i is a maximal semisimple subalgebra of M_i^β, hence we can put $\mathsf{D}_i(\mathbf{M}^\beta) = \mathsf{D}_i$. Modulo I_i^β the solvable radical $\mathsf{S}(M_i^\beta)$ is equal to that of $\langle \mathsf{Gn}_i(\mathbf{L}) \rangle$; by definition

$$\mathsf{N}_i(\mathbf{M}^\beta) = \mathsf{N}(\mathbf{M}^\beta) + \mathsf{I}_i^\beta = [\mathsf{M}_i^\beta, \mathsf{S}(\mathsf{M}_i^\beta)] + \mathsf{I}_i^\beta \ ,$$

hence, it contains $(\mathsf{C}_i, \mathsf{C})$ and, by lemma 2.5, $\mathsf{W}_i^{[0]}$. Hence, $\mathsf{S}(M_i^\beta) = \mathsf{N}_i(\mathbf{M}^\beta) + \mathsf{C}_i$; besides, $\mathsf{N}_i(\mathbf{M}^\beta) \subseteq \mathsf{N}_i$, hence the sum is direct, so we may set $\mathsf{C}_i(\mathbf{M}^\beta) = \mathsf{C}_i$.

Also, by definition

$$\mathsf{M}_i(\mathbf{M}^\beta) = \mathsf{N}_i(\mathbf{M}^\beta)^{[1]} + \mathsf{N}_i(\mathbf{M}^\beta)^2 + \langle \mathsf{C}_i \rangle^2 + \mathsf{I}_i^\beta \ ,$$

hence, attaching W_i, we get $\mathsf{N}_i(\mathbf{M}^\beta)$; besides $\mathsf{M}_i(\mathbf{M}^\beta)$ lies in M_i, hence we can put $\mathsf{W}_i(\mathbf{M}) = \mathsf{W}_i$.

Now consider M_l; let J_0 denote the subalgebra generated by

$$\sum_{\gamma \neq \beta} \mathsf{W}_{l,\gamma} + \mathsf{W}_l^{[1]} \ ,$$

then $S' = \langle \mathsf{C}_l \rangle + J_0 + \mathsf{I}_l^\beta$ is an ideal of M^β (it is closed under multiplication by S^β and $\mathsf{Gn}_i(\mathbf{L})$). Moreover, it is solvable and $J + \mathsf{D}_l = M_l^\beta$, hence, $\mathsf{D}_l(\mathbf{M}^\beta) = \mathsf{D}_l$ and S' is the solvable radical of M_l^β,

$$\mathsf{S}(M_l^\beta) = \langle \mathsf{C}_l \rangle + J_0 + \mathsf{I}_l^\beta \ .$$

Next, $J_0 \subseteq \mathsf{N}(M_l^\beta)$ by lemma 2.5, hence $\mathsf{N}_l(\mathbf{M}^\beta)$ includes

$$N' = \langle \mathsf{C}_i \rangle^2 + J_0 + \mathsf{I}_l^\beta \ ;$$

besides, $N' + \mathsf{C}_i = \mathsf{N}_l(\mathbf{M}^\beta) + \mathsf{C}_i = \mathsf{S}(M_l^\beta)$ and $\mathsf{N}_l(\mathbf{M}^\beta) \subseteq \mathsf{N}_l$, hence, $\mathsf{N}_l(\mathbf{M}^\beta) = N'$ and we can put $\mathsf{C}_l(\mathbf{M}^\beta) = \mathsf{C}_l$.

Now, let us find $\mathsf{W}_l(\mathbf{M}^\beta)$; by definition it is a D-module completing $\mathsf{M}_l(\mathbf{M}^\beta)$ to $\mathsf{N}_l(\mathbf{M}^\beta)$ with simple components lying in the weight subspaces with respect to $\langle \mathsf{C} \rangle$. Recall that

$$\mathsf{M}_l(\mathbf{M}^\beta) = \mathsf{N}_l(\mathbf{M}^\beta)^{[1]} + \mathsf{N}_l(\mathbf{M}^\beta)^2 + \langle \mathsf{C}_l \rangle^2 + \mathsf{I}_l^\beta \ ,$$

obviously it lies in M_l. We choose $\mathsf{W}_l(\mathbf{M}^\beta)$ as the sum of two parts: the complement $\mathsf{M}_l(\mathbf{M}^\beta)$ to $\mathsf{N}_l(\mathbf{M}^\beta) \cap \mathsf{M}_l$ and that of $\mathsf{N}_l(\mathbf{M}^\beta) \cap \mathsf{M}_l$ to $\mathsf{N}_l(\mathbf{M}^\beta)$.

Clearly, $\mathsf{N}_l(\mathbf{M}^\beta) \cap \mathsf{M}_l$ includes

$$M' = \mathsf{W}_l^{[1]} + J_0^2 + \langle \mathsf{C}_l \rangle^2 + \mathsf{I}_l^\beta \ ;$$

on the other hand,

$$M' + \sum_{\gamma \neq \beta} \mathsf{W}_{l,\gamma} = J_0 + \langle \mathsf{C}_l \rangle^2 + \mathsf{I}_l^\beta = \mathsf{N}_l(\mathbf{M}^\beta) \ ,$$

hence,

$$\sum_{\gamma \neq \beta} \mathsf{W}_{l,\gamma} \tag{3.14}$$

is a complement of $\mathsf{N}_l(\mathbf{M}^\beta) \cap \mathsf{M}_l$ to $\mathsf{N}_l(\mathbf{M}^\beta)$.

Next, all $\mathsf{W}_{l,\gamma}^{[1]}$ lie in $\mathsf{W}_l(\mathbf{M}^\beta)^{[1]}$, hence, the complement $\mathsf{M}_l(\mathbf{M}^\beta)$ to $\mathsf{N}_l(\mathbf{M}^\beta) \cap \mathsf{M}_l$ is the direct sum of several D-submodules of $\mathsf{W}_{l,\beta}^{\{1\}}$; they have the form

$$\mathsf{W}_{l,\beta} \cdot (\mathrm{ad}(c) - \lambda(c)) \tag{3.15}$$

for suitable $c \in \mathsf{C}$ and $\lambda \in \Lambda_l$, cf. (2.29). In particular, they are isomorphic to $\mathsf{W}_{l,\beta}$ as M^β-modules, but their height is less than that of $\mathsf{W}_{l,\beta}$.

Lemma 3.6. *Let $\mathcal{B}^\beta = (\mathbf{M}^\beta; V^\beta)$ be the g.Lie submodule of $\mathcal{A}$ generated by $\mathbf{M}^\beta$, then* $\mathrm{type}(\Gamma, \mathcal{B}^\beta) < type(\Gamma, \mathcal{A})$.

Proof. On the contrary, suppose it is untrue; since $\mathsf{D}(\mathbf{M}^\beta) = \mathsf{D}$,

$$\|\mathrm{a}(\Gamma, \mathcal{B}^\beta)\| \le \dim(\mathsf{D}(\mathbf{M}^\beta)) \le \|\mathrm{a}(\Gamma, \mathcal{A})\| \ ,$$

hence, by hypothesis this must be an equality. Similarly, $\mathrm{a}(\Gamma, \mathcal{B}^\beta) = \mathrm{a}(\Gamma, \mathcal{A})$ and by lemma 2.19, $\mathrm{t}(\Gamma, \mathcal{B}^\beta) = \mathrm{t}(\Gamma, \mathcal{A})$.

Next, let us show that $|\mathrm{b}(\Gamma, \mathcal{B}^\beta)| \le |\mathrm{b}(\Gamma, \mathcal{A})|$ and $\mathrm{b}(\Gamma, \mathcal{B}^\beta) \le \mathrm{b}(\Gamma, \mathcal{A})$. Pick $\bar{\mathbf{x}} \in \mathsf{V}(f, \mathcal{B}^\beta)$, where $f \in \Gamma$; each $\bar{x}_j$ is equal to a Lie polynomial h_j in some elements from $\mathsf{Spl}(\mathbf{L})$ (cf. lemma 2.17). Notice that $\mathsf{Spl}(L_i) = \mathsf{Spl}(\mathbf{M}_i^\beta)$ unless $i = l$, hence , $h_j = \bar{x}_j$. From the description of $\mathsf{Gn}_l(\mathbf{M}^\beta)$ given above the same holds for $x_j \in X_l$, unless $\bar{x}_j$ belongs to a module (3.15). Otherwise, $h_j = [\bar{y}_j, \bar{y}_j'] - \lambda \cdot \bar{y}_j$ where $\bar{y}_j \in \mathsf{W}^\beta$, $\bar{y}_j' \in \mathsf{C}$, and $\lambda = \lambda(\bar{y}_j')$. Hence, $f(\bar{\mathbf{x}})$ is equal to a sum of elements of the form $g(\bar{\mathbf{y}})$, where g is multilinear from Γ, $\bar{\mathbf{y}} \in \mathsf{V}(\Gamma, \mathcal{A})$ and

$$\mathrm{b}^{(m,n)}(\bar{\mathbf{y}}) = \mathrm{b}^{(m,n)}(\bar{\mathbf{x}})$$

for all m, n. Hence, $\mathrm{b}(\bar{\mathbf{y}}) = \mathrm{b}(\bar{\mathbf{x}}) = \mathrm{b}(\Gamma, \mathcal{A})$ thanks to (3.2); therefore

$$\mathrm{b}(\Gamma, \mathcal{A}) = \mathrm{b}(\Gamma, \mathcal{B}^\beta) \ .$$

Finally, by (3.2) there is $\bar{y}_j$ lying in $\mathsf{W}_{l,\beta}$, hence one of $\bar{x}_l$ is from a module (3.15) whose height is less than that of $\mathsf{W}_{l,\beta}$, hence $\mathrm{ht}(\bar{\mathbf{x}}) < \mathrm{ht}(\Gamma, \mathcal{A})$. □

We continue proving proposition 3.2. Put $\Upsilon = \Gamma \cap \Psi$, where

$$\Psi = \left(\bigcap_{\alpha \in \mathrm{pr}(\mathsf{D})} T(\mathcal{A}^\alpha) \right) \cap \left(\bigcap_{\beta \in \mathrm{pr}(\mathsf{W})} T(\mathcal{B}^\beta) \right) \ ;$$

we claim that Υ is critical with $\Upsilon^\# \subseteq \Psi$. Actually, if it were untrue, then by lemma 2.15 there would exist a "counter example" of the form $T(\mathcal{A}^\alpha) \subseteq \Gamma'$, where $\Gamma' \subseteq \Gamma + T(\mathcal{A}^\alpha)$ and $(\Gamma')^\# \subseteq \Gamma^\#$, or $T(\mathcal{B}^\beta) \subseteq \Gamma'$, where $\Gamma' \subseteq \Gamma + T(\mathcal{B}^\beta)$ and $(\Gamma')^\# \subseteq \Gamma^\#$. From lemma 3.5 and 3.6 it follows that its type is less than $\mathrm{type}(\Gamma, \mathcal{A})$ which is impossible.

So, we can assume that $\Gamma^\# \subseteq \Psi$; take $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma^\#, \mathcal{A})$ and suppose that (3.2) does not hold. First, if $\pi(\bar{\mathbf{x}}) \neq \mathrm{pr}(\mathsf{D})$, then there is $\alpha \in \mathrm{pr}(\mathsf{D}) \backslash \pi(\bar{\mathbf{x}})$ and $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A}^\alpha)$, which contradicts to the hypothesis $\Gamma \subseteq \Psi$. Similarly, if $\omega(\bar{\mathbf{x}}) \neq \mathrm{pr}(\mathsf{W})$, then $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{B}^\beta)$ for $\beta \in \mathrm{pr}(W) \backslash \omega(\bar{\mathbf{x}})$. □

From (3.2) and lemma 3.4 it follows that

$$\mathrm{a}(\bar{\mathbf{x}}) = \mathrm{a}(\Gamma, \mathcal{A}) \ , \quad \mathrm{b}(\bar{\mathbf{x}}) = \mathrm{b}(\Gamma, \mathcal{A}) \ , \quad \mathrm{ht}(\bar{\mathbf{x}}) = \mathrm{ht}(\Gamma, \mathcal{A}) \tag{3.16}$$

for all $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma^\#, \mathcal{A})$.

Let $\mathbf{L}$ be a k-sorted Lie algebra with a cononical decomposition; put

$$\mathsf{J}_i = \mathsf{J}_i(L) = \mathsf{W}_i^{[1]} + \sum_{t \geq 1} [\mathsf{W}_i^{[0]}, \underbrace{\mathsf{W}^{[0]}, \ldots, \mathsf{W}^{[0]}}_{t}] + \mathsf{I}_i \ ; \tag{3.17}$$

it is an ideal of L; by lemma 2.7,

$$L_i = \mathsf{D}_i + \langle C_i \rangle + \mathsf{W}_i + \mathsf{J}_i \ .$$

PROPOSITION 3.3. *There exists a "counter-example $T(\mathcal{A}) \subseteq \Gamma$ of the smallest type such that*

(1) *$\mathcal{A}$ is linearized ;*
(2) *relations (3.2) hold for all $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$;*
(3) *Let $f(\mathbf{x}) = f(x_1, \ldots, x_n) \in \Gamma^\#$ be multilinear; take some elements $\bar{x}_j \in \bigcup_{i=1}^k L_i$ where at least one of these, say $\bar{x}_p$, belongs to J_i for some i. Then $f(\bar{\mathbf{x}}) = 0$.*

Proof. Note that we have got already a "counter-example" of the smallest type satisfying (1) and (2). Take $\beta \in \mathrm{pr}(\mathsf{W})$, then $\beta \in \mathrm{pr}(\mathsf{W}_l)$ for some l. We define a submodule $\mathcal{C}^\beta = (\mathbf{N}^\beta; U^\beta)$ of $\mathcal{A}$, where $\mathbf{N}^\beta$ is the k-sorted sublagebra of $\mathbf{L}$ generated by $\mathsf{Gn}_i(\mathbf{L})$, where $i \neq l$, and a subset $T^\beta \subseteq L_l$, that is the union of D_l, C_l,

$$\bigcup_{\gamma \neq \beta} \mathsf{W}_{l,\gamma} \tag{3.18}$$

and

$$\bigcup_{t \geq 1} [\mathsf{W}_{l,\beta}^{[0]}, \underbrace{\mathsf{W}_l^{[0]}, \ldots, \mathsf{W}_l^{[0]}}_{t}] \ . \tag{3.19}$$

Like in lemma 3.6 one can define a canonical decomposition such that $\mathsf{Gn}_i(\mathbb{N}^\beta) = \mathsf{Gn}_i(\mathbf{L})$ provided $i \neq l$. Next, put $\mathsf{I}_l^\beta = \mathsf{I}_l(\mathbb{N}^\beta)$; clearly, D_l is a maximal semisimple subalgebra of N_l^β and the radical, $\mathsf{S}(N_l^\beta)$ is the sum of $\langle \mathsf{C}_l \rangle$, (3.18), (3.19) and I_l^β.

Next, $\mathsf{N}_l(\mathbf{N}^\beta)$ contains (3.18) by lemma 2.5; also, (3.19) is invariant under the action by $\langle\mathsf{C}\rangle$, hence all nonzero weight subspaces of (3.19) are contained in $\mathsf{N}_l(\mathbf{N}^\beta)$. Finally, the zero weight subspace of (3.19)is the direct sum of irreducible D-submodules; all non-trivial ones lie in $\mathsf{N}_l(\mathbf{N}^\beta)$, hence a complement of $\mathsf{N}_l(\mathbf{N}^\beta)$ to $\mathsf{S}(N_l^\beta)$ can be chosen as the sum of C_l (because $\mathsf{N}_l(\mathbf{N}^\beta) \subseteq \mathsf{N}_l$) and a trivial D-submodule C'_l from the zero weight subspace of (3.19). Obviously, $\mathsf{C}_l + \mathsf{C}'_l$ and $\sum_{i\neq l}\mathsf{C}_i$ generate a nilpotent subalgebra commuting with D, hence we may put $\mathsf{C}_l(\mathbf{N}^\beta) = \mathsf{C}_l + \mathsf{C}'_l$.

Now, $\mathsf{M}_l(\mathbf{N}^\beta)$ is contained in $\mathsf{M}_l(\mathbf{L})$ and includes all $\mathsf{W}^{[1]}_{l,\gamma}$ for $\gamma \neq \beta$, hence, in order to complete it to $\mathsf{N}_l(\mathbf{N}^\beta)$ we take the the sum of (3.18) and a number of irreducible D-submodules from weight subspaces of (3.19) with respect to $\langle\mathsf{C}(\mathbf{N}^\beta)\rangle$ (as a matter of fact, $\mathsf{C}'_l \subseteq \mathsf{N}_l$ acts nilpotently, hence they are just weight subspaces with respect to $\langle\mathsf{C}\rangle$).

Now we can prove an analog of the lemma above for $\mathcal{C}^\beta$.

LEMMA 3.7. *For all* β $\mathrm{type}(\Gamma, \mathcal{C}^\beta) < type(\Gamma, \mathcal{A})$.

Proof. The argument will be similar to the proof of lemma 3.6. We suppose that the statement is untrue; in absolutely the same way one can show that $\|\mathrm{a}(\Gamma, \mathcal{C}^\beta)\| = \|\mathrm{a}(\Gamma, \mathcal{A})\|$, $\mathrm{a}(\Gamma, \mathcal{C}^\beta) = \mathrm{a}(\Gamma, \mathcal{A})$ and $\mathrm{t}(\Gamma, \mathcal{C}^\beta) = \mathrm{t}(\Gamma, \mathcal{A})$.

Next, take $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathsf{V}(f, \mathcal{C}^\beta)$, where $f \in \Gamma$; then $\bar{x}_j = h_j$, where h_j is a Lie polynomial in elements from $\mathsf{Gn}_i(\mathbf{L})$. Moreover, $h_j = \bar{x}_j$ unless $x_j \in X_l$, otherwise $\bar{x}_j$ belongs to a simple component of $\mathsf{W}_i(\mathbf{N}^\beta)$ from (3.19). In the latter case h_j is a sum of multilinear Lie polynomials in elements from $\mathsf{Gn}_i(\mathbf{L})$, where at least two of them belong to W_l (moreover, one of them must be in $\mathsf{W}_{l,\beta}$). Hence, $f(\bar{\mathbf{x}})$ is equal to a sum of elements of the form $g(\bar{\mathbf{y}})$, where $g \in \Gamma$, $\bar{\mathbf{y}} \in \mathsf{V}(\Gamma, \mathcal{A})$, where

$$|\mathrm{b}(\bar{\mathbf{x}})| + 1 \leq |\mathrm{b}(\bar{\mathbf{y}})| \ .$$

Therefore $|\mathrm{b}(\Gamma, \mathcal{C}^\beta)| < |\mathrm{b}(\Gamma, \mathcal{A})|$, and the lemma is proven. □

Now put

$$\Upsilon = \Gamma \cap \bigcap_{\beta \in \mathrm{pr}(\mathsf{W})} T(\mathcal{B}^\beta) \cap T(\mathcal{C}^\beta) \ , \tag{3.20}$$

by lemma 2.15 Υ must be a critical T-ideal with an apropriate sequence contained in all $T(\mathcal{C}^\beta)$ and $T(\mathcal{B}^\beta)$ (otherwise, thanks to lemma 3.6 and 3.7, we would have a "counter-example" of lesser type).

So, we can assume that Γ coincides with Υ and

$$\Gamma^\# \subseteq \bigcap_{\beta \in \mathrm{pr}(\mathsf{W})} T(\mathcal{B}^\beta) \cap T(\mathcal{C}^\beta) \ ,$$

observe that this replacement is harmless for properties (1), (2), so we need only to check (3).

Suppose $f(\bar{\mathbf{x}}) \neq 0$; arguing like in lemma 2.17, we may replace all $\bar{x}_j$, $j \neq p$, with Lie polynomials in elements from $\mathsf{Spl}(\mathbf{L})$. Hence we may assume that all $\bar{x}_j \in \mathsf{Spl}(\mathbf{L})$ unless $j = p$. Moreover, since f is linear in x_p, the element $\bar{x}_p$ can be taken either from $\mathsf{W}^{[1]}_i$ or

$$[\mathsf{W}_i, \underbrace{\mathsf{W}^{[0]}, \dots, \mathsf{W}^{[0]}}_{t}] \ , \tag{3.21}$$

where $t \geq 1$.

Consider the first case; recall that $\mathsf{W}_i^{[1]}$ is the sum of subspaces

$$\mathsf{W}_{i,\beta} \cdot (\lambda(c_1) - \mathrm{ad}(c_1)) \dots (\lambda(c_p) - \mathrm{ad}(c_q))$$

where $q \geq 1$ and $c_1, \dots, c_q \in \mathsf{C}_i$, see (2.29); hence, we may assume that $\bar{x}_p$ belongs to such a subspace. Since $\mathcal{A}$ is linearized, there are no more $\bar{x}_j$ from $\mathsf{W}_{i,\beta}$; hence, all $\bar{x}_j$, including $j = p$, are elements in $\mathbf{M}^\beta$. It is impossible, because $\Gamma \subseteq T(\mathcal{B}^\beta)$ and $f(\bar{\mathbf{x}}) \neq 0$. Hence, $\bar{x}_p$ lies in (3.21), and we may assume that it is an element (3.19) for some $\beta \in \mathrm{pr}(\mathsf{W}_i)$; hence, all $\bar{x}_j$ belongs to $\mathbf{N}^\beta$ and $f(\bar{\mathbf{x}})$ must vanish. □

Annihilators.

We are going to minimize the set of values taken by polynomials in $\Gamma^\#$ under all possible substitutions.

LEMMA 3.8. *There exists a "counter-example" $T(\mathcal{A}) \subset \Gamma$ of the smallest type, satisfying (1)-(3) from proposition 3.3 and the following condition. Let $\mathcal{A} = (\mathbf{L}, V)$, then the verbal submodule*

$$\Gamma^\#[\mathcal{A}] = \{f(\bar{\mathbf{x}}) |\ f = f(x_1, \dots, x_n) \in \Gamma^\#;\ \bar{x}_j \in \mathbf{L}\}$$

is an irreducible submodule over $\mathsf{En}(\mathcal{A})$ and contained in any nonzero submodule of V.

Proof. Put $V_1 = \Gamma^\#[\mathcal{A}]$; we shall carry out the induction on $\dim(V_1)$. Consider the set of all nonzero submodules of V which do not contain V_1; if it is empty, then V_1 is irreducible and the lemma holds. Otherwise, take a maximal element from this set, V_0, and consider the quotient $\mathcal{A}' = (\mathbf{L}; V/V_0)$.

First, suppose that $\Upsilon = \Gamma \cap T(\mathcal{A}')$ has no an appropriate sequence in $T(\mathcal{A}')$, then, by lemma 2.3, $\Psi = \Gamma + T(\mathcal{A}')$ is critical with the appropriate sequence in $\Gamma^\#$, so $\Psi^\# = \Gamma^\# + T(\mathcal{A}')$. Observe that the first component of $\mathcal{A}$, the k-sorted Lie algebra $\mathbf{L}$ is just that of $\mathcal{A}'$, hence $T(\mathcal{A}') \subset \Psi$ satisfies (1)-(3) from proposition 3.3 and its type is the smallest one. Also, thanks to the maximality of V_0, the module $\Psi^\#[\mathcal{A}'] = \Gamma^\#[\mathcal{A}]/V_0 = V_1 + V_0/V_0$ is irreducible and any nonzero submodule of V/V_0 contains it.

Thus, we may assume that Υ is critical with $\Upsilon^\# \subseteq T(\mathcal{A}')$, hence, $T(\mathcal{A}) \subset \Upsilon$ is a"counter-example" of the smallest type satisfying (1)-(3) and $\Upsilon^\#[\mathcal{A}] \subseteq V_0 \cap \Gamma^\#[\mathcal{A}] \neq V_1$, hence, we can use the induction hypothesis. □

Thus, we assume that the "counter-example" satisfies the hypothesis of the lemma, we call it an *irreducible* one.

Let $M \subseteq L$, then the set of all $v \in V$

$$v \cdot \nu(M) = 0$$

is called the *annihilator of M* and denoted by $\mathrm{Ann}(M)$. Obviously, it is a subspace of V.

LEMMA 3.9. *The verbal submodule $\Gamma^\#[\mathcal{A}]$ is contained in* $\mathrm{Ann}(\mathsf{W})$.

Proof. Suppose it is untrue, then

$$g(\bar{x}_1, \dots, \bar{x}_{n+1}) = f(\bar{\mathbf{x}}) \cdot \mu(\bar{x}_{n+1}) \neq 0$$

for some $\bar{\mathbf{x}}$, $\mu \in \{\rho_l, \delta_l\}$ and $\bar{x}_{n+1} \in \mathsf{W}$. We may assume that f is multilinear and $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma^\#, \mathcal{A})$. Hence, g is a multilinear one from Γ and $\tilde{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_{n+1})$ is in $\mathsf{V}(\Gamma, \mathcal{A})$. Since $\bar{x}_{n+1} \in \mathsf{W}$, $|\mathrm{b}_i(\bar{\mathbf{x}})| = |\mathrm{b}_i(\tilde{\mathbf{x}})| - 1$, which contradicts to (2) of proposition 3.3. □

As a matter of fact, $\Gamma^{\#}[\mathcal{A}]$ annihilates the ideal of L generated by W; actually, $\Gamma^{\#}[\mathcal{A}]$ is a submodule of V over $\mathsf{En}(\mathcal{A})$, hence, if $a \in \mathsf{W}$ and $b \in L$, then by (2.10), (2.11) the operators $\rho_j([a,b])$, $\delta_j([a,b])$ act trivially on $\Gamma[\mathcal{A}]$.

LEMMA 3.10. *Let $\alpha \in \mathrm{pr}(\mathsf{D}_i)$ for some i, then $\mathrm{Ann}(\mathsf{D}_{i,\alpha}) \cap \Gamma^{\#}[\mathcal{A}]$ is a submodule of V over $\mathsf{En}(\mathbf{L},\nu)$.*

Proof. Let $v \in \mathrm{Ann}(\mathsf{D}_{i,\alpha}) \cap \Gamma^{\#}[\mathcal{A}]$, we need to show that for all $a \in L$ and $b \in \mathsf{D}_{i,\alpha}$

$$v \cdot \mu(a)\mu'(b) = 0 \tag{3.22}$$

where $\mu \in \{\rho_l, \delta_l\}$, $\mu' \in \{\rho_{l'}, \delta_{l'}\}$ First all, if $l \neq l'$, then it is trivial by (2.12). So, $\mu, \mu' \in \{\rho_l, \delta_l\}$ for some l.

We may assume that $a \in \mathsf{D}_\beta + \langle \mathsf{C} \rangle$ for some $\beta \neq \alpha$, because $\Gamma^{\#}[\mathcal{A}]$ is contained in the annihilator of the ideal generated by W. Hence, a commutes with $\mathsf{D}_{i,\alpha}$ and, thanks to (2.10), (2.11), we can swap $\mu(a)$ and $\mu'(b)$ and get (3.22). □.

Take $\alpha \in \mathrm{pr}(\mathsf{D}_i)$ and put

$$\mathrm{Ann}(\alpha) = \mathrm{Ann}(\mathsf{D}_{i,\alpha}) \cap \Gamma^{\#}[\mathcal{A}] ;$$

for $\Omega \subseteq \mathrm{pr}(\mathsf{D})$ put let

$$\mathrm{Ann}(\Omega) = \bigcap_{\alpha \in \Omega} \mathrm{Ann}(\alpha) ;$$

it is a submodule of V over $\mathsf{En}(\mathcal{A})$. Now let $\Omega = \Omega(\Gamma, \mathcal{A})$ denote the set of all $\alpha \in \mathrm{pr}(\mathsf{D})$ such that $\mathrm{Ann}(\alpha) \neq 0$; by lemma 3.8, $\Gamma^{\#}[\mathcal{A}]$ is contained in every $\mathrm{Ann}(\alpha)$, hence

$$\Gamma^{\#}[\mathcal{A}] \subseteq \mathrm{Ann}(\Omega) . \tag{3.23}$$

3. Finite dimensional embedding

In this section we consider the case

$$\mathrm{b}(\Gamma, \mathcal{A}) \leq \mathrm{t}(\Gamma, \mathcal{A}) .$$

PROPOSITION 3.4. *Let Υ denote the T-ideal (2.57) where,* $\mathrm{a} = \mathrm{a}(\Gamma, \mathcal{A})$*,* $\mathrm{b} \geq \mathrm{b}(\Gamma, \mathcal{A})$ *and c is the nilpotency index of $\langle \mathsf{C} \rangle$. Then there exists g.Lie module $\mathcal{A}'$ such that $T(\mathcal{A}') = \Upsilon + T(\mathcal{A})$ and $\mathcal{A}'$ is finite dimensional over a field extension of F.*

This statement is an analog of proposition 1.6. At the first stage we prove that Υ is the ideal of identities of a g.Lie module admitting a scalar extension by an associative commutaive affine algebra K, so the resulting g.Lie module satisfies certain finitness conditions. For we need some technical facts describing substitution $x \to \bar{x}$ under which the typical generators of (2.57) do not vanish.

First, take a multilinear $f = f(\mathbf{x}) = f(x_1, \ldots, x_n) \in \Gamma$ that is not an identity of $\mathcal{A}$. Since it is multilinear, $f(\bar{\mathbf{x}}) \neq 0$ under some substitution $x \to \bar{x}$, where

$$\bar{x} \in \mathsf{Spl}_i(\mathbf{L}) \cup \langle \mathsf{C}_i \rangle^2 \cup \mathsf{J}_i$$

provided $x \in X_i$; moreover, by proposition 3.3 $\bar{x} \in \mathsf{Spl}_i^+(\mathbf{L})$, where

$$\mathsf{Spl}_i^+(\mathbf{L}) = \mathsf{Spl}_i(\mathbf{L}) \cup \langle \mathsf{C}_i \rangle^2 . \tag{3.24}$$

This substitution $\bar{\mathbf{x}}$ looks almost like an element in $\mathsf{V}(\Gamma, \mathcal{A})$, the only difference is that some entries $\bar{x}$ may be in $\langle \mathsf{C}_i \rangle^2$. However, it is not important, when we count only representatives from simple components of D and W. Thus, we can define the sets $\pi(\bar{\mathbf{x}})$, $\omega(\bar{\mathbf{x}})$ like for elements of $\mathsf{V}(\Gamma, \mathcal{A})$, see (3.1); also, (3.2) holds for such $\bar{\mathbf{x}}$. Indeed, the variable x with $\bar{x} \in \langle \mathsf{C}_i \rangle^2$ can be replaced with a Lie polynomial from $\mathrm{L}\langle X_i \rangle$ in elements from C_i, so, $\bar{\mathbf{x}}$ can be replaced with $\bar{\mathbf{y}} \in \mathsf{V}(\Gamma, \mathcal{A})$ with $\pi(\bar{\mathbf{x}}) = \pi(\bar{\mathbf{y}})$ and $\omega(\bar{\mathbf{x}}) = \omega(\bar{\mathbf{y}})$.

Now let Γ' denote T-ideal (2.57), where $\mathrm{a} = \mathrm{a}(\Gamma, \mathcal{A})$, $\mathrm{b} \geq \mathrm{b}(\Gamma, \mathcal{A})$ and $\langle \mathsf{C} \rangle^c = 0$. Suppose $\Gamma \not\subseteq T(\mathcal{A})$; let $h(\mathbf{x}) = h(x_1, \dots, x_n)$ be a typical generator (2.66) of Γ' and let $x \to \bar{x}$ be a substitution such that $h(\bar{\mathbf{x}}) \neq 0$ and all $\bar{x}_j \in \mathsf{Spl}^+(\mathbf{L})$, where

$$\mathsf{Spl}^+(\mathbf{L}) = \bigcup_{i=1}^{k} \mathsf{Spl}^+(L_i).$$

The next statement follows immediately from the multiplication table for components of $\mathsf{Spl}_+(L_i)$.

Lemma 3.11. *Let $\phi \in \mathrm{L}\langle X_i \rangle$ and let $x \to \bar{x}$ be a substitution where $\bar{x} \in \mathsf{Spl}_i^+(\mathbf{L})$ and $\bar{\phi} \notin \mathsf{J}_i$. Then we have the following:*

(1) *$\bar{\phi} \in \langle \mathsf{C}_i \rangle + \mathsf{J}_i$ iff $\bar{x} \in \langle \mathsf{C}_i \rangle$ for all $x \in \mathrm{varb}(\phi)$;*
(2) *$\bar{\phi} \in \mathsf{D}_{i,\alpha} + \mathsf{J}_i$ iff $\bar{x} \in \mathsf{D}_{i,\alpha}$ for all $x \in \mathrm{varb}(\phi)$.*
(3) *$\bar{\phi}_x \in \mathsf{W}_{i,\beta} + \mathsf{J}_i$ iff one variable from $\mathrm{varb}(\phi)$ takes value in $\mathsf{W}_{i,\beta}$ and the others are in $\mathsf{D}_i \cup \mathsf{C}_i$ (note that $[\mathsf{W}_i, \langle \mathsf{C}_i \rangle^2] \subseteq \mathsf{J}_i$).*

Consider the Lie polynomials ϕ_x defined in (2.63), (2.64); under $x \to \bar{x}$ they takes values in $\mathsf{Spl}_+(L_i)$ modulo J_i, provided $x \in X_i$. Moreover, $\deg(\phi_x) \geq c$, hence, under $x \to \bar{x}$ ϕ_x takes value in $\mathsf{Spl}(L_i) \backslash \langle \mathsf{C}_i \rangle$ (modulo J_i), i.e., in simple components of D_i and W_i (modulo J_i):

$$\bar{\phi}_x \in \left(\bigcup_{\alpha \in \mathrm{pr}(\mathsf{D}_i)} \mathsf{D}_{i,\alpha} \right) \cup \left(\bigcup_{\beta \in \mathrm{pr}(\mathsf{W}_i)} \mathsf{W}_{i,\beta} \right) \tag{3.25}$$

Like in the associative case, we can describe the location of variables x_j with $\bar{x}_j$ in standard and non-standard submodules of W_i. In fact, we will get even more general statement showing in particular that $\mathrm{t}(\Gamma, \mathcal{A}) \leq \mathrm{b}(\Gamma, \mathcal{A})$ (see corollary 3.2). Here we use notations of the definition of (2.66); besides, put $\mathsf{J} = \sum_{i=1}^{k} \mathsf{J}_i$.

Lemma 3.12. *Suppose that $\Gamma' \not\subseteq T(\mathcal{A})$, then*

$$\mathrm{b} = \mathrm{b}(\Gamma, \mathcal{A}) \ . \tag{3.26}$$

Moreover, let h be a typical generator (2.66) of Γ' and suppose $\bar{h} \neq 0$ under $x \to \bar{x} \in \mathsf{Spl}_+(\mathbf{L})$, then for all $m, l \in \mathbb{N}$ the following statements hold:

$Q_{m,l}$: *take $p \in \eta^{(m,l)}$ and let y_p denote the fixed variable in Θ_p; then modulo J the value of ψ_{y_p} under $x \to \bar{x}$ belongs to a non-standard L-submodule W_β of dimension l satisfying (2.49);*

$Q_{m,0}$: *for all $p \in \eta^{(m,0)}$ there is a variable $x \in \Theta_p$ such that modulo J the value of ϕ_x under $x \to \bar{x}$ belongs to a standard L -submodule W_β satisfying (2.49).*

Proof. Put

$$\omega^{(m,l)} = \{\beta \in \mathrm{pr}(\mathsf{W}) |\ \mathsf{W}_\beta \ is\ non-standard,\ \dim(\mathrm{ad}_\beta(L)) = m;$$

$$\dim(\mathsf{W}_\beta) = l\} \ ,$$

$$\omega^{(m,0)} = \{\beta \in \mathrm{pr}(\mathsf{W}) |\ \mathsf{W}_\beta \ \textit{is standard},\ \dim(\mathrm{ad}_\beta(L)) = m\} \ ;$$

by (3.2), $|\omega^{(m,l)}| = \mathrm{b}^{(m,l)}(\bar{\mathbf{x}})$ for all m, l, hence, equality (3.26) will follow from

$$\mathrm{b}^{(m,l)} = \mathrm{b}^{(m,l)}(\bar{\mathbf{x}}) \ . \tag{3.27}$$

To prove it and $Q_{m,l}$ we carry out the induction on (m, l) with respect to the lexicographical order, where $(m, l) < (m', l')$ if $m < m'$ or $m = m'$ and $l < l'$.

The initial step. Since b is a finite sequence, there exists m_0 such that $\mathrm{b}^{(m,l)} = 0$ for all (m, l) such that $m \geq m_0$. By hypothesis, $\mathrm{b}(\Gamma, \mathcal{A}) \leq \mathrm{b}$, hence, $\mathrm{b}^{(m,l)}(\bar{\mathbf{x}}) \leq \mathrm{b}^{(m,l)} = 0$ for such m and (3.27) takes place. Also, $\eta^{(m,l)}$ is empty, hence $Q_{m,l}$ trivially holds.

Now, suppose that $Q_{m,l}$ and (3.27) are true for all $m' > m$, we need to check these for $m = m'$ and for all l. There exists l_0 such that $\mathrm{b}^{(m,l)} = 0$ for all $l \geq l_0$; since $\mathrm{b}^{(m',l)}(\bar{\mathbf{x}}) = \mathrm{b}^{(m',l)}$ for all l and $\mathrm{b}(\Gamma, \mathcal{A}) \leq \mathrm{b}$, we get $\mathrm{b}^{(m,l)}(\bar{\mathbf{x}}) = \mathrm{b}^{(m,l)}$ for all $l \geq l_0$. Since $\eta^{(m,l)} = \emptyset$, we get $Q_{m,l}$ too.

So, we can suppose that $Q_{m,l}$ and (3.27) hold for all $(m', l') > (m, l)$, we need to check these for (m, l). By the induction hypothesis, for all $p \in \eta^{(m',l')}$ there exists $x \in \Theta_p$ such that $\bar{\phi}_x \in \mathsf{W}_\beta$ for a suitable β (hereafter in the proof it means a relation modulo J). Also, by the induction hypothesis, $\mathrm{b}^{(m',l')}(\bar{\mathbf{x}}) = \mathrm{b}^{(m',l')}$, hence $\mathrm{b}^{(m',l')}(\bar{\mathbf{x}}) = |\eta^{(m',l')}|$ and the total number of such x-s is equal to

$$\sum_{(m',l')>(m,l)} |\eta^{(m',l')}| = \sum_{(m',l')>(m,l)} \mathrm{b}^{(m',l')}(\bar{\mathbf{x}}) = \sum_{(m',l')>(m,l)} |\omega^{(m',l')}| \ ,$$

hence, for every $\beta \in \omega^{(m',l')}$ such x exists. Hence, by lemma 3.11, for all $\beta \in \omega^{(m',l')}$ there exists p and $x \in \Theta_p$ such that $\bar{\phi}_x \in \mathsf{W}_\beta$.

Recall that $\mathcal{A}$ is linearized, hence, if $\bar{\phi}_x \in \mathsf{W}_\gamma$ for some $x \in \Theta_p$ and $p \in \eta^{(m,l)}$, then

$$\gamma \notin \omega^{(m',l')} \tag{3.28}$$

for all $(m', l') > (m, l)$.

Pick $p \in \eta^{(m,l)}$; also, let y_p denote the fixed variable of Θ_p. By definition, the set $\Theta_p \backslash \{y_p\}$ is broken down into a union of disjoint subsets Θ_p^t, $t \geq 1$, where Θ_p^t contains $\mathrm{a}^{(t)} \cdot t$ elements and for every $x \in \Theta_p^t$ the polynomial ϕ_x has form (2.63) (m is replaced by t).

Lemma 3.13. *If $t > m$, then $\bar{\phi}_x$ belongs to a simple component of D of dimension $> m$.*

Proof. Suppose first that $\bar{\phi}_x$ lies in a simple component D_α of dimension $\leq m$, hence by lemma 3.11, $\bar{y} \in \mathsf{D}_\alpha$ for all $y \in \mathrm{varb}(\phi_x)$. Recall that $\psi_x \in L\mathrm{Cap}_{t,1}$ (here we make use the notation of (2.63)), where $t \geq m + 1$, hence $\bar{\phi}_x$ must be equal to 0.

So, we have $\bar{\phi}_x \in \mathsf{W}_\beta$ for some β; by lemma 3.11 there is only one variable $z \in \mathrm{varb}(\phi_x)$ such that $z \in \mathsf{W}_\beta$ and the others are from $\mathsf{D} \cup \mathsf{C}$. By (3.28) $\dim(\mathrm{ad}_\beta(L)) \leq m < t$; since $\psi_x \in L\mathrm{Cap}_{t,1}$, the variable z must be in $\mathrm{varb}(\psi_x)$, hence $\bar{\psi}_x \in \mathsf{W}_\beta$. Therefore, all variables in $\mathrm{varb}(\psi'_x)$ take value in $\mathsf{D} \cup \mathsf{C}$, we may regard them as adjoint operators from $\mathrm{ad}(L)$ acting on W_β. But ψ'_x is skew-symmetric in t variables, hence $\bar{\psi}'_x$ must be equal to 0. □

Put

$$Y_m = \bigcup_{t>m} \Theta_p^t \ ,$$

for all $x \in Y_m$ the element $\bar{\phi}_x$ is in a simple component D_α of dimension $> m$. Next, the order of Y_m is equal to

$$\sum_{t>m} |\Theta_p^t| = \sum_{t>m} \mathrm{a}^{(t)} \cdot t \ ,$$

which is the total dimension of all simple components D_α of dimension $> m$. Denote by D the sum of these; recall the polynomial h_0 (see Step 2 of the definition of (2.66)) is skew-symmetric with respect to the elements in Θ_p, hence, $\{\bar{\phi}_y |\ y \in Y_m\}$ is a basis of the vector space D. So, if $\bar{\phi}_{y_p}$ lies in a simple subalgebra D_α, then its dimension must be $\leq m$.

Now, suppose $l \geq 1$, then the polynomial ϕ_x for $x = y_p$ has form (2.64). In particular, it belongs to $L\mathrm{Cap}_{m+1,1}$, hence its value must be in some W_β. If it were standard, then its dimension would be $\geq m+1$, which is impossible by (3.28). Hence, it is a non-standard one of dimension m; the value of ψ'_{y_p} belongs to W_β, hence $\bar{E}_1, \dots, \bar{E}_{l^2}$ are operators from $\mathrm{End}(\mathsf{W}_\beta)$. Since ϕ_{y_p} is skew-symmetric with respect to these operators, the dimension of W_β must be $\geq l$. On the other hand, it does not exceed l thanks to (3.28), therefore, $Q_{m,l}$ is proven in case $l > 0$. Besides, if $\mathrm{b}^{(m,l)}(\bar{\mathbf{x}}) < \mathrm{b}^{(m,l)} = |\eta^{(m,l)}|$, then there are $p, p' \in \eta^{(m,l)}$ such that the values of ϕ_{y_p} and $\phi_{y_{p'}}$ belong to the same simple component of W, which is impossible, because $\mathcal{A}$ is linearized.

Now suppose $l = 0$; consider the set $\Theta_p^m \cup \{y_p\}$ and pick one of its elements, x. If $\bar{\phi}_x \in \mathsf{D}_\alpha$, then the dimension of D_α must be $\leq m$. Indeed, if it were greater than m, then $\bar{\phi}_x$ would be in D; on the other hand, h_0 is skew-symmetric with respect to Θ_p which contains Y_m and $\{\bar{\phi}_x |\ x \in Y_m\}$ is a basis of D, hence h would vanish.

So, the dimension of D_α is equal to m; denote by D' the sum of all such simple parts of D. Observe that the total number of elements in $\Theta_p^m \cup \{y_p\}$ is equal to $|\mathrm{a}^m| \cdot m + 1$, it is precisely $\dim(D') + 1$, hence there is at least one $x \in \Theta_p^m \cup \{y_p\}$ such that $\bar{\phi}_x$ lies in $\mathsf{W}_{i,\beta}$ for some β.

Suppose that $\dim(\mathrm{ad}_\beta(L)) < m$, then , since $\phi_x = \psi_x \cdot \psi'_x \in L\mathrm{Cap}^2_{m,1}$, $\bar{\phi}_x$ must be equal to 0. Hence, the dimension of $\mathrm{ad}_\beta(L)$ is $\geq m$; by (3.28) it is equal to m and W_β is standard; like in case $l > 0$ it implies immediately (3.26). Lemma 3.12 is proven. □

The lemma gives immediately the following.

COROLLARY 3.2. *Let $T(\mathcal{A}) \subset \Gamma$ be an irreducible "counter-example" of the smallest type, then*

$$\mathrm{t}(\Gamma, \mathcal{A}) \leq \mathrm{b}(\Gamma, \mathcal{A}) \ .$$

We continue the proof of proposition 3.4.

Let W be a unital right module over an associative algebra R, $1 \in R$. It is said to be *representable*, if there exists a finite dimensional vector space W' over a field extension H of F such that

(a) W' is a unital right module over $R_H = H \otimes_F R$;
(b) W is embedded into W' as a module over R.

The next statement is due to A.Anan'in [**3**].

PROPOSITION 3.5. *Let W be a finitely generated module over a finitely generated associative PI-algebra R satisfying a.c.c. for right ideals. Then W is a representable.*

Consider $\mathcal{F}(\mathcal{A}) = (\mathbf{F}(\mathbf{L}); \mathcal{F}(V, \mathbf{L}))$, the g.Lie module of generic elements of $\mathcal{A}$, see (2.18), and put $R = \mathsf{En}(\mathcal{F}(\mathcal{A}))$. Recall that $\mathcal{F}(V, \mathbf{L}) = V\langle X\rangle/T(\mathcal{A})$ is contained in the space of polynomial mappings $\mathrm{Pol}(\vec{\mathbf{L}}; V) \simeq F[\vec{\mathbf{L}}] \otimes_F V$. Put

$$\Upsilon_1 = \Upsilon + T(\mathcal{A}) \tag{3.29}$$

where Υ is the T-ideal in the hypothesis of proposition 3.4). Let $\tilde{\Upsilon}_1 = \Upsilon_1/T(\mathcal{A})$, it is a submodule of $\mathcal{F}(V; \mathbf{L})$ over R.

By proposition 2.2, $\Pi(L)$ is integral over $\mathrm{tr}_U[\mathbf{L}]$, wherte $\mathsf{V} \in \mathcal{Q}(U)$. By some technical reason we replace V with the following module; let d_β denote the dimension of W_β, put

$$U = \mathsf{D} \oplus \bigoplus_\beta \mathrm{d}_\beta \mathsf{W}_\beta \ ,$$

where β runs over all projectors in $\mathrm{pr}(\mathsf{W})$ on *non-standard* submodules. Clearly, every standard submodule is embedded into U, hence V is embedded into the sum of several copies of U, hence, $\mathsf{V} \in Q(U)$. Hence, by proposition 2.2, $\Pi(L)$ is integral over $K = \mathrm{tr}_U(\mathbf{L})$. We can represent U in the following form. Let $\beta \in \mathrm{pr}(\mathsf{W})$, since the module W_β is irreducible, the algebra $A_\beta = \mathrm{End}(\mathsf{W}_\beta)$ is equal to the envelope of $\mathrm{ad}_\beta(L)$. We consider A_β as the regular right module; it is isomorphic to the direct sum of d_β copies of W_β regarded as a right module over A_β. Hence, $A_\beta \simeq d_\beta \mathsf{W}_\beta$ as L-modules and

$$U \simeq \mathsf{D} \oplus \bigoplus_\beta A_\beta \ . \tag{3.30}$$

LEMMA 3.14. *The module $\tilde{\Upsilon}_1$ is invariant under the multiplication by elements in K.*

Proof. Recall that K is generated by elements $\mathrm{tr}_U(E)$ where E is the product of operators $\mathrm{ad}(x)$, $x \in X$. To prove the lemma it suffices to show that for all $h \in \tilde{\Upsilon}$ there exists $h' \in \Upsilon$ such that $h \cdot \mathrm{tr}_U(E) = h'$ under any substitution of elements from $\mathbf{L}$.

Take a typical generator h of Υ of form (2.66) and let $p \in \eta^{(m,l)}$, where $l \geq 1$, and let y_p denote the fixed variable in Θ_p. Recall the polynomial ϕ_{y_p} has form (2.64); put

$$\phi_{y_p} * \mathrm{tr}(E) = \sum_{j=1}^{l^2} \phi_{y_p}|_{E_j \cdot E} \ ,$$

note it is a polynomial of the same form as ϕ_{y_p}. Next, let $h *_{m,l} \mathrm{tr}(E)$ denote the sum $\sum_p h_p$, where p runs over $\eta^{(m,l)}$ and h_p is obtained from h by replacing ϕ_{y_p} with $\phi_{y_p} * \mathrm{tr}(E)$; then $h *_{m,l} \mathrm{tr}(E) \in \Upsilon$.

Also, let $h *_0 \mathrm{tr}(E)$ denote the sum of polynomials h_x, $x \in \Theta_0$, obtained from h under the substitution $\phi_x E$ in place of ϕ_x. Observe that $\phi_x E$ has form (2.63), hence h_x is still a typical generator of Υ. Finally put

$$h' = h *_0 \mathrm{tr}(E) + \sum_{m,l \geq 1} h *_{m,l} \mathrm{tr}(E) \ ,$$

where (m, l) runs over all parameters such that $\mathrm{b}^{(m,l)} \neq 0$; obviously, $h' \in \Upsilon$.

LEMMA 3.15. *The equality*

$$h \cdot \mathrm{tr}_{U_i}(E) = h' \tag{3.31}$$

holds in $\mathrm{Pol}(\vec{\mathbf{L}}; V)$.

Proof. We may assume that E is multilinear and independent from variables in $\mathrm{varb}(h)$, so $h = h(\mathbf{x}) = h(x_1, \dots, x_n)$, $E = E(x_{n+1}, \dots, x_q)$ and therefore $h' = h'(\mathbf{y})$, where $\mathbf{y} = (x_1, \dots, x_q)$. Let us replace the variables x_j with some $\bar{x}_j$ in $\mathbf{L}$; since the both sides are multilinear, we may assume that all $\bar{x}_j \in \mathsf{Spl}^+(\mathbf{L}) \cup \mathsf{J}$.

In addition, suppose that $\bar{h} = h(\bar{\mathbf{x}}) \neq 0$ and $\mathrm{tr}_U(\bar{E}) \neq 0$, then for all $x \in \mathrm{varb}(h)$ the element $\bar{x}$ belongs to $\mathsf{Spl}^+(\mathbf{L})$. Besides, since $\mathrm{tr}_U(\bar{E}) \neq 0$, we have $\bar{x}_j \in \mathsf{D} \cup \mathsf{C}$ for all $j \in \{m+1, \dots, n\}$. By lemma 3.12, for all $m, l \in \mathbb{N}$ and all $p \in \eta^{(m,l)}$ the element $\bar{\phi}_{y_p}$ belongs to a non-standard module W_β of dimension l and satisfying (2.49) (formally speaking, it holds modulo J but it is unimportant, see (3), proposition 3.3. Moreover, ϕ_{y_p} has form (2.64), hence $\bar{\psi}'_{y_p} \in \mathsf{W}_\beta$ and therefore all operators $\bar{E}_j$, $j \in \{1, \dots, l^2\}$, lie in A_β. From lemma 1.11 it follows that

$$\overline{\phi_{y_p} * \mathrm{tr}(E)} = \bar{\phi}_{y_p} \cdot \mathrm{tr}_{A_\beta}(\bar{E}) \ .$$

Next, consider the values of ϕ_x, where $x \in \Theta_0$; by lemma 3.12, the representatives of all W_β, $\beta \in \mathrm{pr}(\mathsf{W})$, among $\bar{x}_j$, occur "outside" $\bigcup_{x \in \Theta_0} \mathrm{varb}(\phi_x)$, hence the values of all variables from $\mathrm{varb}(\phi_x)$ are elements in simple components of D and in $\langle \mathsf{C} \rangle$. Recall that the degree of ϕ_x is not less than the nilpotency index of $\langle \mathsf{C} \rangle$, hence $\bar{\phi}_x$ belongs to D. Also, the order of Θ_0 is equal to $\dim(\mathsf{D})$, hence $\{\bar{\phi}_x | \ x \in \Theta_0\}$ is a basis of the vector space D. So, by lemma 1.11 we get

$$\overline{h *_0 \mathrm{tr}(E)} = \bar{h} \cdot \mathrm{tr}_{\mathsf{D}}(\bar{E}) \ .$$

Summarizing, we have

$$\bar{h}' = \overline{h *_0 \mathrm{tr}(E)} + \sum_{m,l \geq 1} \overline{h *_{m,l} \mathrm{tr}(E)} = \bar{h} \cdot \mathrm{tr}_U(\bar{E}) \ ,$$

so (3.31) holds in case $h(\bar{\mathbf{x}}) \neq 0$; similarly, one can check the case $h'(\bar{\mathbf{y}}) \neq 0$. □

Equality (3.31) holds if we replace some variables with Lie polynomials; also it is linear with respect to h, hence (3.31) takes place for all $h \in \Upsilon$. Clearly, it proves lemma 3.14.

Let m denote the maximum of $\dim(L_i)$, $i \in \{1, \dots, k\}$; take Y_i in the set of generic elements X_i of $\mathcal{A}$ with m variables and put $Y = \bigcup_{i=1}^k Y_i$. Next, let $\mathcal{B} = (\mathbf{M}; W)$ be the g.Lie submodule of $\mathcal{F}(\mathcal{A})$ generated by Y; also, put $\widetilde{\Upsilon}'_1$ to be the set of all polynomials in $\widetilde{\Upsilon}_1$ depending only on Y, i.e.,

$$\widetilde{\Upsilon}'_1 = \widetilde{\Upsilon}_1 \cap W \ ,$$

and put $\hat{\mathcal{B}} = (\mathbf{M}; \hat{W})$, where $\hat{W} = W / \widetilde{\Upsilon}'_1$. Any polynomial in $V\langle X \rangle$, which is linear and skew-symmetric with respect to $m+1$ variables in some X_i, is an identity of $\mathcal{A}$. Hence, from proposition 1.1 it follows that $T(\hat{\mathcal{B}}) = \Upsilon_1$, thus, it suffices to prove the representability of $\hat{\mathcal{B}}$, that is, the existence of a monomorphism from $\hat{\mathcal{B}}$ to a g.Lie module which is finite dimensional over a field extension of F.

Let Φ denote the subalgebra of all polynomials in K depending only on Y; recall that K is generated by $\mathrm{tr}_U(E)$, where $E \in \Pi(U)$. Let $\Pi(U)'$ be the subalgebra of elements depending only on Y, then Φ is generated by $\mathrm{tr}_U(E)$, $E \in \Pi(U)'$. Moreover, from homogeneity of $\Pi(L)$ and Φ it follows that $\Pi(L)'$ is integral over Φ.

LEMMA 3.16. *The algebra Φ is finitely generated over F.*

Proof. Observe that $\Pi(U)'$ is a subalgebra of $\mathrm{Pol}(\vec{\mathbf{L}}; \mathrm{End}(U))$, hence it is an associative PI algebra. Also, it is generated by a finite number of operators $\mathrm{ad}(x)$, $x \in Y_i$; besides, it is integral over Φ. Hence, by Shirshov's theorem about height, e.g. [**34, 37**], $\Pi(U)' \cdot \Phi$ is a finitely generated module over Φ. Let $E_1, \dots , E_d 1\Pi(U)'$ be the generators, then Φ is generated by the elements $\mathrm{tr}_U(E)$; hence Φ is a finitely generated algebra. $\square$

Similarly, one can prove that $\Pi(L)'$ is a finitely generated module over Φ. Let N be the Φ-submodule generated by M in $\mathrm{Pol}(\vec{\mathbf{L}}, L)$, i.e., $N = \Phi \cdot M$; observe that M is generated by Y as a module over $\Pi(L)'$, hence, N is a finitely generated Φ-module. Put $N_i = \Phi \cdot M_i$, then $\mathbf{N} = (N; N_1, \dots , N_k)$ is a k-sorted Lie algebra over Φ, which extends $\mathbf{M}$.

Similarly, let U denote the extension of W by Φ, then $\mathcal{C} = (\mathbf{N}; U)$ is a g.Lie module over Φ. Put $\hat{\mathcal{C}} = (\mathbf{N}; \hat{U})$, where $\hat{U} = U/\tilde{\Upsilon}'_1$, then $\mathcal{B}$ is embedded into $\mathcal{C}$. Let $P = \mathsf{En}(\hat{\mathcal{C}})$; since N is a finitely generated Lie algebra over Φ, P is also a finitely generated associative algebra . Besides, P inherits all polynomial identities of the finite dimensional algebra $\mathsf{En}(\mathcal{A})$, hence, it is PI. Finally, by lemma 2.14, P is a noetherian Φ-algebra, hence we can apply proposition 3.5 to the P-module $\hat{U}$.

Let H be the extension of the field F and let $\hat{U}'$ be a module over $P' = H \otimes_F P$ such that $\dim_H(\hat{U}') < \infty$ and $\hat{U}'$ extends $\hat{U}$ as a module over P. Put $\mathbf{N}' = H \otimes_F \mathbf{N}$ and define a g.Lie representation $\nu : \mathbf{N}' \to \hat{U}'$ over H setting $\mu(\alpha \otimes a) = \alpha\mu(a)$ for all $\mu \in \{\rho_j, \delta_j\}$; this defines a g.Lie module $\mathcal{C}' = (\mathbf{N}'; \hat{U}')$.

Clearly, we may assume that $\mathbf{N}'$ $(\hat{U}')$ is the span of $\mathbf{N}$ (respectively, $\hat{U}$) over H. Hence, the ideal of identities of $\mathcal{C}'$ (regarded as a g.Lie module over F) is equal to $T(\mathcal{C}) = \Upsilon_1$. Note that $\mathrm{Ker}(\nu)$ is an ideal of finite codimension (cf. lemma 2.3), hence we can replace $\mathbf{N}'$ with a finite dimensional k-sorted of Lie algebra over H. Proposition 3.4 is proven. $\square$

So, let $\mathcal{A}' = (\mathbf{L}'; V')$ be a finite dimensional g.Lie module over H whose ideal of identities (as a module over F) is equal to Υ_1.

Consider the variety of g.Lie modules over H of the same type as $\mathcal{A}'$, denote it by $\mathfrak{M}'_{k,s,r}$ to distinguish it from g.Lie modules over F. The free module $\mathcal{M}'\langle X\rangle$ is isomorphic to the scalar extension $H \otimes_F \mathcal{M}\langle X\rangle$, where we have the critical T-ideal $\Gamma' = H \otimes_F \Gamma$ with the same appropriate sequence of polynomials. Besides, $\mathcal{A}'$ satisfies all identities of $\mathcal{A}$, in particular, (2.19) and, therefore, $\mathbf{L}'$ satisfies (2.20). Hence we have a "counter-example"

$$T(\mathcal{A}') \subset \Gamma' .$$

Now, let us notice that the case $\|\mathrm{a}(\Gamma, \mathcal{A})\| = |\mathrm{b}(\Gamma, \mathcal{A})| = 0$ is impossible. Indeed, it means that $\mathsf{D} = 0$ and $\mathsf{W} = 0$ and therefore $L = \langle \mathsf{C} \rangle$ is nilpotent. Consider the g.Lie submodule $\mathcal{B} = (\mathbf{M}; W)$ of $\mathcal{F}(\mathcal{A})$ generated by the finite set $Y = \bigcup_{i=1}^k Y_i$, where $|Y_i| = \dim(L_i)$. By proposition 1.1, chain of T-ideals (2.43) defines a strictly ascending chain of submodules of W over $\mathsf{En}(\mathcal{B})$. On the other hand, M is a finitely generated nilpotent Lie algebra, hence it is finite dimensional and, by lemma 2.14, W is a Noetherian module.

Fix a canonical decomposition of $\mathbf{L}'$.

LEMMA 3.17. *The following inequalities hold:*

$$\|\mathrm{a}(\Gamma, \mathcal{A})\| \geq \|\mathrm{a}(\Gamma', \mathcal{A}')\| \tag{3.32}$$

$$\mathrm{a}(\Gamma, \mathcal{A}) \geq \mathrm{a}(\Gamma', \mathcal{A}') \ . \tag{3.33}$$

Moreover, if $|\mathrm{b}(\Gamma, \mathcal{A})| = 0$*, then* $\|\mathrm{a}(\Gamma, \mathcal{A})\| > 0$ *and the inequality (3.33) is strict.*

Proof. Put $\mathsf{D}' = \mathsf{D}(\mathbf{L}')$, $\mathsf{W}' = \mathsf{W}(\mathbf{L}')$ and so on. First, suppose that (3.32) is not true, then one can find $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma', \mathcal{A}')$, such that $\|\mathrm{a}(\bar{\mathbf{x}})\| > \|\mathrm{a}(\Gamma, \mathcal{A})\|$; take a multilinear $f \in \Gamma$ such that $\bar{\mathbf{x}} \in \mathsf{V}(f, \mathcal{A}')$. To every $\alpha \in \pi(\bar{\mathbf{x}})$ we fix a $x_\alpha \in \mathrm{varb}(f)$ such that $\bar{x}_\alpha \in \mathsf{D}'_\alpha$; also, put $n_\alpha = \dim_H(\mathsf{D}'_\alpha)$.

Let $c \in \mathbb{N}$ such that $\langle \mathsf{C}\rangle^c = 0$ and $c \geq$ than the nilpotency index of the radical of $\mathsf{En}(\mathcal{A})$. By lemma 2.11 and 2.10 there exist Lie polynomials ϕ_α over H and disjoint sets $\lambda^1_\alpha, \dots, \lambda^c_\alpha$ of variables such that

(i) ϕ_α is a sum of multilinear polynomials;

(ii) ϕ_α is multilinear and skew-symmetric with respect to variables of each λ^j_α and $|\lambda^j_\alpha| = n_\alpha$ for all j;

(iii) ϕ_α is linear with respect to x_α and

$$x_\alpha \notin \bigcup_{j=1}^{c} \lambda^j_\alpha \ ;$$

also, $\phi_\alpha \in \mathrm{L}\langle X_i\rangle$ provided $x_\alpha \in X_i$;

(iv) under suitable substitution $x \to \bar{x}$ of elements from D'_α into ϕ_α we get $\bar{\phi}_\alpha = \bar{x}_\alpha$.

The point (iii) provides the following property. If we replace $\bar{y}_1, \dots, \bar{y}_q$ in $\bar{\phi}_\alpha$, where $y_1, \dots, y_q \in \bigcup_{j=1}^{c} \lambda^j_\alpha$, by an element in $\bigcup_{\gamma\neq\alpha} \mathsf{D}'_\gamma$, then we get 0 in place of $\bar{\phi}_\alpha$.

Further, let f_1 denote the polynomial f where every x_α is replaced with ϕ_α; clearly, we have enough variables to assume that $\mathrm{varb}(\phi_\alpha)$ are disjoint sets, hence f_1 is a sum of multilinear polynomials; besides $\bar{f}_1 = \bar{f}$. For every $j \in \{1, \dots, c\}$ put

$$\Theta_j = \bigcup_{\alpha\in\pi(\bar{\mathbf{x}})} \lambda^j_\alpha \ ,$$

and let f_2 denote the alternation of f_1 with respect to the variables in Θ_j for all j; observe that f_2 belongs to Γ' thanks to (ii).

From (i)-(iv) it follows that $\bar{f}_2 = q \cdot \bar{f}_1 \neq 0$ for some $q \in \mathbb{N}$, hence $f_2 \notin T(\mathcal{A}') = \Gamma'$, hence $f_2 \notin H \otimes_F T(\mathcal{A})$. Let g denote a homogeneous component of f_2 which is not in $T(\mathcal{A})$, hence g is multilinear and and skew-symmetric with respect to each Θ_j.

Moreover, since the c-th power of $\mathsf{D}_{i,\alpha}$ is equal to itself, we can replace every element $\bar{y}$, $y \in \lambda^j_\alpha$ with a sum multilinear polynomials of degree $\geq c$ in suitable elements from $\mathsf{D}'_{i,\alpha}$. So we assume that every $y \in \lambda^j_\alpha$ is a multilinear polynomial of degree $\geq c$.

On the other hand, $H \otimes_F T(\mathcal{A}) = T(H \otimes_F \mathcal{A})$; we claim that g is an identity of $H \otimes_F \mathcal{A}$. Actually, suppose it does not vanish under a substitution $x \to \tilde{x}$ of elements from $H \otimes \mathbf{L}$. Since g is multilinear, we may assume that all $\tilde{x} \in \mathbf{L}$. Moreover, f_2 is a linear combination over H of polynomials from Γ, hence, so is g, hence $\tilde{x} \in \mathsf{Spl}^+(\mathbf{L})$. Besides, every $y \in \Theta_j$ is a polynomial of degree $\geq c$, hence we can assume that modulo J

$$\tilde{y} \in \left(\bigcup_{\alpha \in \mathrm{pr}(\mathsf{D})} \mathsf{D}_\alpha \right) \cup \left(\bigcup_{\beta \in \mathrm{pr}(\mathsf{W})} \mathsf{W}_\beta \right) .$$

Arguing in a standard way, one can show that for all $j \in \{1, \dots, c\}$ there is $x \in \Theta_j$ such that $\tilde{x}$ belongs to W. Since $c \geq$ the nilpotency index of the radical of $\mathsf{En}(\mathcal{A})$, we get $\tilde{g} = 0$. So, (3.32)) is proven.

In a similar fashion one may check (3.33)). First of all, recall that

$$\mathrm{a}(\Gamma, \mathcal{A}) = (\mathrm{a}^{(1)}, \dots, \mathrm{a}^{(m)}, \dots) \in \mathbb{N}_0^\infty ,$$

where $\mathrm{a}^{(m)}$ is the number of $\alpha \in \mathrm{pr}(\mathsf{D})$ such that the simple subalgebra D_α has dimension m; let $\mathsf{D}^{(m)}$ denote the sum of such subalgebras. Suppose that (3.33) is untrue, then there is $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma', \mathcal{A}')$ such that $\mathrm{a}(\Gamma, \mathcal{A}) < \mathrm{a}(\bar{\mathbf{x}})$, hence for some $t \in \mathbb{N}$ we have $\mathrm{a}^{(t)} < \mathrm{a}^{(t)}(\bar{\mathbf{x}})$ and $\mathrm{a}^{(m)} = \mathrm{a}^{(m)}(\bar{\mathbf{x}})$ for all $m > t$.

Consider the polynomial g constructed above. Let $\Theta_j^{(m)}$ denote the union of all λ_α^j where the dimension of D'_α is equal to m, then

$$|\Theta_j^{(t)}| > \dim(\mathsf{D}^{(t)})$$

and

$$|\Theta_j^{(m)}| = \dim(\mathsf{D}^{(m)})$$

for all $m > t$. Hence,

$$\sum_{m \geq t} |\Theta_j^{(m)}| > \sum_{m \geq t} \dim(\mathsf{D}^{(m)}) \; ; \tag{3.34}$$

recall that every $y \in \Theta_j^{(m)}$ is supposed to be a multilinear polynomial of degree $\geq c$; we can require an additional property: it is skew-symmetric with respect to m variables.

Now let us prove that g vanishes in $H \otimes_F \mathcal{A}$; consider a substitution $x \to \tilde{x}$. The polynomial g is multilinear, hence we may assume that $\tilde{x} \in \mathsf{Spl}^+(\mathbf{L})$. By (3.34) for all $j \in \{1, \dots, c\}$ there exists y in some $\Theta_j^{(m)}$, $m \geq t$, such that $\tilde{y}$ lies in W. Hence $\tilde{g}$ must be equal to 0, and (3.33) is also proven.

Finally, suppose $|\mathrm{b}(\Gamma, \mathcal{A})| = 0$, then $\mathrm{b}(\Gamma, \mathcal{A}) = (0, \dots, 0, \dots)$; hence, typical generators of (2.57) have a specific form: there are no sets Θ_p for all $p \geq 1$. Suppose that (3.33) is an equality; put $\mathrm{b} = (0, \dots, 0, \dots)$ and $c \in \mathbb{N}$, arguing like in the preceding cases, one can construct a polynomial $g \in \mathcal{S}_{\mathrm{a,b},c}(H \otimes_F \Gamma^{\#}))$ not vanishing under substitution of suitable elements in $\mathbf{L}'$, which is impossible, because $T(\mathcal{A}')$ contains all such g for sufficiently big c. Lemma 3.17 is proven. □

The type of $T(\mathcal{A}') \subset \Gamma'$ must be not less than $\mathrm{type}(\Gamma, \mathcal{A})$, hence, by lemma 3.17 $\mathrm{a}(\Gamma, \mathcal{A}) = \mathrm{a}(\Gamma', \mathcal{A}')$; also, from corollary 3.2 we have $\mathrm{t}(\Gamma, \mathcal{A}) = \mathrm{b}(\Gamma, \mathcal{A})$. On the other hand,

$$\mathcal{S}_{\mathrm{a,b},c}(H \otimes_F \Gamma) \subseteq H \otimes_F \mathcal{S}_{\mathrm{a,b},c}(\Gamma) \subseteq T(\mathcal{A}') ,$$

hence $\mathrm{t}(\Gamma', \mathcal{A}') < \mathrm{t}(\Gamma, \mathcal{A})$ and the type of $T(\mathcal{A}') \subseteq \Gamma'$ is less than $\mathrm{type}(\Gamma, \mathcal{A})$, which is impossible. So, $\mathrm{t}(\Gamma, \mathcal{A}) < \mathrm{b}(\Gamma, \mathcal{A})$ and T-ideal (2.57), where $\mathrm{b} = \mathrm{b}(\Gamma, \mathcal{A})$, is contained in $T(\mathcal{A})$ for some $c \in \mathbb{N}$.

4. Canonical splitting

We are going to classify relations between simple components of D_i and that of W_i for each i; it turns out that there is a natural partition of these sets defined by action of L on W_i, which induces a decomposition of the algebra L_i/J_i into a direct sum of ideals. In fact, we split up ideals of a special form; the semisimple part is a simple algebra and the solvable radical is the direct sum of standard submodules of W_i.

Take $\alpha \in \mathrm{pr}(\mathsf{D}_i)$ and put

$$\omega_{i,\alpha} = \{\beta \in \mathrm{pr}(\mathsf{W}_i) |\ [\mathsf{W}_{i,\beta}, \mathsf{D}_{i,\alpha}] \neq 0\} \ ;$$

in other words, $\omega_{i,\alpha}$ consists of all projectors on simple components of W (not only W_i) with nontrivial action of $\mathsf{D}_{i,\alpha}$. The projector α is called *isolated*, if the following holds:

(I1) $\alpha \in \Omega = \Omega(\Gamma, \mathcal{A})$, see (3.23);

(I2) $\mathsf{W}_{i,\beta}$ is standard for all $\beta \in \omega_{i,\alpha}$.

If α is isolated, then for all $\alpha' \in \mathrm{pr}(\mathsf{D}_i)$, $\alpha' \neq \alpha$, the subalgebra $\mathsf{D}_{i,\alpha'}$ acts trivially on every $\mathsf{W}_{i,\beta}$, $\beta \in \omega_{i,\alpha}$, hence

$$\omega_{i,\alpha} \cap \omega_{i,\alpha'} = \emptyset \ . \tag{3.35}$$

Also,

$$[\mathsf{W}_{i,\beta}, \mathsf{C}_i] = 0 \tag{3.36}$$

modulo $\mathsf{W}_i^{[1]} \subseteq \mathsf{J}_i$.

Let π_i° denote the set of all isolated projectors in $\mathrm{pr}(\mathsf{D}_i)$ and put $\pi_i^\bullet = \mathrm{pr}(\mathsf{D}_i) \backslash \pi_i^\circ$. To each $\alpha \in \pi_i^\circ$ let L_i^α denote the ideal of L generated by

$$\mathsf{D}_{i,\alpha} \cup \bigcup_{\beta \in \omega_{i,\alpha}} \mathsf{W}_{i,\beta} \ ; \tag{3.37}$$

also, put

$$\omega_i^\circ = \bigcup_{\alpha \in \pi_i^\circ} \omega_{i,\alpha} \ , \tag{3.38}$$

it is a disjoint union by (3.35); let

$$\omega_i^\bullet = \mathrm{pr}(\mathsf{W}_i) \backslash \omega_i^\circ \ , \tag{3.39}$$

then for all $\alpha \in \pi_i^\circ$ ($\alpha \in \pi_i^\bullet$) and $\beta \in \omega_i^\bullet$ ($\beta \in \pi_i^\circ$, respectively) we have

$$[\mathsf{W}_{i,\beta}, \mathsf{D}_{i,\alpha}] = 0 \ . \tag{3.40}$$

Now, let $L_i^\bullet$ denote the ideal of L generated by

$$\mathsf{C}_i \cup \bigcup_{\alpha \in \pi_i^\bullet} \mathsf{D}_{i,\alpha} \cup \bigcup_{\beta \in \omega_i^\bullet} \mathsf{W}_{i,\beta} \ , \tag{3.41}$$

then

$$L = \sum_{i=1} L_i = \sum_{i=1}^{k} \sum_{\gamma \in \widetilde{\pi}_i^\circ} L_i^\gamma \ ,$$

where $\widetilde{\pi}_i^\circ = \pi_i^\circ \cup \{\bullet\}$. Put

$$k' = \sum_{i=1}^{k} |\widetilde{\pi}_i^\circ| \ , \tag{3.42}$$

then the formula above defines the structure of k'-sorted Lie algebra

$$\mathbf{L}' = (L; \dots , L_i^\gamma, \dots) \ . \tag{3.43}$$

Note that components of k'-sorting are enumerated by double indices, $i \in \{1, \dots , k\}$ and $\gamma \in \widetilde{\pi}_i$, so the standard notations used in the case of k-sorting are adopted in the same fashion.

The k'-sorted Lie algebra $\mathbf{L}'$ has a canonical decomposition induced by that of $\mathbf{L}$. First of all, by (2.20), (3.40) and (3.36) we have

$$\mathsf{I}_i^\alpha(\mathbf{L}') = \sum_{(j,\gamma)\neq(i,\alpha)} [L_i^\alpha, L_j^\gamma] \subseteq \mathsf{N}(L) \ ,$$

so $\mathbf{L}'$ satisfies the analog of (2.20).

Suppose $\alpha \in \pi_i^\circ$, then $\mathsf{D}_{i,\alpha}$ is the maximal semisimple subalgebra of L_i^α, so $\mathsf{D}_i^\alpha(\mathbf{L}') = \mathsf{D}_{i,\alpha}$; next, all simple components $\mathsf{W}_{i,\beta}$, where $\beta \in \omega_{i,\alpha}$, are irreducible modules over $\mathsf{D}_{i,\alpha}$, hence

$$W = \sum_{\beta\in\omega_{i,\alpha}} \mathsf{W}_{i,\beta}$$

lies in $\mathsf{N}_i^\alpha(\mathbf{L}')$, hence $\mathsf{S}(L_i^\alpha) = \mathsf{N}_i^\alpha(\mathbf{L}')$ and $\mathsf{C}_i^\alpha(\mathbf{L}') = 0$. Next,

$$\mathsf{M}_i^\alpha(\mathbf{L}') = \mathsf{N}_i^\alpha(\mathbf{L}')^{[1]} + \mathsf{N}_i^\alpha(\mathbf{L}')^2 + \mathsf{I}_i^\alpha(\mathbf{L}') \subseteq \mathsf{M}_i \ .$$

On the other hand, it contains $W^{[1]}$ and the square of the algebra generated by $W^{[0]}$, hence, we can put $\mathsf{W}_i^\alpha = \mathsf{W}_i^\alpha(\mathbf{L}')$ equal to W.

Next,

$$\mathsf{D}_i^\bullet = \bigoplus_{\alpha\in\pi_i^\bullet} \mathsf{D}_{i,\alpha}$$

is a maximal semisimple subalgebra of $L_i^\bullet$ and the radical $\mathsf{S}(L_i^\bullet)$ is generated as an ideal of L by C_i and

$$W = \bigoplus_{\beta\in\omega_i^\bullet} \mathsf{W}_{i,\beta} \ .$$

By lemma 2.5 and (3.35), all simple summands of $\mathsf{W}_i^\bullet$ are nontrivial L_i^α-modules (with respect to the action induced by L), hence, $\mathsf{N}_i^\bullet(\mathbf{L}')$ is the ideal generated by W and $[\mathsf{C}_i, \mathsf{C}] + \mathsf{I}_i^\bullet$; hence, we can put $\mathsf{C}_i^\bullet(\mathbf{L}') = \mathsf{C}_i$. Finally, $\mathsf{M}_i^\bullet(\mathbf{L}') \subseteq \mathsf{M}_i$ contains $[\mathsf{C}_i, \mathsf{C}] + \mathsf{I}_i^\bullet$ and $W^{[1]}$, hence we set $\mathsf{W}_i^\bullet(\mathbf{L}') = \mathsf{W}_i^\bullet$ equal to W.

In particular, we have the following decomposition

$$\mathsf{Gn}_i(\mathbf{L}) = \bigoplus_{\gamma\in\widetilde{\pi}_i^\circ} \mathsf{Gn}_i^\gamma(\mathbf{L}') \ . \tag{3.44}$$

Now, put $\mathsf{J}_i^\gamma = \mathsf{J}_i \cap L_i^\gamma$ for all $\gamma \in \widetilde{\pi}_i$; then $\mathsf{J}_i^\gamma(\mathbf{L}') \subseteq \mathsf{J}_i^\gamma$. From (3.40), (3.36) and (3.35) it follows that for all different $\alpha, \beta \in \widetilde{\pi}_i$

$$[L_i^\alpha, L_i^\beta] \subseteq \mathsf{J}_i \ , \tag{3.45}$$

hence, the algebra L_i/J_i is decomposed into the sum of ideals $L_i^\gamma + \mathsf{J}_i/\mathsf{J}_i \simeq L_i^\gamma/\mathsf{J}_i^\gamma$, we claim that it is direct.

First, observe that

$$L_i^\alpha/\mathsf{J}_i^\alpha(\mathbf{L}') \simeq \mathsf{D}_{i,\alpha} + \mathsf{W}_i^\alpha = \mathsf{Gn}_i^\alpha(\mathbf{L}') \subseteq \mathsf{Gn}_i(\mathbf{L})$$

for all $\alpha \in \pi_i^\circ$; since $\mathsf{Gn}_i(\mathbf{L}) \cap \mathsf{J}_i = 0$, we have

$$L_i^\alpha/\mathsf{J}_i^\alpha \simeq \mathsf{Gn}_i(\mathbf{L}) \ ,$$

in particular, $\mathsf{J}_i^\alpha = \mathsf{J}_i^\alpha(\mathbf{L}')$ in this case. Next,

$$L_i^\bullet/\mathsf{J}_i^\bullet \simeq L_i^\bullet + \mathsf{J}_i/\mathsf{J}_i \simeq (\mathsf{D}_i^\bullet \oplus \mathsf{W}_i^\bullet \oplus \mathsf{C}_i \oplus (\langle \mathsf{C}_i \rangle + \mathsf{J}_i))/\mathsf{J}_i \simeq$$
$$\simeq \mathsf{Gn}_i^\bullet(\mathbf{L}') \oplus \left(\langle \mathsf{C}_i \rangle^2 + \mathsf{J}_i/\mathsf{J}_i\right) \ .$$

On the other hand,

$$L_i/\mathsf{J}_i = \mathsf{Gn}_i(\mathbf{L}) \oplus (\mathsf{M}_i/\mathsf{J}_i) \simeq \mathsf{Gn}_i(\mathbf{L}) \oplus \left(\langle \mathsf{C}_i \rangle^2 + \mathsf{J}_i/\mathsf{J}_i\right)$$

which together with (3.44) gives the required decomposition,

$$L_i/\mathsf{J}_i \simeq \bigoplus_{\gamma \in \widetilde{\pi}_i} L_i^\gamma/\mathsf{J}_i^\gamma \ . \tag{3.46}$$

Let μ_i^γ denote the canonical epimorphism $L_i \to L_i/\mathsf{J}_i \to L_i^\gamma/\mathsf{J}_i^\gamma$; we need to extend it to a homomorphism $L \to L_i/\mathsf{J}_i$. Put $\mathsf{J} = \sum_{i=1}^k \mathsf{J}_i$, then

$$L/\mathsf{J} = \sum_{i=1} (L_i + \mathsf{J})/\mathsf{J} \ ; \tag{3.47}$$

if the sum is direct and $\mathsf{J} \cap L_i = \mathsf{J}_i$, then we can extend μ_i^γ to L by replacing it with the composition of

(1) the canonical homomorphism $L \to L/\mathsf{J}$,
(2) the projection of L/J onto the direct summand $L_i + \mathsf{J}/\mathsf{J} \simeq L_i/\mathsf{J}_i$,
(3) μ_i^γ.

Since $\mathsf{J}_i \subseteq \mathsf{J} \cap L_i$, we have a canonical homomorphism from L_i/J_i onto $L_i + \mathsf{J}/\mathsf{J}$. Put $\mathsf{Gn}_i^+(\mathbf{L}) = \mathsf{D}_i + \mathsf{W}_i + \langle \mathsf{C}_i \rangle$, then $L_i = \mathsf{Gn}_i^+(\mathbf{L}) + \mathsf{J}_i$, hence

$$\dim(\mathsf{Gn}_i^+(\mathbf{L})) \geq \dim(L_i/\mathsf{J}_i) \geq \dim(L_i + \mathsf{J}/\mathsf{J}) \ . \tag{3.48}$$

Besides, modulo J the vector space L is equal to the sum

$$\sum_{i=1}^k \mathsf{Gn}_i^+(\mathbf{L}) \ . \tag{3.49}$$

Suppose it is direct, then

$$\sum_{i=1}^k \dim\left(\mathsf{Gn}_i^+(\mathbf{L})\right) = \dim(L/\mathsf{J}) \leq \sum_{i=1}^k \dim(L_i + \mathsf{J}/\mathsf{J})$$

hence, in (3.48) we get equalities which proves the required decomposition of L.

To this end we are going to change slightly $\mathbf{L}$. Let $\mathbf{M} = (M; M_1, \ldots, M_k)$ denote the k-sorted subalgebra of $\mathbf{L}$ generated by $\bigcup_{i=1}^k \mathsf{D}_i \cup \mathsf{W}_i^{[0]}$; then $M + \langle \mathsf{C} \rangle = L$ by lemma 2.7. Recall that M is an ideal of L, hence $\langle \mathsf{C} \rangle$ acts by derivations on M.

Let $q \in \mathbb{N}$ such that $\langle \mathsf{C}\rangle^q = 0$; take a free product of Lie algebras $\langle \mathsf{C}_i\rangle$, $i \in \{1, \dots, k\}$, in the variety of Lie algebras [**5**] and let N denote the quotient of this algebra over its q-th power. Then N is a finite dimensional nilpotent Lie algebra containing every $\langle \mathsf{C}_i\rangle$, where the sum $\sum_{i=1}^{k}\langle \mathsf{C}_i\rangle$ is direct. Moreover, there exists a homomorphism π from N onto $\langle \mathsf{C}\rangle$ restricting to the identity mapping on each $\langle \mathsf{C}_i\rangle$. It induces a representation of N in $\mathrm{Der}(M)$, so $\widetilde{M} = M \oplus N$ is equipped with the structure of Lie algebra, so $\widetilde{\pi} : a \oplus b \to a + \pi(b)$ is an algebra homomorphism of $\widetilde{M}$ onto L.

Let N_i be the ideal of N generated by C_i, then $\widetilde{M}_i = M_i \oplus N_i$ is an ideal $\widetilde{M}$ and $\widetilde{\pi}(\widetilde{M}_i) = L_i$. Thus, $\widetilde{\pi}$ is a homomorphism of the k-sorted Lie algebra $\widetilde{\mathbf{M}} = (\widetilde{M}; \widetilde{M}_1, \dots, \widetilde{M}_k)$ onto $\mathbf{L}$. Its composition with the g.Lie representation $\nu : L \to \mathrm{End}(V)$ is that of $\widetilde{M}$; put $\widetilde{\mathcal{A}} = (\widetilde{\mathbf{M}}; V)$, then $T(\mathcal{A}) = T(\widetilde{\mathcal{A}})$.

We can put $\mathsf{Gn}_i(\widetilde{\mathbf{M}}) = \mathsf{Gn}_i(\mathbf{L})$. Indeed, since the kernel $\mathrm{Ker}(\widetilde{\pi})$ coincides with $\mathrm{Ker}(\pi) \subseteq N$, the subalgebra $\mathsf{D}_i \subseteq \widetilde{M}_i$ is the maximal semisimple one and the radical $\mathsf{S}(\widetilde{M}_i)$ is equal to the pre-image of $\mathsf{S}(L_i) = \mathsf{S}(M_i) + \langle \mathsf{C}\rangle$ with respect to $\widetilde{\pi}$, i.e., $\mathsf{S}(M_i) \oplus N_i$. Next, $\mathsf{N}_i(\widetilde{\mathbf{M}})$ contains $\mathsf{W}_i^{[0]} \subseteq M_i$, hence, $\mathsf{N}_i(\widetilde{\mathbf{M}}) = \mathsf{S}(M_i) \oplus [N_i, N]$ and we can put $\mathsf{C}_i(\widetilde{\mathbf{M}}) = \mathsf{C}_i \subseteq N_i$. Finally, $\mathsf{W}_i + \mathsf{M}_i(\widetilde{\mathbf{M}})$ is equal to $\mathsf{N}_i(\widetilde{M})$, hence we let $\mathsf{W}_i(\widetilde{\mathbf{M}}) = \mathsf{W}_i$.

Notice that the "counter-example" $T(\mathcal{A}') \subset \Gamma$ is irreducible with respect to this canonical decomposition. Indeed, $\widetilde{\pi}$ maps $\mathsf{Gn}_i(\widetilde{\mathbf{M}})$ onto $\mathsf{Gn}_i(\mathbf{L})$, hence, $\mathcal{A}'$ is linearized and (3.2) takes place for all $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A}')$. Also, the image of $\mathsf{J}_i(\mathcal{A}')$ is equal to J_i, hence (3) of proposition 3.3 is valid; finally, the verbal ideal $\Gamma^{\#}[\mathcal{A}']$ is equal to $\Gamma^{\#}[\mathcal{A}]$, hence it is an irreducible submodule of V lying in every nonzero one.

Summarizing, we get the following statement.

LEMMA 3.18. *There exists an irreducible "counter-example" $T(\mathcal{A}) \subset \Gamma$ of minimal type such that $\mathsf{J}_i^\gamma = \mathsf{J}_i \cap L_i^\gamma$ is equal to $\mathsf{J} \cap L_i^\gamma$ for all i, γ and L/J is decomposed into the direct sum of ideals $L_i^\gamma + \mathsf{J}/\mathsf{J}$,*

$$L/\mathsf{J} = \bigoplus_{i=1}^{k} \bigoplus_{\gamma \in \widetilde{\pi}_i^{\circ}} L_i^\gamma / \mathsf{J}_i^\gamma \ . \tag{3.50}$$

So, hereafter we assume that $T(\mathcal{A}) \subseteq \Gamma$ is an irreducible "counter-example", where $\mathbf{L}$ has a canonical decomposition satisfying (3.50). As a matter of fact, this property will be really working only in the last section.

5. Main Lemma

In this section we arrive to the point where the situation differs substantially from the associative case. In fact, Main Lemma describes in details the obstacle for generalization theorem 1.1 about representability of relatively free algebras to the class of Lie algebras. At the same time it helps us to find a "counter-example" of lesser type in another class of g.Lie modules, so it will play the most important role in the rest.

Recall that $\mathrm{t}(\Gamma, \mathcal{A}) < \mathrm{b}(\Gamma, \mathcal{A})$, hence, by corollary 3.1,

$$\mathcal{S}_{\mathrm{a},\mathrm{b},c}(\Gamma^{\#}) \tag{3.51}$$

where $\mathrm{a} = \mathrm{a}(\Gamma, \mathcal{A})$ and $\mathrm{b} = \mathrm{b}(\Gamma, \mathcal{A})$, is contained in $T(\mathcal{A})$ for suitable $c \in \mathbb{N}$.

Take $\bar{\mathbf{x}} \in \mathsf{V}(f, \mathcal{A})$, where $f \in \Gamma^{\#}$; from (3.2) and lemma 3.4 it follows that for each $\alpha \in \mathrm{pr}(\mathsf{D}_i)$ and $\beta \in \mathrm{pr}(\mathsf{W}_i)$ there exist $y_\alpha, z_\beta \in \mathrm{varb}(f)$ such that $\bar{y}_\alpha \in \mathsf{D}_{i,\alpha}$ and $\bar{z}_\beta \in \mathsf{W}_{i,\beta}$; moreover, since $\mathcal{A}$ is linearized, z_β is unique for each β. Put

$$Y_i = \{y_\alpha |\ \alpha \in \pi_i^\circ\} \subseteq \mathrm{varb}_i(f) \tag{3.52}$$

where every y_α is chosen in some way; also, let $Y = \bigcup_{i=1}^k Y_i$. Next, put

$$Z_i^\alpha = \{z_\beta |\ \beta \in \omega_{i,\alpha}\} \quad , \quad Z_i = \bigcup_{\alpha \in \pi^\circ} Z_{i,\alpha} \tag{3.53}$$

and $Z = Z(f, \bar{\mathbf{x}}) = \bigcup_{i=1}^k Z_i$.

Lemma 3.19. (Main Lemma) *For all multilinear $f \in \Gamma^{\#}$, $\bar{\mathbf{x}} \in \mathsf{V}(f, \mathcal{A})$ and $Y, Z \subseteq \mathrm{varb}(f)$ defined above, there exists a substitution $\tilde{\mathbf{x}} \in \mathsf{V}(f, \mathcal{A})$ such that:*

(M1) *if $x \notin Y \cup Z$, then $\tilde{x} = \bar{x}$;*
(M2) *for all i and $\alpha \in \pi_i^\circ$, if $x \in \{y_\alpha\} \cup Z_{i,\alpha}$, then $\tilde{x} \in \mathsf{Spl}_i^\alpha(\mathbf{L}')$;*
(M3) *there exists i and $\alpha' \in \pi_i^\circ$ such that $\tilde{y}_{\alpha'} \in \mathsf{W}_i^{\alpha'}(\mathbf{L}')$.*

Proof. Suppose, the lemma is untrue for some f, $\bar{\mathbf{x}}$ and Y. It means that for all substitutions $x \to \tilde{x} \in \mathsf{Spl}(\mathbf{L})$, satisfying (M1)-(M3) we have

$$f(\tilde{\mathbf{x}}) = 0 \ . \tag{3.54}$$

Using this hypothesis, we will build an element from T-ideal (3.51), which is not an identity of $T(\mathcal{A})$.

By definition, all nonstandard submodules of W_i are contained in $\mathsf{W}_i^\bullet$, let $\omega_i^\star \subseteq \omega_i^\bullet$ denote the corresponding set of projectors. Also, for all $\alpha \in \pi_i^\bullet \cap \Omega$ the set $\omega_{i,\alpha}$ contains the projector on a nonstandard module (otherwise, α would be isolated); we fix one of them, say β; in this case we say that β is *marked for* α. For all $\beta \in \omega_i^\star$ let $\mathbf{m}_\beta$ denote the set of all α for which β being marked; note that $\mathbf{m}_\beta$ can be empty.

Put $q = |\mathrm{pr}(\mathsf{W})|$, by (3.2) it is equal to $|\mathrm{b}|$. Take $\beta \in \omega_i^\star$, where $\dim(\mathrm{ad}_\beta(L)) = m$ and $\mathbf{m}_\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. From lemma 2.10, 2.11 and 2.12 it follows that there exits a Lie polynomial $f_\beta \in \mathrm{L}\langle X_i\rangle$ of the form

$$h\left(\phi_\beta; \phi_{\alpha_1}^{(0)}, \dots, \phi_{\alpha_1}^{(q)}, \dots, \phi_{\alpha_n}^{(0)}, \dots \phi_{\alpha_n}^{(q)} \dots ,\right) \ , \tag{3.55}$$

where

(1) h is a Lie polynomial
$$h\left(y, x_1^{(0)}, \dots, x_1^{(q)}, \dots, x_n^{(0)}, \dots, x_n^{(q)} \dots ,\right) \in \mathrm{L}\langle X_i\rangle$$
which is linear with respect to y and all $x_j^{(p)}$, $1 \le j \le n$, $0 \le p \le q$;
(2) $\phi_\beta(z_\beta, \dots)$ is multilinear and skew-symmetric with respect to $m+1$ variables including z_β (cf. lemma 2.12);
(3) each $\phi_{\alpha_j}^{(p)}$ is linear and skew-symmetric with respect to $\dim(\mathsf{D}_{i,\alpha_j})$ variables;
(2) there is a substitution $x \to \bar{x}$ such that for all $x \in \mathrm{varb}(\phi_{\alpha_j}^{(p)})$ the element $\bar{x}$ lies in D_{i,α_j} and $\bar{\psi}_\beta = \bar{z}_\beta$ modulo J_i.

Note that $\psi_\beta = \phi_\beta$, if $\mathbf{m}_\beta$ is empty; by lemma 2.5, $[\mathsf{W}_{i,\beta}, \underbrace{L_i, \dots, L_i}_{c}] = \mathsf{W}_{i,\beta}$ modulo J_i, hence, there exists a polynomial

$$g_\beta \in [f_\beta, \underbrace{\mathrm{L}\langle X_i\rangle, \dots, \mathrm{L}\langle X_i\rangle}_{c}]$$

and an extension of the substitution $x \to \bar{x}$ such that $\bar{g}_\beta = \bar{z}_\beta$. Moreover, we can choose these polynomials depending on different variables and the substitution $x \to \bar{x}$ extends $\bar{\mathbf{x}}$, so $\bar{f}' = \bar{f}$, where

$$f' = f|_{z_\beta = g_\beta \ \forall \beta \in \omega_i^\star \ \forall i} \ .$$

We cannot require g_β to be multilinear, but we can assume that it is a sum of multilinear polynomials, so is f'; note that every homogeneous component of g_β is linear with rsepect to z_β and each $\lambda_\alpha^{(p)}$.

Next, by definition, for each $\alpha \in \pi_i^\bullet \backslash \Omega$ we have $\mathrm{Ann}(\alpha) = 0$, hence, there are $\xi_\alpha \in \{\rho_j, \delta_j\}$ and $\bar{x}_\alpha \in \mathsf{D}_{i,\alpha}$ such that

$$\bar{f} \cdot \prod_{i=1}^{k} \prod_{\alpha \in \pi^\bullet \backslash \Omega} \xi_\alpha(\bar{x}_\alpha) \neq 0 \ .$$

Like in the case above, for each $\alpha \in \pi_i^\bullet \backslash \Omega$ we build a Lie polynomial

$$f_\alpha = h\left(\mathbf{x}_\alpha, \phi_\alpha^{(0)}, \dots, \phi_\alpha^{(q)}, \dots\right) \ , \tag{3.56}$$

where $h = h(y, x^{(0)}, \dots, x^{(q)}, \dots) \in \mathrm{L}\langle X_i\rangle$ is linear with respect to y and all $x^{(p)}$, each $\phi_\alpha^{(p)}$ is linear and skew-symmetric with respect to $\dim(\mathsf{D}_{i,\alpha})$ variables; and for a substitution of suitable elements in $\mathsf{D}_{i,\alpha}$ we get $\bar{\psi}_\alpha = \bar{x}_\alpha$. Moreover, like in the case above we can find

$$g_\alpha \in [f_\alpha, \underbrace{\mathrm{L}\langle X_i\rangle, \dots, \mathrm{L}\langle X_i\rangle}_{c}] \ , \tag{3.57}$$

which is a sum of multilinear polynomials, and an extension of the substitution $x \to \bar{x}$ such that $\bar{g}_\alpha = \bar{x}_\alpha$. Put

$$f'' = f' \cdot \prod_{i=1}^{k} \prod_{\alpha \in \pi_i^\bullet \backslash \Omega} \xi(g_\alpha) \ ;$$

we assume that the substitution above extends $\bar{\mathbf{x}}$, hence $\bar{f}_2 \neq 0$.

Finally, for all $\alpha \in \pi_i^\circ$ we replace every $y_\alpha \in Y_i$ by a polynomial g_α of form (3.57) and extend the substitution $x \to \bar{x}$ so that $\bar{g}_\alpha = \bar{y}_\alpha$ and the resulting polynomial, f''', does not vanish.

So, for all i, $\alpha \in \mathrm{pr}(\mathsf{D})$ and $p \in \{0, 1, \dots, q\}$ we have built a polynomial $\phi_\alpha^{(p)}$ which is multilinear and skew-symmetric with respect to every set of variables λ_α^p, where $|\lambda_\alpha^p| = \dim(\mathsf{D}_\alpha)$ for all p. Also, we can assume that for each α and p

(*) $\phi_\alpha^{(p)}$ is linear with respect to some variable $x \notin \bigcup_{p=0}^{q} \lambda_\alpha^{(p)}$.

Let $\mathrm{pr}(\mathsf{W}) = \{\beta_1, \dots, \beta_q\}$; for all $\beta = \beta_p \in \mathrm{pr}(\mathsf{W})$ put

$$\Theta_p = \{z_\beta\} \cup \bigcup_{\alpha \in \mathrm{pr}(\mathsf{D})} \lambda_\alpha^{(p)} \ ,$$

where $z_\beta \in Z$; in addition, let
$$\Theta_0 = \bigcup_{\alpha \in \mathrm{pr}(\mathsf{D})} \lambda_\alpha^{(0)} .$$
Notice that every homogeneous component of f''' has the form of f_1 in Step 2 of the definition of (2.66); namely, the role of f is played by
$$f \cdot \prod_{i=1}^{k} \prod_{\alpha \in \pi^\bullet \setminus \Omega} \xi_\alpha(x_\alpha),$$
where ϑ is equal to
$$Z \cup \{x_\alpha |\ \alpha \in \pi_i^\bullet \setminus \Omega\} \cup \{x_\alpha |\ \alpha \in \pi_i^\circ\}$$
and $g_x = g_\beta$ ($g_x = g_\alpha$) if $x = z_\beta$ ($x = x_\alpha$).

Let h_0' denote the alternation of f''' with respect to each Θ_p, $0 \le p \le q$.

LEMMA 3.20. *Under the substitution $x \to \bar{x}$ the polynomial h_0' does not vanish.*

Proof. Let G be the group $\prod_{p=0}^{c} \mathrm{Sym}(\Theta_p)$, then
$$h_0' = \sum_{\sigma \in G} \mathrm{sgn}(\sigma)\sigma \cdot f''' .$$
Take $\sigma \in G$ such that under $x \to \bar{x}$
$$\overline{\sigma \cdot f'''} \neq 0 . \tag{3.58}$$
Let us show that $\sigma(z_\beta) = z_\beta$ for all $\beta \in \mathrm{pr}(\mathsf{W})$.

First, suppose that $\mathsf{W}_{i,\beta}$ is non-standard, i.e., $\beta \in \omega_i^\star$. If $\sigma(z_\beta) \neq z_\beta$ then $y = \sigma^{-1}(z_\beta)$ lies in some $\lambda_j^{(p)}$, hence for all $x \in \mathrm{varb}(\sigma \cdot \phi_\beta)$ the element $\bar{x}$ belongs to $\mathsf{D} \cup C$. Recall that ϕ_β is linear and skew-symmetric with respect to $m+1$-variables, z_β and, say, $x_1, \dots, x_m$, hence, $\sigma \cdot \phi_\beta$ has the same property for $y, x_1, \dots, x_m$.

By definition, $\bar{x} \in \mathsf{D} \cup \mathsf{C}$ for all $x \in \mathrm{varb}(\phi_\beta)$, $x \neq z_\beta$; hence $\bar{x} \in \mathsf{D} \cup \mathsf{C}$ for all $x \in \mathrm{varb}(\sigma \cdot \phi_\beta)$. Suppose, there is a variable among $y, x_1, \dots, x_m$ which belongs to C, then all variables of $\sigma \cdot \phi_\beta$ must take value in C too. Recall that degree of each homogeneous component of ϕ_β is not less than nilpotency index of C, hence $\overline{\sigma \cdot \phi_\beta}$ must be equal to 0 and (3.58) does not hold. So, $\bar{x}_1, \dots, \bar{x}_m$ lies in some $\mathsf{D}_{i,\alpha}$ which is isomorphic to a subalgebra of $\mathrm{ad}_\beta(L)$, hence $\bar{y} \in \mathsf{D}_{i,\alpha}$ too. Besides, $\dim(\mathsf{D}_{i,\alpha}) \le \dim(\mathrm{ad}_\beta(L)) = m$, hence, $\overline{\sigma \cdot \phi_\beta} = 0$ modulo J_i, which is impossible. Thus, σ must fix z_β.

Now suppose that $\mathsf{W}_{i,\beta}$ is standard, then there exists only one $\alpha \in \mathrm{pr}(\mathsf{D}_i)$ such that $[\mathsf{W}_{i,\beta}, \mathsf{D}_{i,\alpha}] \neq 0$, hence, by (*), $\sigma(z_\beta) \in \lambda_\alpha^{(p)}$. We have three cases: $\alpha \in \pi_i^\bullet \cap \Omega$, $\alpha \in \pi_i^\bullet \setminus \Omega$ and $\alpha \in \pi_i^\circ$. In the first case, $\alpha \in \mathbf{m}_{\beta'}$ for some $\beta' \in \omega_j^\star$; hence, $\sigma(z_\beta) \in \mathrm{varb}(f_{\beta'})$. By the previous case, $\sigma(z_{\beta'}) = z_{\beta'}$, hence, $\overline{\sigma \cdot f_{\beta'}}$ belongs to the ideal J_i, and, by lemma 3.3, inequality (3.58) is impossible. Next, if $\alpha \in \pi_j^\bullet \setminus \Omega$, then $\overline{\sigma \cdot \phi_\alpha^{(p)}}$ lies in the ideal generated by W_i, hence, by lemma 3.9, (3.58) does not hold. Finally, if $\alpha \in \pi_j^\circ$, then $\sigma(z_\beta) \in Y$. Hence,
$$\overline{\sigma \cdot f'''} = f(\tilde{x}_1, \dots, \tilde{x}_n) \cdot \prod_{\alpha \in \pi^\bullet \setminus \Omega} \xi_\alpha(\bar{x}_\alpha) ,$$
where $x \to \tilde{x}$ is the substitution satisfying (M1)-(M3), hence, $\overline{\sigma \cdot f'''}$ must equal 0 which contradicts to (3.58).

So, $\sigma(z_\beta) = z_\beta$ for all β; suppose $\beta = \beta_p$, then for all α and $x \in \mathrm{varb}(\sigma \cdot \phi_\alpha^{(p)})$ we have $\bar{x} \in \mathsf{D}_i \cup \mathsf{C}_i$, hence , by (*), $\sigma(\lambda_\alpha^{(p)}) = \lambda_\alpha^{(p)}$ for all α. Similarly, if $\alpha \in \mathrm{pr}(\mathsf{D}_i)$, then $\sigma(\lambda_\alpha^{(0)}) = \lambda_\alpha^{(0)}$. Thus, if $\overline{\sigma \cdot f'''} \neq 0$, then σ belongs to the direct product of groups,

$$G_1 = \prod_{p=0}^{q} \prod_{\alpha \in \mathrm{pr}(\mathsf{D})} \mathrm{Sym}(\lambda_\alpha^{(p)}) \ .$$

Therefore, $\bar{h}_0' = |G_1| \cdot \bar{f}''' \neq 0$, so lemma 3.20 is proven. □

To obtain an element in $\mathcal{S}_{\mathrm{a,b,}c}(\Gamma^\#)\backslash T(\mathcal{A})$ we make the following substitution into h_0': every $x \in \lambda_\alpha^{(p)}$, $p \in \{0, \ldots , q\}$, is replaced with a Lie polynomial ϕ_x defined as follows. By lemmas 2.10 and 2.11, there exists a polynomial $\phi_x \in L\mathrm{Cap}_{m,1}^2$, where $m = \dim(\mathsf{D}_{i,\alpha})$, with homogeneous components of degree $\geq d_i$ such that $\bar{\phi} = \bar{x}$ under some substitution of elements in $\mathsf{D}_{i,\alpha}$. Clearly, it belongs to the T-ideal generated by all Lie polynomials of form (2.63).

Next, every $z_\beta \in \Theta_p$, where $\beta = \beta_p$, plays the role of the fixed element y_p; by lemma 2.12, there exists a polynomial ϕ_{y_p} from the T-ideal generated by (2.64), where $m = \dim(\mathrm{ad}_\beta(L))$ and $l = \dim(\mathsf{W}_\beta)$ such that $\bar{\phi}_{y_p} = \bar{z}_\beta$ under a suitable substitution. Thus, replacing every $x \in \Lambda_p$ with ϕ_x in h_0' we get a polynomial h' from $\mathcal{S}_{\mathrm{a,b,}c}(\Gamma^\#)$ which does not vanish under the substitution $x \to \bar{x}$. □

6. Sorting of variables

Put $\Upsilon = \Gamma^\#$ and take a multilinear $f = f(x_1, \ldots , x_n) \in \Gamma^\#$; modulo $T(\mathcal{A})$ it can be regarded as a function in $\mathrm{Pol}(\vec{\mathbf{L}}; V)$ in generic elements $x_1, \ldots , x_n$. Besides, from (3.46) it follows that modulo J_i every x_j is decomposed into the sum of $\mu_i^\gamma(x_j)$, where γ ranges over $\tilde{\pi}_i$. Hence, by (3) of proposition 3.3, the function f is represented as

$$f(\sum_{\mu_1} \mu_1(x_1), \ldots , \sum_{\mu_n} \mu_n(x_n)) = \sum_{\mu_1, \ldots , \mu_n} f(\mu_1(x_1), \ldots , \mu_n(x_n)) \ , \tag{3.59}$$

where μ_j runs over $\{\mu_i^\gamma, \gamma \in \tilde{\pi}_i^\circ\}$ provided $x_j \in X_i$; note that

$$g(x_1, \ldots , x_n) = f(\mu_1(x_1), \ldots , \mu_n(x_n)) \tag{3.60}$$

is a multilinear mapping in $\mathrm{Pol}(\vec{\mathbf{L}}, V)$.

Let $\mathbf{F}(\mathbf{L})$ be the k-sorted Lie algebra of generic elements of $\mathbf{L}$, see (2.18); it is contained in $\mathrm{Pol}(\vec{\mathbf{L}}; L)$, hence the g.Lie representation $\nu : L \to \mathrm{End}(V)$ induces that of $\mathbf{F}(\mathbf{L})$ in $\mathrm{End}\left(\mathrm{Pol}(\vec{\mathbf{L}}, V)\right)$. A subspace of $\mathrm{Pol}(\vec{\mathbf{L}}, V)$ is said to be fully characteristic, if it is stable under substitutions of $\mathcal{F}(\mathbf{L})_i$ in place of generic elements in X_i. Let $\tilde{\Upsilon}$ denote the fully characteristic subspace of $\mathrm{Pol}(\vec{\mathbf{L}}, V)$ generated by all elements (3.60).

PROPOSITION 3.6. *$\tilde{\Upsilon}$ is a finitely generated fully characteristic submodule over* $\mathsf{En}(\mathbf{F}(\mathbf{L}), \nu)$.

The proposition states the following: there exist functions $g_1, \ldots , g_t$ of form (3.60) such that $\tilde{\Upsilon}$ is equal to the span of elements

$$g_p(u_1, \ldots , u_n) \cdot E \ , \tag{3.61}$$

where $p \in \{1, \ldots , t\}$, $u_j \in \mathcal{F}(\mathbf{L})_i$ provided $x_j \in X_i$ and $E \in \mathsf{En}(\mathbf{F}(\mathbf{L}), \nu)$.

We will rewrite this statement in terms of identities of g.Lie modules in the variety $\mathfrak{N} = \mathfrak{M}_{k',s,r}$, where k' is given by (3.42); then, using Main Lemma, we will pass to another variety with a "counter-example" of type < type$(\Gamma, \mathcal{A})$.

Consider a k'-sorted set of free variables :

$$X' = \bigcup_{i=1}^{k} \bigcup_{\gamma \in \widetilde{\pi}^{\circ}} X_i^{\gamma} \tag{3.62}$$

where $X_i^{\gamma} = \{x^{\gamma} | \; x \in X_i\}$; it can be considered as a k-sorted set of free variables with

$$X_i' = \bigcup_{\gamma \in \widetilde{\pi}^{\circ}} X_i^{\gamma} \; .$$

Let $\mathcal{M}\langle X' \rangle$ be a free g.Lie module in $\mathfrak{M}$ with the set of free generators X'. If I is a T-ideal of $\mathcal{M}\langle X \rangle$, then denote by I' the verbal ideal in $\mathcal{M}\langle X' \rangle$; as a vector space, I is generated by elements of the form

$$f(x_1^{\gamma_1}, \dots , x_n^{\gamma_n}) \; , \tag{3.63}$$

where $f \in I$ and $x_j^{\gamma} \in X_i'$ provided $x_j \in X_i$ (in other words, we replace the set of variables X_i with X_i'). In particular, for all T-ideals I, J of $\mathcal{M}\langle X \rangle$ we have

$$I \nsubseteq J \iff I' \nsubseteq J' \; . \tag{3.64}$$

Put

$$\mathcal{B} = (\mathbf{L}'; V'; \nu, \mathrm{v}_1, \dots , \mathrm{v}_r) \in \mathfrak{N} \; , \tag{3.65}$$

where $\mathbf{L}'$ is the k-sorted Lie algebra (3.43); it is a finite dimensional g.Lie module in $\mathfrak{N}$. Similarly, define a k'-sorting of the Lie algebra $\mathrm{L}\langle X' \rangle$ setting the (i, γ)-component to be the ideal $\mathrm{L}_i^{\gamma}\langle X \rangle$ generated by X_i^{γ}, so

$$\mathbf{L}'\langle X' \rangle = (\mathrm{L}\langle X' \rangle; \dots , \mathrm{L}_i^{\gamma}\langle X \rangle, \dots) \; .$$

Observe that $\mathsf{U}(\mathbf{L}\langle X' \rangle, \nu) = \mathsf{U}(L\langle X' \rangle, \nu) = \mathsf{U}(\mathbf{L}'\langle X' \rangle, \nu)$, hence,

$$\mathcal{N}\langle X' \rangle = (\mathbf{L}'\langle X' \rangle; V\langle X' \rangle) \; , \tag{3.66}$$

where $V\langle X' \rangle$ is the second coordinate of $\mathcal{M}\langle X' \rangle$, is the free g.Lie module in $\mathfrak{N}$. Note that every T-ideal $I \subseteq V\langle X' \rangle$ of $\mathcal{M}\langle X' \rangle$ is that of $\mathcal{N}\langle X' \rangle$. Put $\Upsilon = \Gamma^{\#}$.

Proposition 3.7. *The T-ideal Υ' is finitely generated modulo $T(\mathcal{B})$.*

First of all, we show that it yields proposition 3.6. Define a mapping

$$\theta : \widetilde{\Upsilon} \to \Upsilon'/T(\mathcal{B}) \; ;$$

take $g \in \widetilde{\Upsilon}$; it can be represented in the form

$$g = \sum_{f; \mu_1, \dots , \mu_n} f(\mu_1(x_1), \dots , \mu_n(x_n)) \; , \tag{3.67}$$

where $f \in \Upsilon$ and $\mu_1, \dots , \mu_n \in \{\mu_i^{\gamma} | \; i, \gamma\}$. Put

$$\theta(g) = \sum_{f; \mu_1, \dots , \mu_n} f(x_1^{\gamma_1}, \dots , x_n^{\gamma_n}) \in \Upsilon'/T(\mathcal{B}) \; , \tag{3.68}$$

where γ_j is defined by equality $\mu_j = \mu_i^{\gamma_j}$. Observe that the definition is correct; for it suffices to check that $g = 0$ implies $\theta(g) = 0$. On the contrary, suppose that $\theta(g)$ do not vanish under $x \to \bar{x}$; by defintion, $\bar{x}_j^{\gamma_j}$ must be in $L_i^{\gamma_j}$, where $\gamma_j \in \widetilde{\pi}_i^\circ$. On the othe hand, the projector γ_j has been defined by the equality $\mu_j = \mu_i^{\gamma_j}$. Hence, $\mu_j(\bar{x}_j^{\gamma_j}) = \bar{x}_j^{\gamma_j}$ modulo J_i and

$$\sum_{f;\mu_1,\dots,\mu_n} f(\mu(\bar{x}_1),\dots,\mu(\bar{x}_n)) = \overline{\theta(g)} \neq 0 \ ,$$

which contradicts to the hypothesis.

LEMMA 3.21. *θ is bijective.*

Proof. Clearly, θ is onto; let us show that $\mathrm{Ker}(\theta) = 0$. Take $g \in \widetilde{\Upsilon}$ of form (3.67) and suppose that g does no vanish under $x \to \bar{x}$. If $x_j \in X_i$, then $\bar{x}_j \in L_i$; take $\bar{x}_j^{\gamma_j} \in L_i^{\gamma_j}$ such that $\mu_j(\bar{x}_j) = \bar{x}_j^{\gamma_j}$ modulo J_i. By irreducibility of the "counter-example" (see (3) of proposition 3.3)

$$\sum_{f;\mu_1,\dots,\mu_n} f(\bar{x}_1^{\gamma_1},\dots,\bar{x}_n^{\gamma_n}) = \sum_{f;\mu_1,\dots,\mu_n} f(\mu(\bar{x}_1),\dots,\mu(\bar{x}_n))$$

hence $\theta(g)$ takes a nonzero value under the substitution $x_j^{\gamma_j} \to \bar{x}_j^{\gamma_j}$. □

Let $h_1,\dots,h_t$ be multilinear generators of Υ' modulo $T(\mathcal{B})$, we can assume that they are of form (3.63), where $f_l = f_l(x_1,\dots,x_n) \in \Upsilon$, $l = 1,\dots,t$. Hence, Υ' is spanned modulo $T(\mathcal{B})$ by polynomials

$$f_l(u_1,\dots,u_n)\cdot E \tag{3.69}$$

where $u_j \in \mathrm{L}_i^{\gamma}\langle X'\rangle$ provided $\mu_j = \mu_i^{\gamma}$, and E is a monomial in $\nu(X')$.

To prove proposition 3.6 it suffices to show that the pre-image of (3.69) belongs to the fully characteristic submodule generated by $g_l = f(\mu_1(x_1),\dots,\mu_n(x_n)) = \theta^{-1}(h_l)$, $l \in \{1,\dots,t\}$. Let g be the polynomial obtained from (3.69) by replacing every variable $x^\gamma \in X_i^\gamma$ with $\mu_i^\gamma(x)$,

$$g = f(u_1',\dots,u_n')E',$$

then $\theta(g)$ is equal to (3.69) by definition. Recall that μ_j is a homomorphism from L_i to $L_i^{\gamma_j}/\mathsf{J}_i^{\gamma_j}$, hence, we can replace u_j' with $\mu_j(v_j)$, where $v_j \in \mathrm{L}_i\langle X\rangle$. Also, as $\pi_i^\circ \subseteq \Omega(\Gamma,\mathcal{A})$, in E' every operator $\delta_p(\mu_j(x))$ (or $\rho_p(\mu_j(x))$) can be replaced with $\delta_p(x)$ ($\rho_p(x)$, respectively) for all $x \in X$. Therefore, the pre-image of (3.69) belongs to the fully characteristic submodule generated $g_1,\dots,g_t$, which gives proposition 3.6.

Proof of proposition 3.7. Suppose it is untrue, then there exists a critical T-ideal $\Psi \subseteq \Upsilon' + T(\mathcal{B})$, where $T(\mathcal{B}) \subset \Psi$ and $\Psi^\# \subseteq \Upsilon'$; by (3.45), we have $\Upsilon' \subseteq \mathcal{Q}(\mathcal{B})$, hence, $T(\mathcal{B}) \subset \Upsilon'$ is a "counter-example" in $\mathfrak{N}$; observe that it inherits properties (1)-(3) in proposition 3.3. Let us show that its type is $\le \mathrm{type}(\Gamma,\mathcal{A})$.

Recall that

$$\mathsf{Spl}(\mathbf{L}') = \bigcup_{i=1}^{k}\bigcup_{\alpha\in\widetilde{\pi}_i^\circ}\mathsf{Spl}_i^\alpha(\mathbf{L}') = \bigcup_{i=1}^{k}\mathsf{Spl}_i(\mathbf{L}) = \mathsf{Spl}(\mathbf{L}) \ ,$$

also, (3.2) holds for all $\bar{\mathbf{x}}$, hence

$$\|\mathrm{a}(\Psi,\mathcal{B})\| \le \|\mathrm{a}(\Gamma,\mathcal{A})\| \ , \mathrm{a}(\Psi,\mathcal{B}) \le \mathrm{a}(\Gamma,\mathcal{A}), \tag{3.70}$$

$$|\mathrm{b}(\Psi,\mathcal{B})| \leq |\mathrm{b}(\Gamma,\mathcal{A})| \, , \mathrm{b}(\Psi,\mathcal{B}) \leq \mathrm{b}(\Gamma,\mathcal{A}) \ , \tag{3.71}$$

$$\mathrm{ht}(\Psi,\mathcal{B}) \leq \mathrm{ht}(\Gamma,\mathcal{A}) \ . \tag{3.72}$$

Since the type of the "counter-example" $T(\mathcal{A}) \subset \Gamma$ is the smallest one, (3.70) must be equalities. Now we need to show that

$$\mathrm{t}(\Psi,\mathcal{B}) \leq \mathrm{t}(\Gamma,\mathcal{A}) \ . \tag{3.73}$$

By the definition of (2.57), for all $\mathrm{b} \in (\mathbb{N}_0^\infty)^\infty$ and $c \in \mathbb{N}$

$$\mathcal{S}_{\mathrm{a,b},c}(\Psi) \subseteq \mathcal{S}_{\mathrm{a,b},c}(\Upsilon') \subseteq (\mathcal{S}_{\mathrm{a,b},c}(\Upsilon))' \ ,$$

hence, if $\mathcal{S}_{\mathrm{a,b,c}}(\Psi) \nsubseteq T(\mathcal{B})$, then $(\mathcal{S}_{\mathrm{a,b},c}(\Upsilon))' \nsubseteq T(\mathcal{A})'$, because $T(\mathcal{A})' \subseteq T(B)$; by (3.64)

$$\mathcal{S}_{\mathrm{a,b},c}(\Upsilon) \nsubseteq T(\mathcal{A})$$

which proves (3.73).

To simplify notations, we put $k = k'$, hence $\mathfrak{M}$ becomes $\mathfrak{N}$. Also, now Γ means Ψ and $\mathcal{A}$ denotes $\mathcal{B}$, where $L_1, \dots, L_m$, $m \leq k$, corresponds to L_i^γ with γ running over π_i° and $i \in \{1, \dots, k\}$. Therefore, the other components, $L_{m+1}, \dots, L_k$, are all $L_i^\bullet$. So, for all $i \leq m$ the algebra D_i is simple, $\mathsf{C}_i = 0$ and all $\mathsf{W}_{i,\beta}$ are standard.

Under this change of indices Main Lemma takes the following form. Let $\bar{\mathbf{x}} \in \mathsf{V}(f,\mathcal{A})$, where f is a multilinear polynomial in $\Gamma^\#$; let $Y = \{y_1, \dots, y_m\} \subseteq \mathrm{varb}(f)$ such that $\bar{y}_i \in \mathsf{D}_i$ for all $i \leq m$; also, put $Z_i = \{z_\beta | \ \beta \in \omega_i\}$ and $Z = \bigcup_{i=1}^k Z_i$.

Lemma 3.22. *For any multilinear $f \in \Gamma^\#$, $\bar{\mathbf{x}} \in \mathsf{V}(f,\mathcal{A})$ and $Y \subseteq \mathrm{varb}(f)$ defined above, there exists a substitution $\tilde{\mathbf{x}} \in \mathsf{V}(f,\mathcal{A})$ such that:*

(M1') *if $x_j \notin Y \cup Z$, then $\tilde{x}_j = \bar{x}_j$;*
(M2') *for all $i \leq m$ if $x_j \in \{y_i\} \cup Z_i$, then $\tilde{x}_j \in \mathsf{Spl}_i(\mathbf{L})$;*
(M3') *$\tilde{y}_i \in \mathsf{W}_i$ for some $i \leq m$.*

Proof. To check it we have to use the "old" notations. Recall that Γ means Ψ, since $f \in \Psi^\# \subseteq \Upsilon'$, it has the form $f'(x_1^{\gamma_1}, \dots, x_n^{\gamma_n})$ for some multilinear $f' \in \Upsilon$ (so, $x_1, \dots, x_n$ are different). Next, every $y_\alpha \in Y$, where $\alpha \in \pi_i^\circ$, belongs to X_i^α, i.e., $y_\alpha = x_j^{\gamma_j}$ for some j (where $\gamma_j = \alpha$); put $y'_\alpha = x_j \in X_i$ and let $Y'_i = \{y'_\alpha | \alpha \in \pi_i^\circ\}$ and $Y' = \bigcup_{i=1}^k Y'_i \subseteq \mathrm{varb}(f')$. Observe that f', Y' and the substitution $x_j \to \bar{x}_j^{\gamma_j}$ satisfy the hypothesis of Main Lemma; so there exists $x \to \tilde{x}$ with properties (M1)-(M3). Hence, the substitution $\tilde{x}_j^{\gamma_j} = \tilde{x}_j$ satisfies (M1')-(M3') (translated into "old" notations). □

Now we are going to define a mapping $\mathcal{B} \to \mathcal{B}^\sim$ from $\mathfrak{M}$ to a variety $\mathfrak{M}_{k,s^\sim,r^\sim}$, which gives a "counter-example" of lesser type; in fact, it will be similar to examples given in section 3.

Fix $y_i \in X_i$ for all $i \in \{1, \dots, m\}$, put $Y = \{y_1, \dots, y_m\}$, $X'_i = X_i \backslash \{y_i\}$ and $X' = \bigcup_{i=1}^k X'_i$. Let $V\langle Y; X' \rangle$ denote the subspace of all polynomials from $V\langle X \rangle$ which are linear with respect to each $y_i \in Y$. As a T-ideal Γ is generated by its multilinear elements; moreover, every multilinear $f \in \Gamma^\#$ of degree 0 with respect some X_i, $i \leq m$, must be an identity of $\mathcal{A}$ thanks to irreducibillity of the "counter-example". Hence, modulo $T(\mathcal{A})$ the T-ideal Γ is generated by $\Gamma \cap V\langle Y; X' \rangle$ and if f_n, $n \in \mathbb{N}$, is the appropriate sequence for Γ, then we may assume that each f_n depends on all $y_i \in Y$, i.e. $f_n \in V\langle Y; X' \rangle$.

LEMMA 3.23. *$V\langle Y;X'\rangle$ is spanned by elements of the form*

$$h(u_1,\dots,u_m)\cdot E$$

where u_i belongs to the submodule of $\mathrm{L}\langle X\rangle$ *over* $\mathrm{L}\langle X'\rangle$ *generated by y_i, E is a product of operators in $\nu(X')$ and h runs over the set*

$$\{\mathrm{v}_j\xi_1(y_{i_1})\dots\xi_m(y_{i_m})|\ \xi_l\in\bigcup_{p=1}^{s}\{\rho_p,\delta_p\}\ , \tag{3.74}$$

$$j\in\{1,\dots,r\},\quad \{i_1,\dots,i_m\}=\{1,\dots,m\}\}\ .$$

Proof. Indeed, $V\langle Y;X'\rangle$ is spanned by elements

$$\mathrm{v}_j\cdot w_1\dots w_m E$$

where w_p lies in the submodule over $\nu(\mathcal{M}\langle X'\rangle)\subseteq\nu(\mathcal{M}\langle X\rangle)$ generated by $\xi(y_{i_p})$, $\xi\in\bigcup_{p=1}^{s}\{\rho_p,\delta_p\}$, and E is a product of operators in $\nu(X')$. Using (2.10)-(2.12), we prove the statement. □

Let $h_1,\dots,h_{r^\sim}$ be all elements in (3.74). Take $\mathcal{B}=(\mathbf{M};W)\in\mathfrak{M}$ and define $\mathcal{B}^\sim=(\mathbf{M};W^\sim)\in\mathfrak{M}^\sim=\mathfrak{M}_{k,s^\sim,r^\sim}$, where $s^\sim=s+m$. Let $U=U(\mathcal{B})$ denote the vector space of all multilinear mappings from $M_1\times\dots\times M_m$ to V; every h_j defines a function in U:

$$(a_1,\dots,a_m)\to h_j(a_1,\dots,a_m)\ .$$

For all $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,k\}$ define $\rho_i^\sim : M\to\mathrm{End}(U)$ setting

$$h\cdot\rho_i^\sim(a)=h|_{y_i=[a,y_i]}\ ;$$

also, we put $\delta_i^\sim=0$. Furthermore, for all $i\in\{1,\dots,s\}$, we put $\rho_{i+m}^\sim(x)$ to be the operator $R_{\rho_i(x)}:h\to h\cdot\rho_i(x)$; similarly, $\delta_{i+m}^\sim(x)=R_{\delta_i(x)}$. Finally, let

$$\nu^\sim=(\nu_1^\sim,\dots,\nu_{s^\sim}^\sim)$$

where $\nu_i=(\rho_i^\sim,\delta_i^\sim)$; one may check that all identities (2.10)-(2.11) take place, hence $\nu^\sim$ is a g.Lie representation of the k-sorted Lie algebra $\mathbf{M}$ in $\mathrm{End}(U)$. Let $W^\sim$ denote the submodule over $\mathsf{En}(\mathbf{L},\nu^\sim)$ generated by $\mathrm{v}_1^\sim,\dots,\mathrm{v}_{r^\sim}^\sim$. Then

$$\mathcal{B}^\sim=(\mathbf{M};W^\sim;\nu^\sim;h_1,\dots,h_{r^\sim})$$

is a g.Lie module in $\mathfrak{M}^\sim$.

Let $\mathcal{M}$ be the free g.Lie module $\mathcal{M}\langle X'\rangle$ in the variety $\mathfrak{M}$ with the set of free variables $X'=X\backslash Y$. We identify $V\langle Y;X'\rangle$ with multilinear mappings in $U(\mathcal{M})$; by definition, $V\langle X'\rangle^\sim\subseteq V\langle Y,X\rangle$, hence, by lemma 3.23,

$$V\langle Y;X'\rangle=V\langle X'\rangle^\sim \tag{3.75}$$

Next, let $\mathcal{N}\langle X'\rangle=(\mathbf{L}\langle X'\rangle;W\langle X'\rangle)$ denote the free g.Lie module of $\mathfrak{M}^\sim$; the identity mapping of $\mathbf{L}\langle X'\rangle$ extends to an epimorphism (id,τ) from $\mathcal{N}$ onto $\mathcal{M}^\sim$.

LEMMA 3.24. *Every homomorphism from $\mathcal{N}$ to $\mathcal{B}^\sim$ can be represented as the composition of* (id,τ) *and a suitable homomorphism from $\mathcal{M}^\sim$ to $\mathcal{B}^\sim$.*

Proof. Indeed, let $(\phi, \psi) : \mathcal{N}\langle X'\rangle \to \mathcal{B}^{\sim}$ be a homomorphism, then $\phi : \mathbf{L}\langle X'\rangle \to \mathbf{M}$ is a homomorphism of k-sorted Lie algebras. Define a mapping $\zeta : V\langle Y; X'\rangle \to U(\mathcal{B})$; take $f = f(y_1, \ldots, y_m) \in V\langle Y; X'\rangle$, then $\zeta(f)$ is the following mapping in $U(\mathcal{B})$. Take $\bar{y}_i \in M_i$ for $i \leq m$ and extend ϕ to the homomorphism $\phi' : \mathbf{L}\langle X\rangle \to \mathbf{M}$, letting $\phi'(y_i) = \bar{y}_i$; it induces a homomorphism (ϕ', ψ') from $\mathcal{M}\langle X\rangle$ to $\mathcal{B}$. Put $\zeta(f)(\bar{y}_1, \ldots, \bar{y}_m) = \psi'(f)$; in other words, $\zeta(f)$ is obtained from the polynomial $f = f(y_1, \ldots, y_m, x_1, \ldots, x_n)$ by substitution of $\phi(x_j)$ in place of x_j.

Let us check that $\psi = \zeta \circ \tau$; i.e, for all $f \in W\langle X'\rangle$ we have $\psi(f) = \zeta(\tau(f))$. Recall that $f = \sum_{j=1}^{r^{\sim}} \mathrm{v}_j^{\sim} \cdot E_j$, so we can use the induction on the operator length of E_j. First, if $f = \mathrm{v}_j^{\sim}$, then $\psi(f) = h_j = \zeta(\tau(f))$; next, let $\xi \in \{\rho_i^{\sim}, \delta_i^{\sim}\}$ and $x \in X'$, we need to show that

$$\psi(f \cdot \xi(x)) = \zeta(\tau(f \cdot \xi(x))) \ .$$

Since (ϕ, ψ) is a homomorphism, the left hand side is equal to

$$\psi(f) \cdot \xi(\phi(x)) = \zeta(\tau(f)) \cdot \xi(\phi(x)) \ ;$$

we need to show that

$$\zeta(\tau(f)) \cdot \xi(\phi(x))(\bar{y}_1, \ldots, \bar{y}_m) = \zeta(\tau(f \cdot \xi(x)))(\bar{y}_1, \ldots, \bar{y}_m)$$

for all $\bar{y}_i \in M_i$. For example, let $\xi = \rho_1$, then, by the definition of ζ the left hand side is equal to

$$\zeta(\tau(f))([\phi(x), \bar{y}_1], \ldots, \bar{y}_m) =$$
$$= \psi'(\tau(f))([\phi(x), \bar{y}_1], \ldots, \bar{y}_m) = \psi'(\tau(f))([\phi'(x), \phi'(y_1)], \ldots, \phi'(y_m)) \ ;$$

further, since (ϕ', ψ') is a homomorphism, the latter term can be rewritten as

$$\psi'(\tau(f)([x, y_1], \ldots, y_m) = \psi'(\tau(f) \cdot \xi(x)) = \zeta(\tau(f) \cdot \xi(x))(\bar{y}_1, \ldots, \bar{y}_m) \ .$$

One can check similarly the other cases of ξ. $\square$.

From the lemma it follows that the full pre-image of $T(\mathcal{B}) \cap V\langle Y; X'\rangle$ with respect to τ coincides with $T(\mathcal{B}^{\sim})$ for all $\mathcal{B} \in \mathfrak{M}$. Indeed, if $\tau(f) \in T(\mathcal{B}) \cap V\langle Y; X'\rangle$, then f vanishes under any homomorphism from $\mathcal{N}\langle X'\rangle$ to $\mathcal{B}$.; the reciprocal inclusion is evident.

Since every T-ideal is equal to $T(\mathcal{B})$ for suitable $\mathcal{B}$, the full pre-image $\tau^{-1}(I \cap V\langle Y; X'\rangle)$, is a T-ideal. Put $\Upsilon = \tau^{-1}(\Gamma \cap V\langle Y; X'\rangle)$ and take $g_n \in \tau^{-1}(f_n)$, then g_n, $n \in \mathbb{N}$, is an appropriate sequence for Υ (observe that τ preserves the degree, because it is induced by the identity mapping of $\mathbf{L}\langle X'\rangle$). Besides, $\Upsilon \subseteq Q(\mathcal{A}^{\sim})$, hence $T(\mathcal{A}^{\sim}) \subset \Upsilon$ is a "counter-example" in $\mathfrak{M}^{\sim}$. Moreover, from definition it follows that $\mathrm{type}(\Upsilon, \mathcal{A}^{\sim}) \leq \mathrm{type}(\Gamma, \mathcal{A})$.

Actually, since the first component of $\mathcal{A}^{\sim}$ is $\mathbf{L}$, we have the analog of inequalities (3.70)-(3.72) in this case. Thus, we may assume that

$$\mathrm{a} = \mathrm{a}(\Gamma, \mathcal{A}) = \mathrm{a}(\Upsilon, \mathcal{A}^{\sim})$$

and we only need to show the following: if

$$\mathcal{S}_{\mathrm{a,b},c}(\Upsilon) \not\subseteq T(\mathcal{A}^{\sim}) \tag{3.76}$$

for some b and c, then $\mathcal{S}_{\mathrm{a,b},c}(\Gamma) \not\subseteq T(\mathcal{A})$. Take a typical generator (2.66) of the left hand side of (3.76) and suppose $h \notin T(\mathcal{A}^{\sim})$, then $\tau(h) \neq 0$. On the other hand, $\tau(h) = \tau(\mathcal{S}_{\mathrm{a,b},c}(f)) = \mathcal{S}_{\mathrm{a,b},c}(\tau(f))$; since $\tau(f) \in \Upsilon = \Gamma^{\#}$, we get the required relation.

Take $\beta \in \mathrm{pr}(\mathsf{W})$, hence $\beta \in \mathrm{pr}(\mathsf{W}_l)$ for some l; consider the k-sorted Lie subalgebra $\mathbf{M}^\beta$ of $\mathbf{L}$ generated by (3.13) and $\mathsf{Gn}_i(\mathbf{L})$, $i \neq l$; it generates a g.Lie submodule $\mathcal{C}^\beta$ of $\mathcal{A}^\sim$.

LEMMA 3.25.

$$T(\mathcal{A}^\sim) = \Upsilon \cap \bigcap_{\beta \mathrm{pr}(\mathsf{W})} T(\mathcal{C}^\beta) .$$

Proof. Let Υ' denote the right hand side, then $T(\mathcal{A}^\sim) \subseteq \Upsilon'$, and we need the reciprocal inclusion. Suppose it is untrue; take a multilinear $g \in \Upsilon' \backslash T(\mathcal{A}^\sim)$, then there is $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathsf{V}(g, \mathcal{A}^\sim)$. Put $f = \tau(g) \in \Gamma \cap V\langle Y; X' \rangle$, then $f = f(\mathbf{y}) = (y_1, \dots, y_m, x_1, \dots, x_n)$ and

$$g = g(\bar{\mathbf{x}}) = f(y_1, \dots, y_m, \bar{x}_1, \dots, \bar{x}_n)$$

is a nonzero mapping in $\mathsf{V}^\sim$. Hence, there exists $\bar{y}_i \in L_i$ such that $g(\bar{\mathbf{y}}) \neq 0$; by irreducibilty of the "counter-example" we may assume that each $\bar{y}_i$ lies in $\mathsf{Spl}_i(\mathbf{L})$. By lemma 3.22, there exists $\tilde{\mathbf{y}} \in \mathsf{V}(f, \mathcal{A})$ such that some $\tilde{y}_i$ lies in $\mathsf{W}_{i,\beta}$ for suitable β. Recall that $\mathcal{A}^\sim$ is linearized, hence, all $\tilde{x}_j$ belong to $\mathbf{M}^\beta$, which is impossible, because $f \in T(\mathcal{C}^\beta)$. □

By lemma 3.7

$$\mathrm{type}(\Upsilon, \mathcal{C}^\beta) < \mathrm{type}(\Upsilon, \mathcal{A}^\sim) = \mathrm{type}(\Gamma, \mathcal{A}) ,$$

so, proposition 3.7 follows from lemma 2.15.

7. Final step

Let $g_1, \dots, g_t$ be the functions from proposition 3.6,

$$g_l = f_l(\mu_1(x_1), \dots, \mu_n(x_n)) , \tag{3.77}$$

where the variables x_j and mappings μ_j depend on l as well as the number n, i.e., $x_j = x_j^{(l)}$, $\mu_j = \mu_j^{(l)}$ and $n = n^{(l)}$. Let q denote the maximum of $n^{(l)}$, $l \in \{1, \dots, t\}$. We are going to define a g.Lie module $\mathcal{A}^\diamond$ in the variety $\mathfrak{M}^\diamond = \mathfrak{M}_{k,s^\diamond,t}$, where $s^\diamond = q + s$.

Let y_i denote the projector of qL onto the i-th summand; consider the vector space U of all mappings from the direct sum qL to V that are linear or constant with respect to every y_i. For all $i \in \{1, \dots, q\}$ define linear mappings $\rho_i^\diamond, \delta_i^\diamond : L \to \mathrm{End}(U)$ by formulas (2.8), (2.9)); also, for all $i \in \{1, \dots, s\}$ and $a \in L$ put $\rho_{q+i}^\diamond(a) = R_{\rho_i(a)}$, $\delta_{q+i}^\diamond(a) = R_{\delta_i(a)}$. One may check that identities (2.10)-(2.11) hold, hence,

$$\nu^\diamond = (\nu_1^\diamond, \dots, \nu_{s^\diamond}^\diamond)$$

is a g.Lie representation of the the Lie algebra L; it induces a g.Lie representation of the k-sorted Lie algebra $\mathbf{L}$. Take $l \in \{1, \dots, t\}$ and consider (3.77); recall that every $\mu_j = \mu_i^\gamma$ extends to a homomorphism $L \to L_i^\gamma / \mathsf{J}_i^\gamma$, see (3.50); hence (3.77) defines a function $h_l \in U$.

Let $V^\diamond$ denote the submodule over $\mathsf{En}(\mathbf{L}, \nu^\diamond)$ generated by $h_1, \dots, h_t$ and put

$$\mathcal{A}^\diamond = (\mathbf{L}; V^\diamond; \nu^\diamond; h_1, \dots, h_t) ,$$

it is a f.d. generalized module in $\mathfrak{M}^\diamond$.

Consider the g.Lie module of generic elements of $\mathcal{A}$, see (2.18); let $\mathcal{U}$ be the vector space of mappings from $q \cdot \mathcal{F}(\mathbf{L})$ to $\mathrm{Pol}(\vec{\mathbf{L}}, V)$, which are linear or constant with respect to every generic element of M. For example, every h_l defines a mapping in $\mathcal{U}$:

$$h_l : (u_1, \dots, u_n) \to f_l(\mu_1(u_1), \dots, \mu_n(u_n)) .$$

Like in case of $\mathcal{A}^\diamond$ we define a g.Lie representation $\nu^\diamond : \mathcal{F}(\mathbf{L}) \to \mathrm{End}(\mathcal{U})$. Next, let $W^\diamond$ be the submodule over $\mathsf{En}(\mathbf{F}(\mathbf{L}), \nu^\diamond)$ generated by $h_1, \dots , h_t$, put

$$\mathcal{F}^\diamond = (\mathbf{F}(\mathbf{L}); W^\diamond) = (\mathbf{F}(\mathbf{L}); W^\diamond; \nu^\diamond; h_1, \dots , h_t) \ ,$$

it is a g.Lie module in $\mathfrak{M}^\diamond$.

We identify elements in $\mathrm{Pol}(\vec{\mathbf{L}}, V)$ with constant functions in $\mathcal{U}$; from proposition 3.6 we have

$$\widetilde{U} \subseteq W^\diamond \ .$$

Observe that all $\rho_i^\diamond(a), \delta_i^\diamond(a)$, where $i \le q$, act trivially on constant functions, hence, every submodule of $\widetilde{U}$ over $\mathsf{En}(\mathbf{F}(\mathbf{L}), \nu)$ (for example, $\Gamma/T(\mathcal{A})$) is that of $W^\diamond$ over $\mathsf{En}(\mathbf{F}(\mathbf{L}), \nu^\diamond)$.

Next, a homomorphism from $\mathcal{F}$ to $\mathcal{A}$ means a substitution some elements from $\mathbf{L}$ in place of generic elements. Hence, it induces a homomorphism from $\mathcal{F}^\diamond$ to $\mathcal{A}^\diamond$ and every homomorphism from $\mathcal{F}^\diamond \to \mathcal{A}^\diamond$ is defined by such a substitution. Moreover, for all nonzero $g \in W^\diamond$ there exists a homomorphism to $\mathcal{A}^\diamond$ taking g to a nonzero function. Actually, since $g = g(y_{i_1}, \dots , y_{i_n}; x_1, \dots , x_n) \in W^\diamond$ is a nonzero function, $g(u_1, \dots , u_n; x_1, \dots , x_n) \ne 0$ for some $u_1, \dots , u_n \in \mathbf{M}$. It means that under some substitution $x \to \bar{x}$ we have

$$g(\bar{u}_1, \dots , \bar{u}_n; \bar{x}_1, \dots , \bar{x}_n) \ne 0 \ .$$

It unduces a homomorphism from $\mathcal{F}^\diamond \to \mathcal{A}^\diamond$ which takes g to $g(y_{i_1}, \dots , y_{i_n}; \bar{x}_1, \dots , \bar{x}_n) \in V^\diamond$ which does not vanish under the substitution of $\bar{u}_1, \dots , \bar{u}_n$; therefore, we can identify $\mathcal{F}^\diamond$ with the g.Lie module of generic elements of $\mathcal{A}^\diamond$.

Now, let

$$\mathcal{M}^\diamond\langle X\rangle = (\mathbf{L}\langle X\rangle; V^\diamond\langle X\rangle; \mathrm{v}_1^\diamond, \dots , \mathrm{v}_t^\diamond)$$

be the free g.Lie module in $\mathfrak{M}^\diamond$ and let (π, χ) be the epimorphism from $\mathcal{M}^\diamond$ onto $\mathcal{F}^\diamond$ induced by the canonical homomorphism π from $\mathbf{L}\langle X\rangle$ onto $\mathcal{F}(\mathbf{L})$; then the kernel $\mathrm{Ker}(\chi)$ is equal to $T(\mathcal{A}^\diamond)$. Put $\Xi = \chi^{-1}(\Gamma/T(\mathcal{A}))$; observe that it is a T-ideal. Indeed, $\Gamma/T(\mathcal{A})$ is a submodule of $W^\diamond$ which is closed under the substitution of elements in $\mathbf{F}(\mathbf{L})$ in place of generic elements. Since every endomorphism of $\mathcal{M}^\diamond\langle X\rangle$ is defined by such substitution, Ξ is a fully characteristic submodule of $V^\diamond\langle X\rangle$.

Next, let f be a multilinear element in $\widetilde{U}$, then there exists a multilinear $f' \in V^\diamond\langle X\rangle$ such that $\chi(f') = f$; clearly, $\deg(f') = \deg(f)$. Now, let f_n, $n \in \mathbb{N}$, be the appropriate sequence for Γ; since $\Gamma^{\#}/T(\mathcal{A}) \subseteq \widetilde{U}$, there is a multilinear $f'_n \in \chi^{-1}(f_n/T(\mathcal{A}))$ with $\deg(f'_n) = \deg(f_n)$. Clearly, $f'_n \notin \Xi$ and $(f'_n)^{\ddagger}_{\dagger} \subseteq \Xi$ for all $n \in \mathbb{N}$; so, Ξ is a critical T-ideal. Besides, $\Xi^{\#} \subseteq \mathcal{Q}(\mathcal{A}^\diamond)$, therefore, $T(\mathcal{A}^\diamond) \subset \Xi$ is a “counter-example”.

PROPOSITION 3.8. *There exists a finite family of finite dimensional g.Lie modules $\mathcal{B}^\xi \in \mathfrak{M}^\diamond$, $\xi \in J$, such that*

(1)

$$\Xi \cap \bigcap_{\xi\in J} T(\mathcal{B}^\xi) \subseteq T(\mathcal{A}^\diamond) \ ; \tag{3.78}$$

(2) $\mathrm{type}(\Xi, \mathcal{B}^\xi) < \mathrm{type}(\Gamma, \mathcal{A})$ *for all* $\xi \in J$.

Proof. For each $\alpha \in \pi_i^\circ$ there exists a polynomial G^α in adjoint operators such that under some substitution of variables $x \to \bar{x}$ it becomes the identity mapping of $\mathrm{End}(\mathsf{D}_{i,\alpha})$. Moreover, for all $\beta \in \omega_{i,\alpha}$ the L-module $\mathsf{W}_{i,\beta}$ is standard, hence, as a module over $\mathsf{D}_{i,\alpha}$, it is isomorphic to the regular $\mathsf{D}_{i,\alpha}$-module, hence $\bar{G}^\alpha$ is the identity mapping of $\mathsf{Gn}(L_i^\alpha)$. Moreover, we may assume that G^α is linear with respect to at least one variable, y_α, and it is the sum of multilinear polynomials. Thus, the set of all homogeneous components of G^α consists of multilinear polynomials being linear with respect to y_α; we denote this set by $\mathcal{H}_\alpha$. Also, put

$$Y = \{y_\alpha | \alpha \in \pi^\circ\} \ , \tag{3.79}$$

where $\pi^\circ = \bigcup_{i=1}^k \pi_i^\circ$. Now for each $\alpha \in \pi^\circ$ take $H_\alpha \in \mathcal{H}_\alpha$ and let η denote finite sequence $(H^\alpha |\ \alpha \in \pi^\circ)$. Take $h_l \in W^\diamond$, see (3.77); since (3.2) holds for all $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$, for every $\alpha \in \pi_i^\circ$ there exists $j \in \{1, \dots , n\}$ such that μ_j is equal to $\mu^\alpha = \mu_i^\alpha$, the homomorphism from L onto

$$L_i^\alpha / \mathsf{J}_i^\alpha \simeq \mathsf{D}_{i,\alpha} \oplus \bigoplus_{\beta \in \omega_{i,\alpha}} \mathsf{W}_{i,\beta} \ .$$

We fix one of these, so

$$h_l = f_l(\dots , \mu_\alpha(x), \dots , \mu_\gamma(z), \dots) \ , \tag{3.80}$$

where $\{\alpha, \dots , \gamma\}$ is π°. Next, put

$$h_l^\eta = f_l(\dots , \mu_\alpha(x \cdot H^\alpha), \dots , \mu_\gamma(z \cdot H^\gamma), \dots) \ ;$$

it can be rewritten in the form

$$h_l^\eta = g_l \cdot H_l^\eta \ ;$$

where H_l^η is a multilinear polynomial in $\mathsf{En}(\mathbf{M}, \nu^\diamond) \subseteq \mathrm{End}(\mathcal{U})$; Observe that $\mathrm{varb}(H_l^\eta)$ is independent from l and equal to

$$Y^\eta = \bigcup_{\alpha \in \pi^\circ} \mathrm{varb}(H^\alpha) \ ,$$

Let $\alpha \in \pi_i^\circ$, fix a basis of the vector space $\mathsf{W}_{i,\beta}$ for all $\beta \in \omega_{i,\alpha}$ and let B_α denote the union of these. Take $x \in Y^\eta$, then $x \in \mathrm{varb}(G^\alpha)$ for some α; let $x \to \bar{x}$ be that substitution under which $\bar{G}^\alpha$ is the identity mapping of $\mathsf{D}_{i,\alpha}$. Besides, $Y \subseteq Y^\eta$; consider a substitution $x \to \tilde{x}$ for elements $x \in Y^\eta$ such that

($\circ$) $\tilde{y}_\alpha \in B_\alpha$ for some variables $y_\alpha \in Y \subseteq Y^\eta$, and $\tilde{x} = \bar{x}$ for the other variables $x \in Y^\eta$.

Now we are prepared to define the set of indices J, it consists of pairs $\xi = (\eta, x \to \tilde{x})$, where η is as above and $x \to \tilde{x}$ means a substitution for variables in Y^η satisfying ($\circ$); obviously, J is finite. Take $\xi = (\eta, x \to \tilde{x}) \in J$ and let us define a g.Lie module $\mathcal{B}^\xi$. First, let $\widetilde{H}_l^\eta$ denote the value of H_l^η under the substition $x \to \tilde{x}$; it is an operator in $\mathsf{En}(\mathbf{L}, \nu^\diamond) \subseteq \mathrm{End}(U)$. Put

$$h_l^\xi = h_l \cdot \widetilde{H}_l^\eta \ ,$$

and let V^ξ be the submodule of U generated by $h_1^\xi, \dots , h_t^\xi$ over $\mathsf{En}(\mathbf{L}, \nu^\diamond)$; then

$$\mathcal{B}^\xi = (\mathbf{L}; V^\xi; \nu^\diamond; h_1^\xi, \dots , h_t^\xi)$$

is a finite dimensional g.Lie module in $\mathfrak{M}^\diamond$.

LEMMA 3.26. *For all* $\xi \in J$

$$\mathrm{type}(\Xi, \mathcal{B}^\xi) < \mathrm{type}(\Gamma, \mathcal{A}) \ .$$

Proof. By irreducibility of $T(\mathcal{A}) \subset \Gamma$, $\|\mathrm{a}(\Xi, \mathcal{B}^\xi)\| \leq \|\mathrm{a}(\Gamma, \mathcal{A})\|$, $\mathrm{a}(\Xi, \mathcal{B}^\xi) \leq \mathrm{a}(\Gamma, \mathcal{A})$, and we may assume that these are equalities (otherwise, the lemma is proven).

Next, let us show that $\mathrm{t}(\Xi, \mathcal{B}^\xi) \leq \mathrm{t}(\Gamma, \mathcal{A})$; let $\mathrm{a} = \mathrm{a}(\Gamma, \mathcal{A}) = \mathrm{a}(\Xi, \mathcal{B}^\xi)$, take some $\mathrm{b} \in (\mathbb{N}_0^\infty)^\infty$, $c_i, d_i \in \mathbb{N}$ and put

$$\mathcal{S} = \mathcal{S}_{\mathrm{a},\mathrm{b},c} \ .$$

Suppose that $\mathcal{S}(\Xi) \not\subseteq T(\mathcal{B}^\xi)$, take a typical generator $h = \mathcal{S}(w) \notin T(\mathcal{B}^\xi)$, then $g = \chi(h) \neq 0$. By defintion, $w = \sum_{l=1}^t \mathrm{v}_l^\diamond \cdot E_l$ for some $E_l \in \mathsf{En}(\mathcal{M}^\diamond\langle X\rangle)$ and $\chi(w) \in \widetilde{\Upsilon}$, hence

$$\chi(w) = \sum_{l=1}^{t} \sum_{u_1,\dots,u_n,E} f_l(\mu_1(u_1), \dots , \mu_n(u_n)) \cdot E \ ,$$

where $u_1, \dots , u_n \in \mathbf{F}(\mathbf{L})$ and $E \in \mathsf{En}(\mathbf{M}, \nu)$.

Next, since χ is induced by the identity mapping, we have $\chi(h) = \chi(\mathcal{S}(w)) = \mathcal{S}(\chi(w))$. Take l, $u_1, \dots , u_n, E$ such that

$$g = \mathcal{S}(f_l(\mu_1(u_1), \dots , \mu_n(u_n)) \cdot E) \tag{3.81}$$

differs from 0; for all j there exist i, γ such that $\mu_j = \mu_i^\gamma$; as μ_j kills all elements from $L_{i'}$, $i' \neq i$, we can assume that $u_j \in \mathcal{F}(\mathbf{L})_i$.

Recall that for all $\alpha \in \pi_i^\diamond \subseteq \pi^\diamond$ we have an operator $G^\alpha = G_i^\alpha$ such that under some substitution of elements in $\mathsf{D}_{i,\alpha}$ the operator $\bar{G}_i^\alpha$ is equal to the identity mapping of $L_i^\alpha/\mathsf{J}_i^\alpha$; also, it acts trivially on the others direct summands of decomposition (3.46), so it is equal to the projector from L_i/J_i onto $L_i^\alpha/\mathsf{J}_i^\alpha$. Also, put $G_i^\bullet = 1 - \sum_{\alpha\in\pi_i^\diamond} G_i^\alpha$, where 1 is the identity mapping of L_i/J_i, then $\bar{G}_i^\bullet$ is the projector onto $L_i^\bullet/\mathsf{J}_i^\bullet$. Hence, the function (3.81) can be written as

$$\begin{aligned} g &= \mathcal{S}(f_l(\mu_1(u_1) \cdot \bar{G}_1, \dots , \mu_n(u_n) \cdot \bar{G}_n) \cdot E) \\ &= \mathcal{S}(f_l(u_1 \cdot \bar{G}_1, \dots , u_n \cdot \bar{G}_n) \cdot E) \ , \end{aligned}$$

where $G_j = G_i^\gamma$ provided $\mu_j = \mu_i^\gamma$. We can choose G_j being independent from variables of g, hence, the element above is obtained from

$$\mathcal{S}(f_l(u_1 \cdot G_1, \dots , u_n \cdot \ G_n) \cdot E) \in \mathcal{S}(\Gamma^\#)/T(\mathcal{A})$$

by substitution $x \to \bar{x}$ for all $x \in \mathrm{varb}(G_j)$; that means that $\mathcal{S}(\Gamma^\#) \not\subseteq T(\mathcal{A})$ as required.

Let us show that $|\mathrm{b}(\Xi, \mathcal{B}^\xi)| < |\mathrm{b}(\Gamma, \mathcal{A})|$; take $\bar{\mathbf{x}} \in \mathsf{V}(f, \mathcal{B}^\xi)$, where $f \in \Xi$. Then $f(\bar{\mathbf{x}})$ is a nonzero function

$$g = \sum_{i_1,\dots,i_m} g_{i_1,\dots,i_m}(y_{i_1}, \dots , y_{i_m}) \ ;$$

there exist $\bar{x}_{i_j} \in \bigcup_{i=1}^k L_i$ such that

$$\bar{g} = \sum_{i_1,\dots,i_m} g_{i_1,\dots,i_m}(\bar{x}_{i_1}, \dots , \bar{x}_{i_m}) \neq 0 \ .$$

We can assume that all $x_{i_j} \in X$ are not in $\mathrm{varb}(f)$, so the substitution $x_{i_j} \to \bar{x}_{i_j}$ extends $\bar{\mathbf{x}}$. Hence, $\bar{g} = \bar{f}'$, where

$$f' = f \cdot \delta_{i_1}(x_{i_1}) \dots \delta_{i_m}(x_{i_m}) \in \Xi \ ,$$

thus, we can replace f with f' and assume that $\bar{f}$ is a constant, $\bar{f} \in V \subseteq V^\diamond$.

On the other hand, $f = \sum_{l=1}^{t} \mathrm{v}_l^{\diamond} \cdot E_l$, hence

$$f(\bar{\mathbf{x}}) = \sum_{l=1}^{t} h_l^{\xi} \cdot \bar{E}_l = \sum_{l=1}^{t} h_l \cdot \widetilde{H}_l^{\eta} \bar{E}_l \ .$$

Take l such that $h_l \cdot \widetilde{H}_l^{\eta} \bar{E}_l \neq 0$; by definition, $g = h_l \cdot H_l^{\eta} E_l$ is equal to

$$\sum_{u_1, \dots, u_n, E} f_l(\mu_1(u_1), \dots, \mu_n(u_n)) \cdot E \ ,$$

where $u_1, \dots, u_n \in M = \mathcal{F}(\mathbf{L})$ and $E \in \mathsf{En}(\mathbf{M}, \nu)$. We may assume that H_l^{η} and E_l depend on disjoint set of variables, hence g is multilinear and

$$\mathrm{varb}(g) = \mathrm{varb}(E) \cup \bigcup_{i=1}^{n} \mathrm{varb}(u_i) = \mathrm{varb}(H_l^{\eta}) \cup \mathrm{varb}(E_l) \ . \tag{3.82}$$

Observe that the joint substitution, $x \to \widehat{x}$, of $x \to \widetilde{x}$ (for variables of H_l^{ξ}) and $x \to \bar{x}$ (for E_l) takes some $f_l(\mu_1(u_1), \dots, \mu_n(u_n))$ to a nonzero element,

$$f_l(\mu_1(\widehat{u}_1), \dots \mu_n(\widehat{u}_n)) \ .$$

Since $f_l \in \Gamma^{\#}$, the total number of variables in (3.82) such that $\widehat{x} \in \mathsf{W}$ is equal to $|\mathrm{b}(\Gamma, \mathcal{A})|$. Also, by definition, there exists $x \in \mathrm{varb}(H_l^{\eta})$ such that $\widetilde{x} \in \mathsf{W}$, hence $|\mathrm{b}(\bar{\mathbf{x}})| < |\mathrm{b}(\Gamma, \mathcal{A})|$. □

So, part (2) of the proposition is proven. Let I denote the left hand side of (3.78) and suppose that it does not hold. Take $\bar{\mathbf{x}} \in \mathsf{V}(w, \mathcal{A}^{\diamond})$ for some $w \in I \subseteq \Xi$. Hence, $0 \neq w(\bar{\mathbf{x}}) = \chi(w)(\bar{\mathbf{x}})$, where $\chi(w) \in \Gamma$, hence $\bar{\mathbf{x}} \in \mathsf{V}(\Gamma, \mathcal{A})$. On the other hand, $w = \sum_{j=1}^{t} \mathrm{v}_l^{\diamond} \cdot E_l$ and

$$\chi(w) = \sum_{l=1}^{t} h_l \cdot E_l \ .$$

Consider the operator polynomials G^{α}, $\alpha \in \pi^{\circ}$, defined above; recall that $\bar{G}_{\alpha}$ is the identity mapping of $L_i^{\alpha}/\mathsf{J}_i^{\alpha}$. Using (3.80), we can define an operator $G_l \in \mathsf{En}(\mathbf{L}\langle X \rangle, \nu^{\diamond})$ so that

$$h_l \cdot G_l = h_l(\dots, \mu_{\alpha}(x \cdot G^{\alpha}), \dots, \mu_{\gamma}(z \cdot G^{\gamma}), \dots) \ .$$

Under the substitution $x \to \bar{x}$ for variables of G_{α} for all α, we get $\bar{G}_l$ which is equal to the identity mapping, hence,

$$\chi(w)(\bar{\mathbf{x}}) = \sum_{l=1}^{t} g_l \cdot \bar{G}_l \bar{E}_l \ .$$

Observe that G_l is the sum of homogeneous components H_l^{η}, hence, there is η such that the same substitution gives

$$\sum_{l=1}^{t} h_l \cdot \bar{H}_l^{\eta} \bar{E}_l \neq 0 \ .$$

Put $h = \sum_{l=1}^{t} h_l \cdot H_l E_l$; obviously, we can assume that H_l and E_l depend on different variables, hence h is multilinear. Moreover, $\xi(w) \in \Gamma / T(\mathcal{A}) \subseteq \widetilde{U}$ is a constant function, hence

$$h = \sum_{l=1}^{t} \sum_{(u_1, \dots, u_n), E} f_l(\mu_1(u_1), \dots, \mu_n(u_n)) \cdot E$$

where $(u_1, \dots, u_n)$ is an n-tuple of multilinear Lie polynomials and $E \in \mathsf{En}(\mathbf{F}(\mathbf{L}), \nu)$. Although the parameters of the second sum, $u_1, \dots, u_n, E$, depends on l as well as the number n and $\mu_1, \dots, \mu_n$, the set of variables $\mathrm{varb}(E) \cup \bigcup_{j=1}^{n} \mathrm{varb}(u_j)$ is independent from l and equal to $\mathrm{varb}(H_l) \cup \mathrm{varb}(E_l) = \mathrm{varb}(h)$ for all l.

Recall that each μ_j is the homomorphism μ_i^γ from L onto $L_i^\gamma / \mathsf{J}_i^\gamma$ for some $\gamma \in \tilde{\pi}_i^\circ$. If there is $x \in \mathrm{varb}(u_j)$ such that $\bar{x} \notin L_i^\gamma$, then $\mu_j(\bar{u}_j) = 0$. Otherwise $\mu_j(\bar{u}_j) = \bar{u}_j$, hence

$$\bar{h} = \sum_{l=1}^{t} \sum_{u_1', \dots, u_n', E} h_l(\bar{u}_1', \dots, \bar{u}_n') \cdot \bar{E} \ ,$$

where $(u_1', \dots, u_n')$ runs other the subset of all $(u_1, \dots, u_n)$ satisfying the following condition for all l :

(*) $\bar{x} \in L_i^\gamma$ for all $x \in \mathrm{varb}(u_j)$ provided $\mu_j = \mu_i^\gamma$.

Now we are prepared to use Main Lemma; put

$$f = \sum_{l=1}^{t} \sum_{(u_1', \dots, u_n', E)} f_l(u_1', \dots, u_n') \cdot E \ ,$$

it is a multilinear polynomial which does not vanish under the substitution $x \to \bar{x}$. Subset of variables (3.79) plays the role of Y from Main Lemma:

$$Y \subseteq \mathrm{varb}(H_l^\eta) \subseteq \bigcup_{j=1}^{n} \mathrm{varb}(u_j')$$

for all l and $(u_1', \dots, u_n')$, hence, $Y \subseteq \mathrm{varb}(f)$. Also, the subset $Z = Z(f, x \to \bar{x})$, see (3.53), is contained in $\mathrm{varb}(E_l)$ for all l, because each $x \in \mathrm{varb}(H_l^\eta)$ belongs to $\mathrm{varb}(G^\alpha)$ for some $\alpha \in \pi_i^\circ$ and, therefore, takes value in $\mathsf{D}_{i,\alpha}$.

By Main lemma there exists a substitution $x \to \tilde{x}$ such that

(a) if $\bar{x} \in L_i^\gamma$, then $\tilde{x} \in L_i^\gamma$ for all $x \in \mathrm{varb}(f)$;

(b) if $x \notin Y \cup Z$, then $\tilde{x} = \bar{x}$;

(c) there exist $y = y_\alpha \in Y$ such that $\tilde{y}$ belongs to $\mathsf{W}_{i,\beta}$ for some $\beta \in \omega_{i,\alpha}$.

From (a) it follows that for all j we have $\mu_j(\tilde{u}_j) = \tilde{u}_j$, if u_j satisfies (*), and $\mu_j(\tilde{u}_j) = 0$, otherwise. Under this substitution

$$\widetilde{h} = \sum_{l=1}^{t} \sum_{u_1, \dots, u_n} f_l(\mu_1(\tilde{u}_1), \dots, \mu_n(\tilde{u}_n)) \cdot \widetilde{E} =$$

$$= \sum_{l=1}^{t} \sum_{u_1', \dots, u_n')} f_l(\tilde{u}_1', \dots, \tilde{u}_n') \cdot \widetilde{E} \neq 0 \ .$$

In other terms, $\widetilde{h} = \sum_{l=1}^{t} h_l \cdot \widetilde{H_l^\eta} \widetilde{E}_l \neq 0$; since h is linear with respect to each $y_\alpha \in Y$, we may assume that $\widetilde{y}_\alpha \in B_\alpha$, see ($\circ$). It means that $w \notin T(\mathcal{B}^\xi)$, where $\xi = (\eta, x \to \tilde{x})$; indeed, its value under the substitution $x \to \tilde{x}$ for all $x \in \mathrm{varb}(w) = \mathrm{varb}(E_l)$ is equal to

$$\sum_{l=1}^{t} h_l^\xi \cdot \tilde{E}_l = \widetilde{h} \neq 0 \ ,$$

which contradicts to the choice of w. Therefore, the proposition is proven. $\square$

Thus,

$$\Xi \cap \bigcap_{\xi \in J} T(\mathcal{B}^{\xi}) = T(\mathcal{A}') \cap \bigcap_{\xi \in J} T(\mathcal{B}^{\xi}) \ ;$$

the right hand side is the ideal of identities of the finite dimensional g.Lie module

$$\mathcal{A}' \oplus \bigoplus_{\xi \in J} \mathcal{B}^{\xi} \ ;$$

besides, $\Xi \subseteq \mathcal{Q}(\mathcal{B}^{\xi})$ for all ξ. Hence, by proposition 2.15 there must exist a "counter-example" of lesser type, which is impossible. Therefore, "counter-examples" do not exist; theorem 2.1 is proven.

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