On generators of the algebra of polynomial invariants of group F_4

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Abstract

We describe the algebra of polynomial invariants of a system of 3 vectors of an irreducible 26-dimensional representation of a simple algebraic group of type F_4 (the ground field is algebraically closed of characteristic zero). Generators are given in the form of trace polynomials of the Albert algebra (an exceptional simple Jordan algebra); a system of homogeneous parameters and a basis over it are also found.

1 Introduction

The problem of finding generators of the algebra of invariants of a system of vectors with respect to a linear algebraic group is classical (see, e.g., [17]); this paper is devoted to the problem of extending the results of [14, 15] on invariants of a simple algebraic group of type G_2 to that of F_4.

In the sequel the ground field F is algebraically closed of characteristic zero.

Notations.

- $\mathrm{End}_F V$ is the algebra of linear transformations of a vector space V;
- $\mathrm{vect}_F X$ is the subspace spanned by a subset X of a vector space over F;
- $\mathrm{alg}_F X$ is the subalgebra generated by a subset X of an algebra over F;
- $F[V]$ (respectively, $F(V)$) is the algebra (field) of polynomial (rational) functions from a vector space V to F;
- $V : G$ is a G-module V, i.e., V is a vector space and G is closed subgroup of the group $GL(V)$ of non-singular linear transformations of V;
- $F[V]^G$ $(F(V)^G)$ is the subalgebra (the subfield) of polynomial (rational) invariants of $V : G$;
- $kV = \underbrace{V \oplus \ldots \oplus V}_{k}$ is the direct sum of k copies of a vector space V;

- $\circ$ $\mathrm{Pol}(V, W) \simeq F[V] \otimes_F W$ is the space of polynomial mappings from a vector space V to a vector space W.

It is well-known that the minimal linear representations of exceptional algebraic groups are connected with some non-associative algebras. Thus, the automorphism group $G = Aut(\mathbf{O})$ of the Cayley-Dickson algebra $\mathbf{O}$ is a simple algebraic group of type G_2 [13]. It is known that every element $x \in \mathbf{O}$ satisfies the quadratic identity

$$x^2 - t(x)x + n(x) = 0 \ ,$$

where $t(x)$ is the trace of x. As a G-module $\mathbf{O}$ is the direct sum of a 1-dimensional trivial submodule and an irreducible 7-dimensional submodule $\mathbf{O}' = \{a \in \mathbf{O} | \ t(a) = 0\}$.

In [14, 15] it was proven that the algebra of invariants $F[k\mathbf{O}]^G$ with respect to the diagonal action (the system of k vectors of $\mathbf{O} : G$) is generated by trace polynomials, which can be defined in the following way.

Let X_i is the projector of $k\mathbf{O}$ onto the i-th summand (a generic element of the algebra $\mathbf{O}$), then $X_1, \ldots, X_k$ lie in the algebra $\mathrm{Pol}(k\mathbf{O}, \mathbf{O})$ with respect to the pointwise operations. The subalgebra $\mathcal{F}_k(\mathbf{O})$ generated by $X_1, \ldots, X_k$ is called *the algebra of generic elements of* $\mathbf{O}$ of rank k; for every mapping $u \in \mathcal{F}_k(A)$ we can consider a polynomial function $t(u) \in F[k\mathbf{O}]$ which is the composition of u with $t(x)$. Moreover, $t(u)$ is a G-invariant, and the subalgebra $t[k\mathbf{O}]$ generated by 1 and all such polynomials is called *the algebra of trace polynomials of* $\mathbf{O}$ and it is equal to $F[k\mathbf{O}]^G$ by the result cited above. Respectively, $F[k\mathbf{O}']^G$ is generated by trace polynomials in generic elements X_i' with $t(X_i') = 0$.

A similar conjecture makes sense in the case of the Albert algebra $\mathbf{A}$ (for definition see, e.g., [8, 16] or section 3); it is known that $G = Aut(\mathbf{A})$ is a simple algebraic group of type F_4 (e.g., [7]); also, every element $x \in \mathbf{A}$ satisfies a cubic equation

$$x^3 - \mathrm{tr}(x)x^2 + S(x)x - N(x) = 0 \ , \tag{1}$$

where $\mathrm{tr}(x)$ is the trace of x. The G-module $\mathbf{A}$ is decomposed into the sum of a 1-dimensional one and 26-dimensional irreducible module $\mathbf{A}' = \{a \in \mathbf{A} | \ \mathrm{tr}(a) = 0\}$.

Denote by $\mathrm{tr}[k\mathbf{A}]$ ($\mathrm{tr}[k\mathbf{A}']$) the subalgebra of $F[k\mathbf{A}]^G$ ($F[k\mathbf{A}']^G$) of trace polynomials, i.e., the subalgebra generated by $\mathrm{tr}(u)$, where u is a monomial in $X_1, \ldots X_k$ ($X_1', \ldots, X_k'$). It is known that that $F[k\mathbf{A}]^G$ ($F[k\mathbf{A}']^G$) is integral over $\mathrm{tr}[k\mathbf{A}]$ ($\mathrm{tr}[k\mathbf{A}']$), their spectra are homeomorphic and their fields of fractions coincide; in particular, for $k = 1, 2$ they are equal [4].

The case $k = 3$ is special: the action of G on $3\mathbf{A}$ is almost free, that is, the stabilizer of a point in general position is trivial. Indeed, the algebra $\mathbf{A}$ can be generated by 3 elements [16, p.58,ex.12], hence, the set of points $(x, y, z) \in 3\mathbf{A}$ such that x, y, z generate $\mathbf{A}$ is non-empty and, obviously, it is open in $3\mathbf{A}$. Besides, if $\phi \in G$ fixes x, y, z, then ϕ is the identity mapping.

The main result of this paper is the following theorem.

Theorem 1 *Let* $\mathbf{A}$ *be the Albert algebra over an algebraically closed field* F *of characteristic* 0, *let* G *be the automorphism group of* $\mathbf{A}$. *Then the algebra of polynomial invariants* $F[3\mathbf{A}]^G$ *coincides with the algebra of trace polynomials* $\mathrm{tr}[3\mathbf{A}]$.

Recall that the subspace $\mathbf{A}'$ of zero trace elements is an irreducible G-module of minimal dimension and $3\mathbf{A}' : G$ is almost free ($\mathbf{A}$ can be generated by 3 elements in $\mathbf{A}'$ [16, p.58, ex.12]).

Corollary 1 *The algebra* $F[3\mathbf{A}']^G$ *is generated by* 1 *and trace polynomials* $\mathrm{tr}(u)$*, where* u *ranges over all monomials in the generic elements with zero trace.*

As a matter of fact, to prove it we need first to study the structure of $\mathrm{tr}[3\mathbf{A}]$ in detail; in particular, we find a system of homogeneous parameters and a basis over it for $\mathrm{tr}[3\mathbf{A}]$ (that is, $\mathrm{tr}[3\mathbf{A}]$ is a Cohen-Macauley algebra) and then we prove theorem 1; thus, they become eventually such a system and a basis for $F[3\mathbf{A}]^G$ (theorem 2); finally, it yields a similar decomposition for $F[3\mathbf{A}']^G$.

2 System of homogeneous parametres

Let x, y, z be the generic elements of $\mathbf{A}$; denote by Q the subalgebra of $\mathrm{tr}[3\mathbf{A}]$ generated by 1 and the following polynomials:

(G1) $\mathrm{tr}(x^i)$, $\mathrm{tr}(y^i)$, $\mathrm{tr}(z^i)$, where $i = 1, 2, 3$;

(G2) $\mathrm{tr}(x^i y^j)$, $\mathrm{tr}(y^i z^j)$, $\mathrm{tr}(z^i x^j)$, where $1 \le i, j \le 2$;

(G3) $\mathrm{tr}(x^i y^j \cdot z^k)$, where $1 \le i, j, k \le 2$

The following statement is due to I.P.Shestakov (an oral communication); also, cf. [9, lemma 2].

Proposition 1 *The algebra of trace polynomials* $\mathrm{tr}[3\mathbf{A}]$ *is generated by* Q *and*

$$\omega = \mathrm{tr}(((zx)y)^2 \cdot (zy)x) \tag{2}$$

We will divide the proof into several steps.

Recall that the Albert algebra $\mathbf{A}$ is a Jordan one, i.e., it is commutative and it satisfies the Jordan identity:

$$(a^2 b)a = a^2(ba) \; ; \tag{3}$$

from the definition it follows that $\mathcal{F}_k(\mathbf{A})$ is also a Jordan algebra for all k.

Now let A be a Jordan algebra, for $a \in A$ denote by R_a the operator of the (right) multiplication : $b \to ba$; therefore, $R_a \in \mathrm{End}_F(A)$. Let B be a subalgebra of A, put $R^A(B) = \mathrm{alg}_F\{R_b |\; b \in B\} \subseteq \mathrm{End}(A)$.

From the commutative law and (3) one can get the following identities [8, p.34]:

$$R_a R_b R_c + R_c R_b R_a + R_{(ac)b} = R_{ab} R_c + R_{bc} R_a + R_{ca} R_b \; ; \tag{4}$$

$$[R_{ab}, R_c] + [R_{bc}, R_a] + [R_{ca}, R_b] = 0 \; , \tag{5}$$

where $[s, t]$ is the commutator $st - ts$.

Lemma 1 *[16, p.67] Let* A *be a Jordan algebra and let* B *be the subalgebra generated by a subset* $X \subseteq A$*. Then* $R^A(B) = \mathrm{alg}_F\{R_a, R_{ab} |\; a, b \in X\}$*.*

Let A be a Jordan algebra satisfying the identity

$$a^3 = 0 \; . \tag{6}$$

Lemma 2 *The following identities hold in A:*

$$R_{a^2} = -2R_a^2 \; ; \; R_{ab} = -R_aR_b - R_bR_a \; ; \tag{7}$$

$$R_{a^2}R_a = 0 \; ; \; R_{ab}R_c + R_{bc}R_a + R_{ca}R_b = 0 \; ; \tag{8}$$

$$R_aR_bR_a = -1/2R_{ba^2} \; ; \tag{9}$$

$$R_{a^2}R_b = -2R_{ab}R_a \; ; \tag{10}$$

$$2(ab)^2 = -a^2b^2 \tag{11}$$

$$2(ab)(ac) = -a^2(bc) \tag{12}$$

Proof. The identity (7) is a consequence of the linearized (6). The first identity in (8) follows from (3) and (7); the second one is its linearization. Identity (9) follows from (4) and (8); next, (10) is a consequence of (8). Finally, (11) follows from the previous one and (12) is a linearization of (11). □

Notation: let $v_1, \ldots, v_m, u_1, \ldots, u_n$ be elements in a vector space, we write

$$v_1, \ldots, v_m \to u_1, \ldots, u_n$$

if $v_1, \ldots, v_m$ belong to the subspace spanned by $u_1, \ldots, u_n$.

Now we suppose that A is a free algebra in the variety of Jordan algebras defined by (6) in the free generators x, y, z [16, chapter 1].

Lemma 3 *Let $u = u(x, y, z) \in A$, then*

(i)

$$u \to xU_1, yU_2, zU_3 \; ,$$

where $U_1, U_2, U_3 \in \mathrm{alg}_F\{R_x, R_y, R_z\}$,

(ii) if $\deg(u) \geq 4$, then

$$u \to x^2U_1, y^2U_2, z^2U_3 \; ,$$

where $U_1, U_2, U_3 \in \mathrm{alg}_F\{R_x, R_y, R_z\}$.

Proof. Part (i) follows from (7). Next, thanks to (7) we may assume that the initial part of u is equal to $xR_yR_zR_x$, hence, by (7), (12)

$$xR_yR_zR_x \to xR_yR_xR_z, (zx)(xy) \to x^2R_yR_z, x^2R_{zy} \to x^2R_yR_z, x^2R_zR_y \; .$$

□

Now we are going to describe homogeneous generators of the vector space A. Let $U(k)$ be the subspace of A spanned by all monomials in x, y, z of the total degree k. For polynomials u, v we write $u \simeq v$, if u is equal to v where some variables are swept. Denote by $V(i, j, k)$ the subspace of homogeneous polynomials f, where $\deg_x(f) = i$, $\deg_y(f) = j$, $\deg_z(f) = k$. Also, let

$$V(r) = \sum_{i+j+k=r, i \geq j \geq k} V(i, j, k) \; .$$

Notice that for every monomial $u \in U(k)$ there is $v \in V(k)$ such that $u \simeq v$.

Lemma 4 *The subspace $V(k)$ is spanned by a set of monomials $M(k)$, where*

(a1) $M(1) = \{x\}$;

(a2) $M(2) = \{x^2,\ xy\}$;

(a3) $M(3) = \{x^2y,\ xR_yR_z,\ xR_zR_y\}$;

(a4) $M(4) = M(2,2,0) \cup M(2,1,1)$, *where*

$$M(2,2,0) = \{x^2R_{y^2}\}\ ,\ \ M(2,1,1) = \{x^2R_yR_z,\ x^2R_zR_y\}\ ;$$

(a5) $M(5) = M(3,1,1) \cup M(2,2,1)$, *where*

$$M(3,1,1) = \{x^2R_yR_zR_x\ \}\ ,\quad M(2,2,1) = \mathrm{vect}\{x^2R_{y^2}R_z\ ,\ x^2R_zR_{y^2}\}\ ;$$

(a6) $M(6) = M(3,2,1) \cup M(2,2,2)$, *where*

$$M(3,2,1) = \{x^2R_zR_{y^2}R_x\},$$

$$M(2,2,2) = \{x^2R_{y^2}R_{z^2},\ x^2R_{z^2}R_{y^2}\}\ ;$$

(a7) $M(7) = M(3,3,1) \cup M(3,3,2)$, *where*

$$M(3,3,1) = \{zR_{x^2y^2}R_xR_y\}\ ,\quad M(3,2,2) = \mathrm{vect}\{z^2R_{x^2y^2}R_x,\ y^2R_{x^2z^2}R_x\}\ ;$$

(a8) $M(8) = \{z^2R_{x^2y^2}R_xR_y,\ zR_{x^2y^2}R_xR_yR_z\}$;

(a9) $M(9) = \{z^2R_{x^2y^2}R_xR_yR_z,\ y^2R_{x^2z^2}R_xR_zR_y\}$.

(a10) $M(k) = 0$ *for all* $k \geq 10$.

Proof. We will check it by the induction on k. Notice that (a1), (a2) are trivial and (a3) follows easily from (7). By lemma 3, the subspace $U(k+1)$ is spanned by

$$v \in \{uR_x,\ uR_y,\ uR_z\},$$

where u runs over monomials in $U(k)$. Also, all such v satisfying

$$\deg_x(v) \geq \deg_y(v) \geq \deg_z(v) \tag{13}$$

generate $V(k+1)$. To get all them it suffices to pick $u \in M(k)$ (by induction, $M(k)$ generates $V(k)$), then to multiply it by $t \in \{x,y,z\}$ and to check that every monomial $v \simeq uR_t$ with property (13) belong to $M(k)$. It is equivalent to the following conditions:

(*) for every $v \in M(k+1)$ and for every permutation σ of the variables if $\sigma(v)$ satisfies (13), then $\sigma(v) \in \mathrm{vect}_F M(k+1)$;

(**) for all $u \in M(k)$ and $t \in \{x,y,z\}$ there is a suitable permutation of the variables x,y,z which takes the monomial uR_t to $\mathrm{vect}_F M(k+1)$.

For the rest of the proof of lemma 4 the relation $\equiv$ means the equality modulo $\mathrm{vect}_F M(k+1)$.

(a4)

We have $k = 3$; the condition (*) for M(4) is evident (it follows from the commutative law).

Next, let us check (**): consider $u = x^2 R_y$; by (8) we have

$$uR_x = x^2 y R_x = 0;\ uR_y = x^2 R_y R_y = -1/2 x^2 y^2 \equiv 0\ ;\ uR_z = x^2 y R_z \equiv 0\ ;$$

if $u = (xy)z$, then by (9), (7)

$$uR_x = (xy)zR_x \to (x^2 z)y \equiv 0;\quad (xy)zR_y \simeq (xy)zR_x \equiv 0\ ;$$

$$(xy)zR_z \to z^2(xy) \simeq x^2(zy) \to x^2 R_y R_z,\ x^2 R_z R_y \equiv 0\ .$$

In a similar way one can check the case $u = (xz)y$.

(a5)

Now $k = 4$; for $M(5)$ condition (*) follows from (7), (8):

$$x^2 R_z R_y R_x = -x^2 R_{yz} R_x - x^2 R_y R_z R_x = -x^2 R_y R_z R_x \equiv 0;$$

$$y^2 R_z R_{x^2} \to x^2 R_z R_{y^2} \equiv 0, x^2 R_{y^2} R_z \equiv 0\ .$$

Next, let $u = x^2 y^2$, then, by (8),

$$x^2 y^2 R_x = 0\ ;\ x^2 y^2 R_y = 0\ ;\ x^2 y^2 R_z \equiv 0\ ;$$

if $u = x^2 R_y R_z$, then by (9),

$$uR_x = x^2 R_y R_z R_x \equiv 0;\ uR_y = x^2 R_y R_z R_y \to x^2 y^2 R_z \equiv 0\ ;$$

$$uR_z = x^2 R_y R_z R_z \to x^2 R_y R_{z^2} \simeq x^2 R_z R_{y^2} \equiv 0\ ;$$

for $u = x^2 R_z R_y$ it can be checked similarly.

(a6)

We have $k = 5$; the property (*) follows from (7). Next, let $u = x^2 R_y R_z R_x$, then

$$uR_x = x^2 R_y R_z R_x R_x \to x^2 R_y R_z R_{x^2} = y R_{x^2} R_z R_{x^2} = 0;$$

$$uR_y = x^2 R_y R_z R_x R_y \to x^2 R_y R_{zx} R_y \to x^2 R_{y^2} R_{zx},\ x^2 R_{zx} R_{y^2} \to$$

$$\to x^2 R_{y^2} R_z R_x,\ x^2 R_z R_x R_{y^2} \to x^2 R_z R_{y^2} R_x \equiv 0\ ;$$

$$uR_z = x^2 R_y R_z R_x R_z \to x^2 R_y R_{z^2} R_x \simeq x^2 R_z R_{y^2} R_x \equiv 0\ .$$

Now let $u = x^2 y^2 R_z$:

$$uR_x = x^2 y^2 R_z R_x \to x^2 R_z R_{y^2} R_x \equiv 0\ ;$$

$$uR_y = x^2 y^2 R_z R_y \simeq x^2 R_{y^2} R_z R_x \to x^2 R_z R_{y^2} R_x \equiv 0\ ;$$

$$uR_z = x^2 R_z R_{y^2} R_z \to x^2 R_{y^2} R_{z^2} \equiv 0,\ x^2 R_{z^2} R_{y^2} \equiv 0\ .$$

(a7)

The case $k=6$; let us check (*): by (7) and (11),

$$zR_{y^2x^2}R_yR_x \to zR_{(xy)^2}R_{xy}\ ,\ zR_{x^2y^2}R_xR_y \equiv 0\ .$$

Next, let $u=x^2R_zR_{y^2}R_x$, then

$$uR_x = x^2R_zR_{y^2}R_xR_x \to zR_{x^2}R_{y^2}R_{x^2} = 0;$$

$$uR_y = x^2R_zR_{y^2}R_xR_y \to zR_{x^2}R_{y^2}R_xR_y \to zR_{x^2y^2}R_xR_y \equiv 0;$$

$$uR_z = x^2R_zR_{y^2}R_xR_z \to x^2R_zR_{y^2}R_zR_x,\ x^2R_zR_{y^2}R_{zx} \to$$

$$\to x^2R_{y^2z^2}R_x = 0, xR_{y^2(zx)^2} \to xR_{y^2(x^2z^2)} = z^2R_{x^2}R_{y^2}R_x \to z^2R_xR_{y^2}R_{x^2} \equiv 0\ ;$$

for $u=x^2R_{y^2}R_{z^2}$:

$$uR_x = x^2R_{y^2}R_{z^2}R_x \to z^2R_{x^2y^2}R_x \equiv 0\ ;$$

$$uR_y \simeq x^2R_{y^2}R_{z^2}R_x \equiv 0\ ;$$

$$uR_z = 0\ ;$$

the case $u=x^2R_{y^2}R_{z^2}$ can be checked in a similar way.

(a8)

Now $k=7$; the condition (*) follows from (12) and (7). Let $u=zR_{x^2y^2}R_xR_y$, then

$$uR_x = zR_{x^2y^2}R_xR_yR_x \to zR_{x^2}R_{y^2}R_{yx^2} \to zR_{x^2}R_{y^2}R_yR_{x^2} = 0, zR_{x^2}R_{y^2}R_{x^2}R_y = 0;$$

$$uR_y = zR_{x^2y^2}R_xR_yR_y \to zR_{x^2}R_{y^2}R_xR_{y^2} = 0;$$

$$uR_z = zR_{x^2y^2}R_xR_yR_z \to zx^2R_{y^2}R_xR_yR_z \to zx^2R_{y^2}R_zR_yR_x\ ,\ zx^2R_{y^2}R_{y(zx)} \to$$

$$\to z^2R_{y^2}R_{x^2}R_yR_x\ ,\ (zx)xR_{y(zx)}R_{y^2} \to z^2R_{y^2}R_{x^2}R_yR_x, (zx)^2(yx)R_{y^2} \to$$

$$\to z^2R_{y^2}R_{x^2}R_yR_x,\ z^2R_{x^2}R_{y^2}R_xR_y \to z^2R_{x^2y^2}R_yR_x \to z^2R_{x^2y^2}R_xR_y \equiv 0\ .$$

Let $u=z^2R_{x^2y^2}R_x$, then

$$uR_x = z^2R_{x^2y^2}R_xR_x = 0;$$

$$uR_y = z^2R_{x^2y^2}R_xR_y \equiv 0;$$

$$uR_z = z^2R_{x^2y^2}R_xR_z \simeq y^2R_{x^2z^2}R_xR_y \equiv 0\ .$$

The case $u=y^2R_{x^2z^2}R_x$ is similar.

(a9)

Now $k=8$; let us check condition (*). It suffices to show that $V'(9)$ is invariant under two transpositions: (yz) and (xy). The first case is trivial; the latter one: by (7)

$$z^2R_{x^2y^2}R_yR_xR_z \equiv 0\ ;$$

next, by (7) and (9),

$$x^2R_{y^2z^2}R_yR_zR_x = z^2R_{y^2}R_{x^2}R_yR_zR_x \to z^2R_{x^2y^2}R_yR_zR_x \to$$

$$\to z^2R_{x^2y^2}R_{z(xy)}, z^2R_{x^2y^2}R_xR_zR_y \equiv 0 \to z^2R_{(xy)^2}R_{z(xy)} = 0 \ .$$

Now, let $u = z^2R_{x^2y^2}R_xR_y$, then

$$uR_x = z^2R_{x^2y^2}R_xR_yR_x \to z^2R_{x^2}R_{y^2}R_xR_yR_x \to$$

$$\to z^2R_{x^2}R_{y^2}R_{yx^2} \to z^2R_{x^2}R_{y^2}R_{x^2}R_y = 0 \ ;$$

$$uR_y = z^2R_{x^2y^2}R_xR_yR_y \to z^2R_{x^2}R_{y^2}R_xR_{y^2} = 0;$$

$$uR_z = z^2R_{x^2y^2}R_xR_yR_z \equiv 0 \ .$$

(a10)

Finally, $k = 9$; since $V'(10) = 0$, condition (*) is trivial.

Next, let $u = z^2R_{x^2y^2}R_xR_yR_z$, then

$$uR_x = z^2R_{x^2y^2}R_xR_yR_zR_x \to (z(xy))^2R_xR_yR_zR_x \to (z(xy))^2R_xR_{z(xy)} = 0 \ ;$$

the case uR_y is similar; further,

$$uR_z = z^2R_{x^2y^2}R_xR_yR_zR_z \to z^2R_{x^2y^2}R_xR_yR_{z^2} \to (z^2x^2)R_{y^2}R_xR_yR_{z^2} \to$$

$$\to (z^2x^2)R_{y^2}R_{z^2}R_yR_x = 0, \ (z^2x^2)R_{y^2}R_{y(z^2x)} \to (zx)^2y^2R_{y((zx)z)} = zR_{x_1}R_yR_{x_1^2y^2}$$

where $x_1 = zx$; by (7), (9) (or by (a7)), we have

$$zR_{x_1}R_yR_{x_1^2y^2} \to zR_{x_1^2}R_{y^2}R_{x_1}R_y = 0 \ ,$$

because $zR_{x_1^2} = z(zx)^2 \to z(z^2x^2) = 0$. So, lemma 4 is proven. □

Proof of proposition 1. Let $B = \mathcal{F}_3(\mathbf{A})$; denote by P the subalgebra of tr[3**A**] generated by Q and ω. We need to show that $\mathrm{tr}(u) \in P$ for every monomial $u \in B$; we will carry out the induction on the degree of u.

If $u \in \{x, y, z\}$, then it is trivial; next, $u = vt$, where v, t are monomials of lesser degree. Also, since

$$\mathrm{tr}((a, b, c)) = 0 \ , \tag{14}$$

where $(a, b, c) = (ab)c - a(bc)$, we may assume that t is a generator in $\{x, y, z\}$.

Now if u, v are homogeneous elements in B of equal degree, then $\mathrm{tr}(u) \equiv \mathrm{tr}(v)$ means that for some non-zero $\alpha \in F$ the element $\mathrm{tr}(u) - \alpha\mathrm{tr}(v)$ belongs to P modulo the subalgebra generated by elements of the form $\mathrm{tr}(w)$, where w runs over all monomials in B of degree being less than $\deg(u) = \deg(v)$.

Let I be the ideal of B generated by a^3, where a ranges over all elements in B; from (1) it follows that for all $c \in I$ and $b \in B$ we have $\mathrm{tr}(cb) \equiv 0$. Hence, by lemma 4, it suffices to show that $\mathrm{tr}(vt) \equiv 0$, where v runs over the generators in (a1)-(a9).

The cases (a1), (a2) are trivial; consider (a3): for example, if $v = xR_yR_z$ and $t = x$, then, by (13) and (a2),

$$\mathrm{tr}(vt) = \mathrm{tr}((xy)zR_x) = \mathrm{tr}(((xy)x)z) \equiv \mathrm{tr}((x^2y)z) \equiv 0 \ .$$

Let us check some more cases (the others are left as an exercise for the reader).

(a5)

Let $v = x^2 R_y R_z R_x$; if $t = x$, then

$$\mathrm{tr}(vt) = \mathrm{tr}(x^2 R_y R_z R_{x^2}) = \mathrm{tr}(x^2 R_z R_{x^2} R_z) \equiv 0 \; ;$$

if $t = y$, then

$$\mathrm{tr}(vt) = \mathrm{tr}(x^2 R_y R_z R_x R_y) \equiv \mathrm{tr}(x^2 R_z R_y R_x R_y) = \mathrm{tr}(x^2 R_z R_y R_y R_x) \equiv$$

$$\equiv \mathrm{tr}(x^2 R_z R_{y^2} R_x) = \mathrm{tr}(z R_{x^2} R_x R_{y^2}) \equiv 0 \; .$$

(a7)

Let $v = z R_{x^2 y^2} R_x R_y$ and $t = z$, then

$$\mathrm{tr}(vt) = \mathrm{tr}((x^2 y^2) R_z R_x R_y R_z) = \mathrm{tr}((x^2 y^2) R_z R_x R_z R_y) \equiv$$

$$\equiv \mathrm{tr}((x^2 y^2) R_{xz^2} R_y) = \mathrm{tr}((x^2 y^2) R_y R_{xz^2}) \equiv 0 \; .$$

(a8)

Let $v = z^2 R_{x^2 y^2} R_x R_y$ and $t = z$, then

$$\mathrm{tr}(vt) = \mathrm{tr}(z^2 R_{x^2 y^2} R_x R_y R_z) \equiv \mathrm{tr}(z^2 R_{x^2} R_{y^2} R_x R_y R_z) \equiv$$

$$\equiv \mathrm{tr}((((zx)y)^2 R_x R_y R_z) = \mathrm{tr}(((zx)y)^2 ((zy)x)) \equiv 0 \; .$$

Let $v = z R_{x^2 y^2} R_x R_y R_z$; if $t = x$, then

$$\mathrm{tr}(vt) \equiv \mathrm{tr}(z R_{x^2 y^2} R_y R_x R_z R_x) = \mathrm{tr}(z R_{x^2 y^2} R_y R_x R_x R_z) \equiv$$

$$\equiv \mathrm{tr}(z y^2 R_{x^2} R_{y^2} R_{x^2} R_z) \equiv 0 \; ;$$

if $t = z$, then

$$\mathrm{tr}(vt) = \mathrm{tr}(z R_{x^2 y^2} R_x R_y R_z R_z) = \mathrm{tr}(z^2 R_y R_x R_{x^2 y^2} R_z) \equiv \mathrm{tr}(z^2 R_{x^2 y^2} R_x R_y R_z) \; ,$$

by the previous case, the latter element belongs to P.

Finally, let $v = z^2 R_{x^2 y^2} R_x R_y R_z$ and let $t = x$, then

$$\mathrm{tr}(vt) = \mathrm{tr}(z^2 R_{x^2} R_{y^2} R_x R_y R_x R_z) \equiv \mathrm{tr}(z^2 R_{x^2} R_{y^2} R_{x^2 y} R_z) \equiv$$

$$\equiv \mathrm{tr}(z^2 R_{x^2} R_{y^2} R_{x^2} R_y R_z) \equiv 0 \; .$$

□

Proposition 2 *The element (2) is integral over Q of degree 2.*

Proof. Put $a = (zx)y$ and $b = (zy)x$, then $\omega = \mathrm{tr}(a^2b)$. We will check first that $\mathrm{tr}(a^2b) + \mathrm{tr}(b^2a) \in Q$. Notice that $\mathrm{tr}(u) \in Q$ for every monomial $u \in B$ of degree less then 9; also, $\mathrm{tr}(a^3), \mathrm{tr}(b^3) \in Q$. Denote by $\equiv$ the equality modulo Q; from (10) and (7) we have

$$\mathrm{tr}(a^2b) = \mathrm{tr}(((zx)y)^2 \cdot (zy)x) \equiv 1/4\mathrm{tr}((z^2x^2)y^2 \cdot (zy)x) \equiv$$

$$\equiv -1/4\mathrm{tr}(z^2(x^2y^2) \cdot (zy)x) \equiv 1/4\mathrm{tr}(z^2(x^2y^2) \cdot (zx)y) \equiv -1/4\mathrm{tr}((z^2y^2)x^2 \cdot (zx)y) \equiv$$

$$\equiv -\mathrm{tr}(((zy)x)^2 \cdot (zx)y) = -\mathrm{tr}(b^2a) \; .$$

Now consider $f = \mathrm{tr}(a^2b \cdot b^2a)$; by (8) and (1)

$$f = \mathrm{tr}(a^2 \cdot (b \cdot b^2a)) \equiv \mathrm{tr}(a^2b)\mathrm{tr}(b^2a) \; .$$

Furthermore,

$$f = \mathrm{tr}(a^2 \cdot b^2(ba)) = \mathrm{tr}(a^2b^2(ab)) \; ;$$

by lemma 4, (a6)

$$(ab) \to (x^2y^2)z^2, (x^2z^2)y^2$$

modulo identity (6), hence, modulo $Q+Q\omega$ the polynomial f is equal to $\mathrm{tr}(w(x^2,y^2,z^2))$, where $w = w(x,y,z)$ is a polynomial in x, y, z. We claim that

$$\mathrm{tr}(w(x^2,y^2,z^2)) = \sum_j \mathrm{tr}(u_j)h_j \; , \tag{15}$$

where $\deg(u_j) \leq 8$ and $h_j \in \mathrm{tr}[3\mathbf{A}]$ for all j.

Indeed, by lemma 1 we may assume that w is equal to w_0U, where w_0 is a monomial in x, y, z of degree 4 or 5 and U is a product of operators of right multiplication. If $\deg(w_0) = 4$, then we make use of lemma 3; othewise, $w_0 = (pq)rR_{st}$, where $p, q.r, s, t \in \{x, y, z\}$. If the initial segment of w_0 is of the form x^2 or xR_{yx} (up to the order of the variables), then the claimed statement (15) follows from (7). Hence, we may assume that $w_0 = (xy)zR_{yz}$ or $w_0 = (xy)zR_{xy}$; the first case is reduced to the previous ones by (3); next, thanks to (7) we may change the order of the first three multiples and get also in the previous situation.

So, (15) is proven; since $\deg(f) = 18$ we get $f \in Q + \omega Q$; on the other hand, $\omega^2 = \omega(\mathrm{tr}(a^2b) + \mathrm{tr}(b^2a)) - \mathrm{tr}(a^2b)\mathrm{tr}(b^2a)$, hence,

$$\omega^2 = p + q\omega \tag{16}$$

for suitable $p, q \in Q$; so, the proposition is proven. $\square$.

From this proposition it follows that the transcendence degree of Q over F is equal to that of K; also, since $F(3\mathbf{A})^G$ is the field of fractions of K, we have $\mathrm{tr.deg}_F\, Q = 29$. Comparing it with the number of elements in (G1), (G2) and (G3), we get the following statement.

Corollary 2 *The elements in (G1),(G2),(G3) are algebraically independent generators of the algebra Q.*

3 Proof of the main statement

It is known (e.g., [11, sect.3.1]) that the field of fractions of $F[3\mathbf{A}]^G$ is equal to the field of rational invariants $F(3\mathbf{A})^G$. By [5, 4], it is generated by trace polynomials, therefore, for every $f \in F[3\mathbf{A}]^G$ there are $g, h \in \mathrm{tr}[3\mathbf{A}]$ such that $h \neq 0$ and $f = g/h$. Now we will show that h of a special form can be found.

Proposition 3 *For every $f \in F[3\mathbf{A}]^G$ there is a nonzero trace polynomial $h \in Q$ such that $f \cdot h \in \mathrm{tr}[3\mathbf{A}]$ and h does not depend on z.*

The underlying idea of the proof comes from the slice method (e.g., [2]).

Let V be a G-module and let E be a subspace of V; denote by $N_G(E)$ the normalizer of E in G:

$$N_G(E) = \{g \in G |\ g(E) \subseteq E\} \ .$$

Denote by $f|_E$ the restriction to E of a function f defined on V, then the mapping

$$\phi(f) \to f|_E \tag{17}$$

is a homomorphism from $F[V]^G$ to $F[E]^{N(E)}$. It is common that ϕ is not an isomorphism; however, if the orbit $G(E) = \{g(v) |\ g \in G\ ,\ \ v \in E\}$ is a dense subset of the affine space $V = F^{\dim V}$, then ϕ is obviously a monomorphism, hence, it can be extended to a homomorphism from the field $F(V)^G$ to $F(E)^{N(E)}$. Moreover, this mapping is an isomorphism, if E is a *slice*, that is, the following conditions are satisfied [2]:

(b1) the orbit $G(E)$ is a dense subset of the affine space $V = F^{\dim V}$;

(b2) there is an open subset $E' \subseteq E$ such that for every $g \in G$ and every $v \in E'$ if $g(v) \in E$, then $g \in N_G(E)$.

Now, let E be a slice of $3\mathbf{A} : G$ and let $f \in F[3\mathbf{A}]^G$; since the field $F(3\mathbf{A})^G$ is generated by trace polynomials, $\phi(f) \in F[E]^{N(E)}$ is equal to $\phi(g/h) = \phi(g)/\phi(h)$, where g, h are trace polynomials. Our task is to find E where $\phi(h)$ does not depend z.

Since we make use of some arguments from [6], we will keep notations of that paper.

The elements of a Cayley-Dickson algebra $\mathbf{O}$ have the form

$$\begin{pmatrix} \alpha & v \\ v^* & \beta \end{pmatrix} \ ,$$

where $\alpha, \beta \in F$ and $v, v^* \in F^3 = F \times F \times F$. The multiplication is defined as follows:

$$\begin{pmatrix} \alpha_1 & v_1 \\ v_1^* & \beta_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 & v_2 \\ v_2^* & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + (v_1, v_2^*) & \alpha_1 v_2 + \beta_2 v_1 - v_1^* \times v_2^* \\ \alpha_2 v_1^* + \beta_1 v_2^* + v_1 \times v_2 & (v_1^*, v_2) + \beta_1\beta_2 \end{pmatrix} \ ,$$

where (v,w), $v\times w$ are the scalar and vector product in F^3 respectively. The algebra $\mathbf{O}$ is equipped with the involution

$$\overline{\left(\begin{array}{cc}\alpha & v\\ v^* & \beta\end{array}\right)} = \left(\begin{array}{cc}\beta & -v\\ -v^* & \alpha\end{array}\right) ,$$

where the trace $t(x)$ of a matrix $x\in\mathbf{O}$ equals to $x+\bar{x}\in 1\cdot F\simeq F$.

Let $\mathbf{O}_3=\mathbf{O}\otimes_F F_3$ be the algebra of 3×3-matrices over $\mathbf{O}$; a linear mapping $a\otimes b\to\bar{a}\otimes b^\top$, where $b\to b^\top$ is the matrix transposition, is an involution of $\mathbf{O}_3$; the subspace $H(\mathbf{O}_3)$ of symmetric elements with the multiplication $a\circ b=1/2(ab+ba)$ is a central simple Jordan algebra $\mathbf{A}$ of dimension 27 [8, 16], which is called an Albert algebra.

Let $e_i=e_{ii},e_{ij}$ $(i,j\in\{1,2,3\},\ i\neq j)$ be the elementary matrices in $\mathbf{O}_3$ (we identify $1\otimes F_3$ with F_3), then an arbitrary element in $\mathbf{A}$ can be written in the form

$$\sum_{i=1}^{3}\alpha_i e_i+\sum_{(ijk)}[c_i]_{jk}, \tag{18}$$

where $\alpha_i\in F$, $c_i\in\mathbf{O}$ and $[c]_{jk}$ means $c\cdot e_{jk}+\bar{c}\cdot e_{kj}$, and (ijk) runs over the subgroup A_3 of even permutations.

The subspace of elements (18), where

$$c_1,c_2,c_3\in\left(\begin{array}{cc}F & 0\\ 0 & F\end{array}\right)\subseteq\mathbf{O} ,$$

is a subalgebra of $\mathbf{A}$ isomorphic to $F_3^{(+)}$ (the matrix algebra F_3 with the symmetrized multiplication $\circ b$). Also, denote by C the subalgebra of diagonal matrices $Fe_1+Fe_2+Fe_3$.

The automorphism group $G=Aut(\mathbf{A})$ contains the following subgroups:

(c1) the symmetric subgroup S_3 whose action is induced by permutation of the idempotents e_1,e_2,e_3 [8, p.389,ex.3];

(c2) the subgroup $\mathbf{Z}_2$ generated by an automorphism of the second order [6, sect.2]:

$$\sigma: e_i\to e_i\ \forall i\in\{1,2,3\} ;$$

$$\sigma:\left[\begin{array}{cc}\alpha & v\\ v^* & \beta\end{array}\right]_{ij}\to\left[\begin{array}{cc}\beta & -v^*\\ -v & \alpha\end{array}\right]_{ij}$$

for all $(ijk)\in A_3$;

(c3) the subgroup $T_2=\{\xi=(\xi_1,\xi_2,\xi_3)\in F^*\times F^*\times F^*|\ \xi_1\xi_2\xi_3=1\}$ acting in the following way:

$$\xi: e_i\to e_i\ \forall i\in\{1,2,3\} ;$$

$$\xi:\left[\begin{array}{cc}\alpha & v\\ v^* & \beta\end{array}\right]_{ij}\to\left[\begin{array}{cc}\xi_i\xi_j^{-1}\alpha & \xi_k^{-1}v\\ \xi_k v^* & \xi_i^{-1}\xi_j\beta\end{array}\right]_{ij}$$

for all $(ijk)\in A_3$;

(c4) the subgroup M of automorphisms of $\mathbf{A}$ which keep the elements in $F_3^{(+)}$ being fixed; by [8, p.386, th.8], it is isomorphic to $SL_3(F)$. More precisely, if $\psi \in SL_3(F)$, then we can define an automorphism $\hat{\psi} \in M$ in following way (it suffices to consider only elements in the complement of $F_3^{(+)}$ to $\mathbf{A}$):

$$\hat{\psi} : \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}_{ij} \to \begin{bmatrix} 0 & \psi(v) \\ (\psi^\top)^{-1}(v^*) & 0 \end{bmatrix}_{ij} ,$$

where $(ijk) \in A_3$. Then $\psi \to \hat{\psi}$ is an isomorphism of $SL_3(F)$ onto M.

Note that the product H of the subgroups S_3 and $\mathbf{Z}_2$ is direct (cf. [6]); also, it is easy to check that T_2 normalizes M and $M \cap T_2 = \{1\}$; finally, H normalizes the (internal) semidirect product $H_0 = M \rtimes T_2$ and $H_0 \cap H = \{1\}$.

Put $E = C \oplus F_3^{(+)} \oplus \mathbf{A} \subseteq 3\mathbf{A}$.

Lemma 5 (cf. [1, lemma 4.5]) *The subspace E is a slice of $3\mathbf{A} : G$, where $N_G(E) = H_0 \rtimes H$.*

Proof. Let U be the subset of points (x, y, z) in the affine space $3\mathbf{A}$ such that

1) the characteristic polynomial of x has 3 distinct eigen-values,

2) x, y generate a subalgebra isomorphic to $F_3^{(+)}$.

From [9, lemma 2] it follows that U is open. In addition, by [8, p.370, th.3] for every $(x, y, z) \in M$ there is $g_1 \in G$ such that $g_1(x), g_1(y) \in F_3^{(+)}$. The subalgebra generated by $g_1(x)$ is spanned by three orthogonal idempotents, hence, thanks to the Coordinatization Theorem [8, p.137] there is an automorphism g_2 of $F_3^{(+)}$ such that $g_2(g_1(x)) \in C$. By [8, p.370, th.3] we may assume that $g_2 \in G$, so, $g = g_2 \circ g_1$ sends (x, y, z) into E, therefore, the orbit of E is dense.

Next, let $E' = U \cap E$, then for every $g \in G$ and $(x, y, z) \in E'$ if $(g(x), g(y), g(z)) \in E$, then the element $g(x)$ generates C and $g(x), g(y)$ generate $F_3^{(+)}$, hence, since g is an automorphism, we have $g(C) = C$ and $g(F_3^{(+)}) = F_3^{(+)}$, therefore, $g(E) = E$. $\square$

Also, from the proof it follows that the subspace $E_0 = C \oplus F_3^{(+)} \subset E$ is a slice of $2\mathbf{A} : G$, where the normalizer $N_G(E_0)$ is equal to $N_G(E)$. Hence, the restriction of ϕ to $F(2\mathbf{A})^G$ is an isomorphism onto $K = F(E_0)^{N_G(E)} \subseteq F(E)^{N_G(E)}$. On the other hand, by [6, lemma 9], $F(2\mathbf{A})^G$ is generated by trace polynomials in x, y. Put $S = \phi^*(\mathrm{tr}[3\mathbf{A}]) \subseteq F[E]^{N_G(E)}$ and let $\tilde{S}$ be the subalgebra over K generated by S. Obviously, $\tilde{S} = \mathrm{vect}_K S$ and its pre-image in $F[3\mathbf{A}]^G$ lies in the subspace over $F(2\mathbf{A})^G$ spanned by $\mathrm{tr}[3\mathbf{A}]$; hence, to prove proposition 3 it suffices to show that

$$F[E]^{N_G(E)} \subseteq \tilde{S} . \tag{19}$$

We wiil do it in the following way: we will extend $\tilde{S}$ by some elements in $F(E_0)$ such that the extension will contain $F[E]^{N_G(E)}$ and, on the other hand, we will show that the subalgebra of all elements in the extension which are fixed under the action of a subgroup of $N_G(E)$ coincides with $\tilde{S}$.

The restriction $f \to f|_E$ defines a homomorphism from $\mathcal{F}_3(\mathbf{A})$ to the algebra $\mathrm{Pol}(E, \mathbf{A})$ of polynomial mappings from E to $\mathbf{A}$. We also denote it by ϕ; by [6, lemma 1]), ϕ is a monomorphism; denote by B_3 the image $\mathrm{alg}_F\{\phi(x), \phi(y), \phi(z)\}$.

The algebra $\mathrm{vect}_{F(E)}\mathbf{A} \simeq F(E) \otimes_F \mathbf{A}$ is a scalar extension of $\mathbf{A}$, where the trace function is extended by linearity. Hence, the algebra S is generated by 1 and $\mathrm{tr}(u)$, where u ranges over elements in B_3.

In the standard coordinates (18) of $\mathbf{A}$ the images of the generic elements have the following form:

$$\phi(x) = \sum_{i=1}^{3} x_i e_i \ ,$$

$$\phi(y) = \sum_{i=1}^{3} y_i e_i + \sum_{(ijk)} [\mathbf{y}_i]_{jk},$$

$$\phi(z) = \sum_{i=1}^{3} z_i e_i + \sum_{(ijk)} [\mathbf{z}_i]_{jk},$$

where

$$\mathbf{y}_i = \begin{pmatrix} y_{jk} & 0 \\ 0 & y_{kj} \end{pmatrix}, \ \mathbf{z}_i = \begin{pmatrix} z_{jk} & v_i \\ v_i^* & z_{kj} \end{pmatrix}$$

and $v_i = (v_{i1}, v_{i2}, v_{i3})$, $v_i^* = (v_{i1}^*, v_{i2}^*, v_{i3}^*)$.

Let us extend the field K: consider $\hat{K} = K(x_1, x_2, x_3, \alpha)$ and let $\hat{S} = \mathrm{vect}_{\hat{K}}\tilde{S} = \mathrm{vect}_{\hat{K}}S$. Respectively, let $\hat{B}_3 = \hat{S} \cdot B_3 \subseteq \mathrm{vect}_{F(E)}\mathbf{A}$, Notice that $\mathrm{tr}(u) \in \hat{S}$ for all $u \in \hat{B}_3$ and, therefore, $\hat{S} = \mathrm{alg}_F\{\mathrm{tr}(u) | \ u \in \hat{B}_3\}$. Now we will show that $F[E]^{N_G(E)} \subseteq \hat{S}$; in fact, it will be more convinient to prove the same relation for some extension of $F[E]^{N_G(E)}$.

The next statement follows from a classical result on invariants of $SL_n(F)$ (e.g., [17, 11]).

Lemma 6 *The subalgebra $D_1 = F[E]^M$ is generated by 1 and the following elements:*

(d1) $x_i, \ y_i, \ z_i$ *(*$i \in \{1, 2, 3\}$*);*

(d2) $y_{ij}, \ z_{ij}$ *(*$i, j \in \{1, 2, 3\}, \ i \neq j$*);*

(d3) (v_i, v_j^*) *(*$i, j \in \{1, 2, 3\}$*);*

(d4) $\Delta := \det(v_1, v_2, v_3), \ \bar{\Delta} := -\det(v_1^*, v_2^*, v_3^*)$.

Obviously, $D_2 = F[E]^{M \leftthreetimes T_2} = (F[E]^M)^{T_2} = D_1^{T_2}$. Now, let L be the subalgebra of $F(E)$ generated over $\hat{K}$ by 1 and the following elements :

(e1) z_i $(i \in \{1, 2, 3\})$;

(e2) $z_{ij} y_{ji}$ $(i, j \in \{1, 2, 3\}, i \neq j)$;

(e3) (v_i, v_i^*) $(i \in \{1, 2, 3\})$;

(e4) $(v_i, v_j^*) y_{ij}$, $(i, j \in \{1, 2, 3\}, i \neq j)$.

(e5) $\Delta, \bar{\Delta}$;

Lemma 7 $L \subseteq \hat{S}$.

Proof. Arguing like in [6, lemma 3 and 4], we will show simultaneously the following:

(i) the field $\hat{K}$ includes y_k, $t_k = y_{ij}y_{ji}$ ($(ijk) \in A_3$) and $\bar{\alpha}$;

(ii) elements (e1)-(e5) belong to $\hat{S}$;

(iii) the algebra $\hat{B}_3$ contains the following elements:

(f1) e_1, e_2, e_3;

(f2)
$$\begin{bmatrix} y_{ij} & 0 \\ 0 & 0 \end{bmatrix}_{ij}, \quad \begin{bmatrix} 0 & 0 \\ 0 & y_{ji} \end{bmatrix}_{ji},$$
where $(ijk) \in A_3$;

(f3)
$$\begin{bmatrix} 0 & v_i \\ 0 & 0 \end{bmatrix}_{jk}, \quad \begin{bmatrix} 0 & 0 \\ v_i^* & 0 \end{bmatrix}_{jk},$$
where $(ijk) \in A_3$;

(f4)
$$\begin{bmatrix} 0 & v_j y_{ji} \\ 0 & 0 \end{bmatrix}_{jk}, \quad \begin{bmatrix} 0 & 0 \\ v_j^* y_{ij} & 0 \end{bmatrix}_{jk}$$
where $(ijk) \in S_3$.

First, since $\phi(x) \in B_3$ and $x_1, x_2, x_3 \in \hat{K}$ are the roots of its characteristic polynomial, we have $e_1, e_2, e_3 \in \hat{B}_3$. Applying the Peirce decomposition [8, 16], we get $y_i \in \hat{K}$, then $[\mathbf{y}_i]_{jk} \in \hat{B}_3$ and, therefore, $y_{jk}y_{kj} = tr([\mathbf{y}_i]_{jk}^2) \in \hat{K}$ for all $(ijk) \in A_3$. Also, $z_i \in \hat{S}$ and $[\mathbf{z}_i]_{jk} \in \hat{B}_3$.

Next, $\bar{\alpha} = y_{13}y_{32}y_{21} = y_{12}y_{21} \cdot y_{23}y_{32} \cdot y_{31}y_{13}/\alpha \in \hat{K}$. Hence,

$$\begin{bmatrix} y_{ij} & 0 \\ 0 & 0 \end{bmatrix}_{ij} = \frac{1}{\alpha - \bar{\alpha}}(\alpha[\mathbf{y}_k]_{ij} - 2(y_{ij}y_{ji})([\mathbf{y}_i]_{jk} \circ [\mathbf{y}_j]_{ki}))$$

belongs to $\hat{B}_3$. Also, $\begin{bmatrix} y_{ji} & 0 \\ 0 & 0 \end{bmatrix}_{ji} = [\mathbf{y}_k]_{ij} - \begin{bmatrix} y_{ij} & 0 \\ 0 & 0 \end{bmatrix}_{ij} \in \hat{B}_3$. Further, $z_{ij}y_{ji} = tr([\mathbf{z}_k]_{ij} \circ \begin{bmatrix} y_{ji} & 0 \\ 0 & 0 \end{bmatrix}_{ji}) \in \hat{S}$; therefore, $z_{ij}/y_{ij} = (z_{ij}y_{ji})/(y_{ij}y_{ji}) \in \hat{S}$ and

$$\begin{bmatrix} 0 & v_i \\ v_i^* & 0 \end{bmatrix}_{jk} = [\mathbf{z}_i]_{jk} - \frac{z_{jk}}{y_{jk}} \begin{bmatrix} y_{jk} & 0 \\ 0 & 0 \end{bmatrix}_{jk} - \frac{z_{kj}}{y_{kj}} \begin{bmatrix} 0 & 0 \\ 0 & y_{kj} \end{bmatrix}_{jk}$$

lies in $\hat{B}_3$. Hence, $(v_i, v_i^*) = -\frac{1}{2}tr(\left[\begin{array}{cc} 0 & v_i \\ v_i^* & 0 \end{array}\right]_{jk}^2) \in \hat{S}$. Next,

$$\left[\begin{array}{cc} 0 & 0 \\ -v_j^* y_{ij} & 0 \end{array}\right]_{jk} = 2\left[\begin{array}{cc} 0 & v_j \\ v_j^* & 0 \end{array}\right]_{ki} \circ \left[\begin{array}{cc} y_{ij} & 0 \\ 0 & 0 \end{array}\right]_{ij} \in \hat{S} ,$$

and all elements (f4) can be obtained in this way. Furthermore,

$$(v_j, v_k^*) y_{ji} y_{ik} = -tr(\left[\begin{array}{cc} 0 & v_j y_{ji} \\ 0 & 0 \end{array}\right]_{jk} \circ \left[\begin{array}{cc} 0 & 0 \\ v_k^* y_{ik} & 0 \end{array}\right]_{jk}) \in \hat{S} ,$$

and since $y_{ji} y_{ik} / y_{jk} = \bar{\alpha}/t_i \in \hat{K}$, we have also $(v_j, v_k^*) y_{jk} \in \hat{S}$. Using an appropriate pair of elements (f4), we prove all elements (e4) to lie in $\hat{S}$.

Finally,

$$\left[\begin{array}{cc} 0 & v_i y_{ik} \\ 0 & 0 \end{array}\right]_{ij} \circ \left[\begin{array}{cc} y_{ki} & 0 \\ 0 & 0 \end{array}\right]_{ki} = \frac{-y_{ik} y_{ki}}{2} \left[\begin{array}{cc} 0 & v_i \\ 0 & 0 \end{array}\right]_{jk} ,$$

So, $\left[\begin{array}{cc} 0 & v_i \\ 0 & 0 \end{array}\right]_{jk} \in \tilde{B}_3$. Moreover, $\left[\begin{array}{cc} 0 & 0 \\ v_j \times v_k & 0 \end{array}\right]_{jk} = -2\left[\begin{array}{cc} 0 & v_j \\ 0 & 0 \end{array}\right]_{ki} \circ \left[\begin{array}{cc} 0 & v_k \\ 0 & 0 \end{array}\right]_{ij}$ is an element in $\hat{B}_3$, and

$$\Delta = (v_j \times v_k, v_i) = -tr(\left[\begin{array}{cc} 0 & 0 \\ v_j \times v_k & 0 \end{array}\right]_{jk} \circ \left[\begin{array}{cc} 0 & v_i \\ 0 & 0 \end{array}\right]_{jk}) \in \hat{S}.$$

The lemma is proven. □

From the next lemma it follows that $F[E]^{N_G(E)}$ is contained in $\hat{S}$.

Lemma 8 *$D_2 \subseteq L$.*

Proof. Indeed, let $\tilde{L} = \mathrm{alg}_F L \cup \{y_{12}, y_{23}, y_{12}^{-1}, y_{23}^{-1}\}$, then $y_{ij}, y_{ij}^{-1} \in \tilde{L}$ for all $i, j \in \{1,2,3\}$, $i \neq j$; hence, $D_1 \subseteq \tilde{L}$. On the other hand, $\tilde{L}^{T_2} = L$, hence, $D_2 \subseteq L$. □

Recall that $\hat{S} = \mathrm{vect}_{\hat{K}} S$; also, using the argument of [6, lemma 6], one can show that $\hat{K}^H = K$, hence, since S consists of invariants of $H \subset N_G(E)$, we have

$$\hat{S}^H = \mathrm{vect}_{\hat{K}^H} S = \tilde{S} .$$

Therefore, by lemma 8, $F[E]^{N_G(E)} = D_2^H \subseteq L^H \subseteq \hat{S}^H = \tilde{S}$ and proposition 3 is proven. □

Lemma 9 *The element (2) does not belong to the field of fractions of Q.*

Proof. Let T be the subfield of $F(E)$ generated by $\phi(Q)$ and x_1, x_2, x_3, α, then, arguing like in the proof of lemma 8 (or [6, lemma 3 and 4]), one can show that K_0 contains the elements

(g1) x_i, y_i, z_i, $i \in \{1,2,3\}$;

(g2) t_i, $i \in \{1,2,3\}$;

(g3) $y_{ij} z_{ji}$, $i, j \in \{1,2,3\}$, $i \neq j$;

(g4) (v_i, v_i^*), $i \in \{1,2,3\}$;

(g5) $(v_i, v_j^*) y_{ij}$, $i, j \in \{1,2,3\}$, $i \neq j$;

(g6) $\Delta + \bar{\Delta}$.

(g7) α .

Moreover, one can easily check that every generator of $\phi(Q)$ is a polynomial in (g1)-(g7), hence, these elements generate T over F. The total number of them is 29; it is precisely the transcendence degree of $F(3\mathbf{A})^G$ over F [6]. Recall that $F[3\mathbf{A}]^G$ is integral over Q and T is a finite field extension of $\phi(Q)$, hence, the trascendence degree of T over F is also equal to 29, therefore, (g1)-(g7) are algebraically independent.

Suppose that $\omega \in Q$, then Q must be equal to P, and, by lemma 8 (or [6, lemma 3]), $\Delta \in T$.

Let B be the subalgebra of T generated over F by (g1)-(g7). Notice that

$$\bar{\Delta}\Delta t_1 t_2 t_3 \alpha = \det((v_i, v_j^*)|_{i,j=1}^3) = (v_1, v_1^*)(v_2, v_2^*)(v_3, v_3^*)t_1 t_2 t_3 \alpha +$$
$$+((v_1, v_2^*)y_{12})((v_2, v_3^*)y_{23})((v_3, v_1^*)y_{31})t_1 t_2 t_3 +$$
$$+((v_2, v_1^*)y_{21})((v_3, v_2^*)y_{32})((v_1, v_3^*)y_{13})\alpha^2 -$$
$$-((v_3, v_1^*)y_{31})(v_2, v_2^*)((v_1, v_3^*)y_{13})\alpha t_3 t_1 -$$
$$-((v_1, v_2^*)y_{21})(v_3, v_3^*)((v_2, v_1^*)y_{21})\alpha t_1 t_2 -$$
$$-((v_2, v_3^*)y_{23})(v_1, v_1^*)((v_3, v_2^*)y_{32})\alpha t_2 t_3,$$

belongs to B. Therefore, $a = \Delta\alpha t_1 t_2 t_3$ satisfies a quadratic equation over B:

$$a^2 - a(\Delta + \bar{\Delta})\alpha t_1 t_2 t_3 + \bar{\Delta}\Delta\alpha^2 t_1^2 t_2^2 t_3^2 = 0 \;;$$

since B is integrally closed, we get $a \in B$. On the other hand, comparing degrees in the coordinates of E, we see that it is impossible (Δ would be equal to $\beta(\Delta + \bar{\Delta})$ for some $\beta \in F$). $\square$

We immediately get the following.

Corollary 3 *The P is a free Q-module with basis* 1, ω*.*

Proof of theorem 1. From [11, th.3.19] it follows that the algebra $N = F[3\mathbf{A}]^G$ is a Cohen-Macaulay one. Since the homogeneous generators (G1)-(G3) of Q are algebraically idependent and N is a finitely generated module over Q, the elements (G1)-(G3) are homogeneous parameters of N and, therefore, N is a free Q-module. Let $1, u_1, \ldots, u_m$ be a basis of N over Q, we may assume that every u_i is homogeneous (cf. [11, p.178]). Moreover, by the preceding statement, P is a free Q-module of rank 2, on the other hand, from proposition 3 it follows that $\forall i\ u_i h \in Q$ for some nonzero $h \in F[2\mathbf{A}]^G \subseteq Q$, hence, $m = 1$.

The following assesment is evident.

Lemma 10 *Let $u \in N \backslash Q$ be a homogeneous polynomial of degree* $\deg(u_1)$*. then $u = a + \alpha u_1$, where $a \in Q$ and $\alpha \in F$, $\alpha \neq 0$.*

Since Q is homogeneous in all variables x, y, z, we may assume that u_1 is homogeneous in x, y, z (we may replace u_1 with a homogeneous component of it, by the previous lemma). Moreover, every permutation of the variables x, y, z induces an automorphism τ of N which keeps Q being invariant; by the previous lemma, $\tau(u_1) = a + \alpha u_1$, hence, $\deg_x(u_1) = \deg_y(u_1) = \deg_z(u_1)$.

Next, $\omega = a + bu_1$ for some $a, b \in Q$; recall that ω and u_1 are homogeneous in x, y, z, hence, we may assume that a, b are the same and, since $\deg_x(\omega) = \deg_y(\omega) = \deg_z(\omega) = 3$, for every $t \in \{x, y, z\}$

$$\deg_t(b) = \deg_t(\omega) - \deg_t(u_1) = \deg_z(\omega) - \deg_z(u_1) = \deg_z(b) .$$

On the other hand, by proposition 3, $u_1 h = c + d\omega$, where $h, c, d \in Q$, $h \neq 0$ and h does not depend on z. Since ω and u_1 are homogeneous in x, y, z, we may assume that h, c, d have the same property and $\deg_z(h) = 0$. Hence, $\omega h = ah + b(u_1 h) = ah + bc + bd\omega$; this yields $h = bd$ and, since $h \neq 0$, we have $\deg_z(b) = 0$. Hence, $\deg_x(b) = \deg_y(b) = \deg_z(b) = 0$, that is, b is a non-zero scalar in F. Therefore, $u_1 \in \mathrm{tr}[3\mathbf{A}]$ and the theorem is proven. $\square$

From proposition 2 and corollories 2 3 we get the following statement.

Theorem 2 *The elements* (G1)-(G2) *form a system of homogeneous parametries of* $F[3\mathbf{A}]^G$*, where* 1 *and polynomial (2) are a basis over it.*

Note that the elements p, q in (16) can be found explicitly as polynomials in (G1)-(G3) and this yields a syzygy for the generators of $F[3\mathbf{A}]^G$; obviously, it is unique. Also, from the proof of proposition 2 it follows that this relation is a consequence of commutivity, the Jordan identity (3) and the Cayley-Hamilton identity (1) (cf. Second Main Theorem for G_2 [15]).

If we replace x, y, z in (G1)-(G3) and (2) by the generic elements with zero trace, then we will get generators of $F[3\mathbf{A}']^G$. Moreover, since the elements $\mathrm{tr}(x)$, $\mathrm{tr}(y)$ and $\mathrm{tr}(z)$ vanish, the rest of elements obtained from (G1)-(G3) will form a system of homogeneous parameters of the algebra $F[3\mathbf{A}']^G$. In the same way from (15) on can get the unique syzygy for this set of generators.

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