LAPLACE OPERATOR AND POLYNOMIAL INVARIANTS

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ABSTRACT. Let G be a simple linear algebraic group of type F_4 over an algebraically closed field of characteristic zero. We describe generators of the algebra of invariants of a system of vectors of a minimal irreducible representatioin of G in terms of trace polynomials of an exceptional simple Jordan algebra (the Albert algebra) and Laplace operators.

1. PRELIMINARIES

Let F denote an algebraically closed field of characteristic zero.
Notations.

- $\mathrm{End}_F V$ is the algebra of linear transformations of a vector space V;
- $F[V]$ (respectively, $F(V)$) is the algebra (field) of polynomial (rational) functions from a vector space V to F; all subalgebras of $F[V]$ are supposed to be with 1;
- $V : G$ is a module V over a group G;
- $F[V]^G$ $(F(V)^G)$ is the subalgebra (the subfield) of polynomial (rational) invariants of $V : G$;
- $kV = \underbrace{V \oplus \ldots \oplus V}_{k}$ is the direct sum of k copies of a vector space V;
- $\mathrm{Pol}[V, W] \simeq F[V] \otimes_F W$ is the space of polynomial mappings from a vector space V to a vector space W;
- R_a (L_a) the operator of right (left) multiplication by an element a, i.e., $R_a : x \to xa$ $(L_a : x \to ax)$.

It is well-known that minimal linear representations of exceptional algebraic groups are connected with some non-associative algebras. Thus, the automorphism group $G = Aut(\mathbf{O})$ of the Cayley-Dickson algebra $\mathbf{O}$ is a simple algebraic group of type G_2 (cf. [12, Prop. 1.10]). It is known that every element $x \in \mathbf{O}$ satisfies the quadratic identity

$$x^2 - \mathrm{t}(x)x + n(x) = 0 \ , \tag{1}$$

where $\mathrm{t}(x)$ is the trace of x. As a G-module, $\mathbf{O}$ is the direct sum of a 1-dimensional trivial submodule $F \cdot 1$ and an irreducible 7-dimensional submodule $\mathbf{O}' = \{a \in \mathbf{O} |\ \mathrm{t}(a) = 0\}$.

In [12, 13] G.Schwarz proved that the algebra of invariants $F[k\mathbf{O}]^G$ with respect to the diagonal action (the system of k vectors of $\mathbf{O} : G$) is generated by trace polynomials which can be defined in the following way.

Let $x^{(i)}$ be the projector of $k\mathbf{O}$ onto the i-th summand (a generic element of the algebra $\mathbf{O}$), then $x^{(1)}, \ldots, x^{(k)}$ lie in the algebra $\mathrm{Pol}[k\mathbf{O}, \mathbf{O}]$ with respect to the pointwise operations. The subalgebra $\mathcal{F}_k(\mathbf{O})$ generated by $x^{(1)}, \ldots, x^{(k)}$ is called *the algebra of generic elements of* $\mathbf{O}$ of rank k; for all $u \in \mathcal{F}_k(A)$ we can consider a polynomial function $\mathrm{t}(u) \in F[k\mathbf{O}]$ which is the composition of u with $\mathrm{t}(x)$. Moreover, $\mathrm{t}(u)$ is a polynomial G-invariant, and the subalgebra

$\mathrm{t}_k[\mathbf{O}]$ generated by all such polynomials is called *the algebra of trace polynomials of* $\mathbf{O}$ *of rank* k; it is equal to $F[k\mathbf{O}]^G$ by Schwarz's theorem. Respectively, $F[k\mathbf{O}']^G$ is generated by trace polynomials in generic elements $x_*^{(i)}$ with $\mathrm{t}(x_*^{(i)}) = 0$.

A similar conjecture makes sense in the case of the Albert algebra $\mathbf{A}$ (an exceptional simple Jordan algebra $H(\mathbf{O}_3)$ [9, ch. IX]); it is known that $G = Aut(\mathbf{A})$ is a simple algebraic group of type F_4 (e.g., [8]); also, every element $x \in \mathbf{A}$ satisfies a cubic equation

$$x^3 - \mathrm{tr}(x)x^2 + S(x)x - N(x) = 0\ , \tag{2}$$

where $\mathrm{tr}(x)$ is the trace of x. The G-module $\mathbf{A}$ is decomposed into the sum of a 1-dimensional one and 26-dimensional irreducible module $\mathbf{A}' = \{a \in \mathbf{A} |\ \mathrm{tr}(a) = 0\}$ [8].

Denote by $\mathrm{tr}_k[\mathbf{A}]$ ($\mathrm{tr}_k[\mathbf{A}']$) the subalgebra of $F[k\mathbf{A}]^G$ ($F[k\mathbf{A}']^G$) of trace polynomials, i.e., the subalgebra generated by $\mathrm{tr}(u)$, where u is a monomial in $x^{(1)}, \dots x^{(k)}$ ($x_*^{(1)}, \dots , x_*^{(k)}$). It is known that $F[k\mathbf{A}]^G$ ($F[k\mathbf{A}']^G$) is integral over $\mathrm{tr}_k[k\mathbf{A}]$ ($\mathrm{tr}_k[\mathbf{A}']$), their spectra are homeomorphic and their fields of fractions coincide; in particular, for $k \leq 3$ they are equal [5, 6]. However, for $k \geq 5$ we exhibit an invariant which does not belong to $\mathrm{tr}_k[\mathbf{A}]$ (proposition 8).

Let V be a vector space over F; pick $a \in \mathrm{Pol}[kV, V]$ and let $x = x^{(r)} \in \mathrm{Pol}[kV, V]$ be a projector from kV onto the r-th summand (we also call it a generic element); denote by $\mathrm{D}_x(a)$ the following derivation of $F[kV]$ [15]:

$$f \cdot \mathrm{D}_x(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} a_i$$

where x_i, a_i are the coordinates with respect to a basis of V. In fact, the defintion does not depend on the choice of the basis; besides, if V is an algebra and a is another generic element, then $\mathrm{D}_x(a)$ can be regarded as an operator of partial linearization in a more general context [14, ch.1]. Thus, if $u \in \mathcal{F}_k(\mathbf{A})$ is a monomial of degree n in x, then $\mathrm{tr}(u)\mathrm{D}_x(a) = \sum_{i=1}^n \mathrm{tr}(u_i)$, where u_i is obtained from u by replacing the i-th occurrence of x with a.

Now suppose that V is endowed with a nondegenerate symmetric bilinear form $\langle x, y\rangle$; fix an orthonormal basis $E_1, \dots , E_n$ and for generic elements $x = \sum_{i=1}^n x_i E_i$, $y = \sum_{i=1}^n y_i E_i$ define a linear operator $\Delta(x, y)$ on $F[kV]$: for all $f \in F[kV]$ put

$$f\Delta(x, y) = \sum_{i=1}^{n} f \cdot D_x(E_i) D_y(E_i) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial y_i}\ . \tag{3}$$

We call it a *Laplace operator*, because the case $x = y$ will be the most important one in the proof.

By lemma 7, the definition of $\Delta(x, y)$ does not depend on the choice of orthonormal basis. Besides, if V is a G-module and the form $\langle x, y\rangle$ is G-invariant, then the algebra of polynomial invariants $F[kV]^G$ is closed under Laplace operators .

It is convinient to consider the algebra of invariants with an arbitrary number of vector variables. Namely, let $\vec{V}$ denote the direct sum of a countable number of copies of a G-module V; we consider it as $\lim_{k\to\infty} kV$. Therefore, $F[\vec{V}]$ ($F[\vec{V}]^G$) is the sum of $F[kV]$ ($F[kV]^G$).

In the case $V = \mathbf{A}$ and $G = Aut(\mathbf{A})$ we put $\langle x, y\rangle = \mathrm{tr}(xy)$, it is a nondegenerate associative symmetric bilinear form of $\mathbf{A}$ [9, p.240, th.5]. We also denote by $\mathrm{tr}[\mathbf{A}]$ the sum of $\mathrm{tr}_k[\mathbf{A}]$, $k \in \mathbf{N}$ and $\mathcal{F}(\mathbf{A}) = \lim_{k\to\infty} \mathcal{F}_k(\mathbf{A})$ is generated by $x^{(r)}$, $r \in \mathbf{N}$.

Theorem 1. *Let S be a set of homogeneous in each $x^{(r)}$ generators of* $\mathrm{tr}[\mathbf{A}]$*, then the algebra of invariants $F[\vec{\mathbf{A}}]^{Aut(\mathbf{A})}$ is generated by elements*

$$f\Delta_1 \dots \Delta_t \ , \tag{4}$$

where $f \in S$, $t \geq 0$ and $\Delta_r = \Delta(x^{(i_r)}, x^{(j_r)})$ for all $r \in \{1, \dots, t\}$.

By lemma 6 we can estimate the degree of generators.

Corollary 1. *The algebra $F[\vec{\mathbf{A}}]^{Aut(\mathbf{A})}$ is generated by homogeneous invariants of degree ≤ 85.*

Probably, this estimate is not precise; besides, it is not clear how many Δ-s are necessary in (4), the example from section 6 shows that $t \geq 1$.

Using the projection from $\mathbf{A}$ onto the submodule $\mathbf{A}'$, we get similar statements on generators of $F[\vec{\mathbf{A}}']^G$. Actually, the restriction of $\mathrm{tr}(xy)$ to $\mathbf{A}'$ is nondegenerate, hence, we may choose $E_p \in \mathbf{A}'$ for all $p > 1$ and $E_1 = 1/\sqrt{3}$. Next, for a generic element $x^{(r)}$ the projector $x_*^{(r)}$ is equal to $x^{(r)} - 1/3\mathrm{tr}(x^{(r)})$. Put

$$f \cdot \mathrm{D}_{x_*}(a) = \sum_{i=2}^{27} \frac{\partial f}{\partial x_i} a_i$$

then $\mathrm{D}_{x_*}(a) = \mathrm{D}_{x_*}(a_*)$, where $a_* = \sum_{i=2}^{27} a_i E_i$. Also, for all $f \in F[\vec{\mathbf{A}}']$

$$f \cdot \mathrm{D}_x(a) = f \cdot \mathrm{D}_{x_*}(a) = f \cdot \mathrm{D}_{x_*}(a_*) \ ,$$

hence, a Laplace operator $\Delta(x, y)$ acts on $F[\vec{\mathbf{A}}']$ as

$$\Delta(x_*, y_*) = \sum_{i=2}^{27} \mathrm{D}_{x_*}(E_i)\mathrm{D}_{y_*}(E_i) \ . \tag{5}$$

Corollary 2. *Suppose that* $\mathrm{tr}[\mathbf{A}']$ *is generated by a subset S' of homogeneous polynomials. Then the algebra of invariants $F[\vec{\mathbf{A}}']^{Aut(\mathbf{A})}$ is generated by*

$$f\Delta_1 \dots \Delta_t \ , \tag{6}$$

where $f \in S'$, $t \geq 0$ and $\Delta_r = \Delta(x_*^{(i_r)}, x_*^{(j_r)})$ for all $r \in \{1, \dots, t\}$.

Also, $F[\vec{\mathbf{A}}']^{Aut(\mathbf{A})}$ is generated by polynomials of degree ≤ 85.

Proof. Put $S_0 = \{\mathrm{tr}(\mathbf{x}^{(r)}) |\ r \in \mathbf{N}\}$, then $\mathrm{tr}[\mathbf{A}]$ is generated by $\mathrm{tr}[\mathbf{A}']$ and S_0. Hence, $S = S' \cup S_0$ is a set of homogeneous generators of $\mathrm{tr}[\mathbf{A}]$ and, therefore, $F[\vec{\mathbf{A}}]^{Aut(\mathbf{A})}$ is generated by elements (4). Obviously, $\Delta(x, y)$ kills $\mathrm{tr}(x^{(r)})$, hence, by (5) the set of elements (4) consists of (6) and $\mathrm{tr}(x^{(r)})$; using the projection on $\mathbf{A}'$, we prove the statement. □

The underlying idea of the proof was taken from Luna's slice method (e.g., [2]) that is very efficient in birational Invariant Theory.

Let V be a G-module and let W be a subspace of V; denote by $N_G(W)$ the normalizer of W:

$$N_G(W) = \{\sigma \in G |\ W \cdot \sigma \subseteq W\} \ .$$

Let $f|_W$ be the restriction to W of a function f defined on V, then the mapping

$$\phi : f \to f|_W \tag{7}$$

is a homomorphism from $F[V]^G$ to $F[W]^{N_G(W)}$. It is common that ϕ is not an isomorphism; however, if the orbit $W \cdot G = \{w \cdot \sigma |\ w \in W, \sigma \in G\}$ of W is a dense subset of the affine space $V = F^{\dim V}$, then ϕ is obviously a monomorphism and it extends to a monomorphism from the field of fractions to $F(W)^{N_G(W)}$. Moreover, this mapping is an isomorphism between $F(V)^G$ and $F(W)^{N_G(W)}$, if W is a *slice*, that is, the following conditions are satisfied [2]:

1. the orbit $W \cdot G$ is a dense subset of the affine space $V = F^{\dim V}$;
2. there is an open subset $W' \subseteq W$ such that for every $\sigma \in G$ and every $v \in W'$ if $v \cdot \sigma \in W$, then $\sigma \in N_G(W)$.

Now, let $G = Aut(\mathbf{A})$ and let W be a "good" slice of $(k+2)\mathbf{A} : G$. Pick $f \in F[k\mathbf{A}]^G$, since the field $F((k+2)\mathbf{A})^G$ is generated by trace polynomials, $f = g/h$ for some $g, h \in \mathrm{tr}_{k+2}[\mathbf{A}]$. In fact, h can be found in a special form if we pass to $F[W]^{N_G(W)}$ and consider $\phi(f)$. In particular, h does not depend on $x^{(r)}$, $r \leq k$. Then we apply the operators $\Delta(x^{(k+1)}, x^{(k+1)})$, $\Delta(x^{(k+2)}, x^{(k+2)})$ several times to the equalty $fh = g$; since f does not depend on $x^{(k+1)}, x^{(k+2)}$, the operators act only on h and make it a nonzero constant. On the other hand, g becomes an element of the algebra generated by (4).

In the next section we use this method in the case of G_2 and give another proof of Schwarz's theorem. The rest of the paper is devoted to invariants of F_4; in section 3 using the slice technique, we find a suitable representation of invariants in the form of a quotient of trace polynomials; then, in section 4, we describe some properties of the Laplace operator and reduce the proof of theorem 1 to some technical statements proven in section 5. In the last section we give an example of an invariant of degree 5 which does not belong to $\mathrm{tr}[\mathbf{A}]$.

2. Invariants of G_2

The Cayley-Dickson algebra $\mathbf{O}$ can be represented as the set of matrices (e.g., [14, ch.2])

$$\begin{pmatrix} \alpha & v \\ w & \beta \end{pmatrix},$$

where $\alpha, \beta \in F$ and $v, w \in F^3 = F \times F \times F$. The multiplication is defined in the following way:

$$\begin{pmatrix} \alpha_1 & v_1 \\ w_1 & \beta_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 & v_2 \\ w_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + (v_1, w_2) & \alpha_1 v_2 + \beta_2 v_1 - w_1 \times w_2 \\ \alpha_2 w_1 + \beta_1 w_2 + v_1 \times v_2 & (w_1, v_2) + \beta_1\beta_2 \end{pmatrix},$$

where (v, w), $v \times w$ are the scalar and vector product in F^3, respectively. The algebra $\mathbf{O}$ is equipped with an involution

$$\overline{\begin{pmatrix} \alpha & v \\ w & \beta \end{pmatrix}} = \begin{pmatrix} \beta & -v \\ -w & \alpha \end{pmatrix},$$

where the trace $\mathrm{t}(x)$ of a matrix $x \in \mathbf{O}$ is equal to $x + \bar{x} \in 1 \cdot F \simeq F$. One can easily check that $\mathbf{O}$ satisfies the Cayley-Hamilton identity (1), where $\delta(x) = \bar{x}x = 1/2(\mathrm{t}(x)^2 - \mathrm{t}(x^2))$.

Besides, $\mathbf{O}$ is an alternative algebra, this means that

$$x^2 y = x(xy)\ ;\quad yx^2 = (yx)x \tag{8}$$

for all $x, y \in \mathbf{O}$.

Throughout this section G denotes $Aut(\mathbf{O})$.

Theorem 2. *([12, 13]) The algebra of polynomial invariants $F[k\mathbf{O}]^G$ is equal to $\mathrm{t}_k[\mathbf{O}]$.*

We start the proof with the following lemma.

Lemma 1. *The algebra* $\mathrm{t}[k\mathbf{O}]$ *is generated by the following elements*
(a1) $\mathrm{t}(a)$, $\mathrm{t}(a^2)$,
(a2) $\mathrm{t}(ab)$, $\mathrm{t}(ab\cdot c)$, $\mathrm{t}((ab)c\cdot d)$,
where a,b,c,d *are different generic elements from* $\{x^{(1)},\dots,x^{(k)}\}$.

Proof. Put $Q=\mathrm{t}_k[\mathbf{O}]$ and let A be the Q-module generated by $\mathcal{F}_k(\mathbf{O})$ (in $\mathrm{Pol}[k\mathbf{O},\mathbf{O}]$). Denote by Q_+ the subspace spanned by homogeneous elements of positive degree; it is an ideal of Q and $I=Q_+\cdot\mathcal{F}_k(\mathbf{O})$ is an ideal of the Q-algebra A.

By (1) $\bar{A}=A/I$ is an alternative algebra of nil index 2. Hence, by (8) $\bar{A}$ satisfies the identities

$$ab=-ba\ ;(ab)c=-(ac)b\ .$$

Hence, it is spanned by $u=x_iR_{x_j}\dots R_{x_r}$ and u is skew symmetric in $x_i,x_j,\dots x_r$; in particular, a nonzero u is multilinear. Observe that $xy=1/2[x,y]$, hence, by (8) $v=(xy)z\cdot t=-z(xy)\cdot t=(zt)(xy)=1/8[[x,y],[z,t]]$. Also, since u is skew symmetric,

$$v=\frac{1}{8\cdot 4!}\sum_{\sigma\in S_4}\mathrm{sgn}(\sigma)\sigma\cdot[[x,y],[z,t]]$$

where σ acts by permutation of the variables. Obviously, the sum is equal to 0 (it is skew symmetric under an even permutation (12)(34)).

Thus, $\bar{A}$ is spanned by skew-symmetric monomials of degree ≤ 3. Let u be a monomial from A, we may assume that $\deg(u)\ge 3$, hence, $u=vx$, where x is a variable. If $\deg(u)>4$, then $v\in I$, that is, $v=\sum_p v_p\phi_p$, where $\phi_p\in Q_+$. We may assume that u_p, ϕ_p are homogeneous and $\deg(u_p)+\deg(\phi_p)=\deg(v)$. Therefore, to prove the lemma we can use the induction on $\deg(u)$. Also, v does not depend on x : if $v=ay$, where a depends on x, then $\mathrm{t}(u)=\mathrm{t}((xa)y)$, hence, $xa\in I$ and we can reduce the degree of u. Thus, y must be equal to x and $\mathrm{t}(u)=\mathrm{t}(vx^2)=\mathrm{t}(vx)\mathrm{t}(x)-\mathrm{t}(v)n(x)=\mathrm{t}(vx)\mathrm{t}(x)-1/2\mathrm{t}(v)(\mathrm{t}(x)^2-\mathrm{t}(x^2))$. $\square$

Let $R(x)$ denote the descriminant of the characteristic polynomial (1),

$$R(x)=2\mathrm{t}(x^2)-\mathrm{t}(x)^2\ . \tag{9}$$

Proposition 1. *For all* $f\in F[k\mathbf{O}]^G$ *there exists* $m\in\mathbf{N}$ *such that* $f\cdot R(x_{k+1})^m$ *belongs to* $\mathrm{t}_{k+1}[\mathbf{O}]$.

Proof. Consider a subspace $W=k\mathbf{O}\dot{+}F^2\subseteq(k+1)\mathbf{O}$, where F^2 is the subalgebra of the diagonal matrices. The mapping $\theta:f\to f|_W$ is an algebra homomorphism from $F[(k+1)\mathbf{O}]\to F[W]$. Moreover, it is monic on the subalgebra of invariants $F[(k+1)\mathbf{O}]^G$. Indeed, the subset U of elements from $\mathbf{O}$ which are conjugate to a diagonal matrix is Zariski dense (U is defined by the condition $R(x)\neq 0$). Hence, the orbit of every element from the Zariski dense subset $k\mathbf{O}\times U\subseteq(k+1)\mathbf{O}$ meets W and if $f|_W=0$ for an invariant f, then $f=0$ everywhere.

Similarly, $f\to f|_W$ defines a monomorphism from the algebra of generic elements $\mathcal{F}_{k+1}(\mathbf{O})$ to $\mathrm{Pol}[W,\mathbf{O}]$; we also denote it by θ.

Thus $x=\theta(x_{k+1})$ is a diagonal matrix

$$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$$

and

$$\theta(x^{(r)}) = x^{(r)} = \begin{pmatrix} \alpha_r & v_r \\ w_r & \beta_r \end{pmatrix} ,$$

for all $r \leq k$.

The normalizer $N_G(W) = \{\sigma \in G | \ W\sigma \subseteq W\}$ is a semidirect product of the group $\mathbf{Z}_2$ and the special linear group SL_3. The generator of $\mathbf{Z}_2$ is an automorphism

$$\sigma : \begin{pmatrix} \alpha & v \\ w & \beta \end{pmatrix} \to \begin{pmatrix} \beta & -w \\ -v & \alpha \end{pmatrix} ;$$

the action of SL_3 is defined in the following way: let P be a matrix from SL_3, then

$$\begin{pmatrix} \alpha & v \\ w & \beta \end{pmatrix} \cdot P = \begin{pmatrix} \alpha & vP \\ w(P^\top)^{-1} & \beta \end{pmatrix}.$$

Indeed, for all $\psi \in N_G(W)$ one can find $i \in \{0, 1\}$ and $P \in SL_3$ such that $\psi\sigma^i P = \mathrm{id}$.

By the classical result on invariants of SL_3 (cf. [10, p. 254]) the algebra $F[W]^{SL_3}$ is generated by

(b1) α, β, α_r, β_r;

(b2) (v_p, w_q),

(b3) $\det(v_p, v_q, v_r)$, $\det(w_p, w_q, w_r)$.

Let $B = \theta(\mathcal{F}_{k+1}(\mathbf{O}))$, it is the subalgebra of $\mathrm{Pol}[W, \mathbf{O}]$ generated by $x^{(r)}$, $r \leq k$ and x; also, consider the subalgebra $\tilde{B}$ generated by B and the idempotents

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

Lemma 2. *The subalgebra $P \subseteq F[W]$ generated by $\mathrm{t}(u)$, $u \in \tilde{B}$, contains $F[W]^{SL_3}$.*

Proof. For all r we have $\alpha_r = \mathrm{t}(e_1 x^{(r)} e_1) \in \tilde{B}$; similarly, $\beta_r, \alpha, \beta \in \tilde{B}$. Also,

$$(v^{(p)}, w^{(q)}) = \mathrm{t}((e_1 x^{(p)} \cdot e_2)(e_2 x^{(q)} \cdot e_1)) \in P$$

and

$$\det(v^{(p)}, v^{(q)}, v^{(r)}) = \mathrm{t}((e_1 x^{(p)} \cdot e_2)(e_1 x^{(q)} \cdot e_2) \cdot (e_1 x^{(r)} \cdot e_2)) \in P;$$

similarly, $\det(w^{(p)}, w^{(q)}, w^{(r)}) \in P$. $\square$

Proof of proposition 1. Put $R = R(x^{(k+1)})$ and pick $f \in F[(k+1)\mathbf{O}]^G$, we may assume that f is homogeneous of positive degree. Since θ is monic on $F[(k+1)\mathbf{O}]^G$, it extends to an isomorphism from the field of fractions (in fact, it is equal to the field of invariants $F((k+1)\mathbf{O})^G$ [10, p.165]) into $F(W)$. Hence, it suffices to show that $\theta(f)\theta(R)^m \in \theta(\mathrm{t}_{k+1}[\mathbf{O}])$ for some m.

By the previous lemma, $\theta(f) \in P$, hence $\theta(f)$ is equal to a trace polynomial $g \in \mathrm{t}[\mathbf{O}]$ where some variables are replaced by e_1, e_2 and x. Observe that

$$e_1 = \frac{(x - \beta \cdot 1)(\alpha - \beta)}{(\alpha - \beta)^2} \quad , e_2 = \frac{(x - \alpha \cdot 1)(\beta - \alpha)}{(\alpha - \beta)^2} ;$$

we replace each occurrence of e_i in $\theta(f)$ by the corresponding element and get

$$\theta(f) = \frac{1}{(\alpha - \beta)^{2m}} \sum_p \theta(f_p) h_p$$

for some $m \geq 0$, $f_p \in \mathrm{t}_{k+1}[\mathbf{O}]$ and $h_p \in F[\alpha, \beta]$.

Since $\theta(f), \theta(f_p)$ are $\mathbf{Z}_2$-invariants, we may assume that h_p is a symmetric polynomial in two variables α, β. Hence, it is a polynomial in $\alpha + \beta = \theta(\mathrm{t}(x^{(k+1)}))$ and ω. Finally,

$$\theta(R) = 2\mathrm{t}(x^2) - \mathrm{t}(x)^2 = 2(\alpha^2 + \beta^2) - (\alpha + \beta)^2 = (\alpha - \beta)^2$$

hence, $\sum_p \theta(f_p) h_p \in \theta(\mathrm{t}_{k+1}[\mathbf{O}])$ and $\theta(f)\theta(R)^m \in \theta(\mathrm{t}_{k+1}[\mathbf{O}])$. $\square$

The next lemma is quite trivial but very useful.

Lemma 3. *Let V be a vector space over F with a nondegenerate symmetric bilinear form $\langle x, y\rangle$ and an orthonormal basis $e_1, \dots, e_n$. Then*

$$\sum_{i=1}^{n} \langle a e_i \rangle \langle b e_i \rangle = \langle a, b \rangle \tag{10}$$

for all $a, b \in V$

Proof. Indeed, $a = \sum_{i=1}^{n} a_i e_i$ and $b = \sum_{j=1}^{n} b_j e_j$, hence

$$\langle a, b\rangle = \langle \sum_{i=1}^{n} a_i e_i, \sum_{j=1}^{n} b_j e_j \rangle =$$

$$= \sum_{i,j=1}^{n} a_i b_j \langle e_i, e_j \rangle = \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} \langle a, e_i \rangle \langle b, e_i \rangle \ .$$

$\square$

It is well-known (e.g.[14, ch.2]) that the form $\mathrm{tr}(xy)$ is nondegenerate. Let $e_1, e_2, \dots, e_8$ be an orthonormal basis of $\mathbf{O}$, by (10)

$$\sum_{i=1}^{8} \mathrm{t}(a e_i) \mathrm{t}(b e_i) = \mathrm{t}(ab) \tag{11}$$

for all $a, b \in \mathbf{O}$.

Now put $x = x^{(k+1)}$ and

$$\Delta = \sum_{i=1}^{8} \mathrm{D}_x(e_i)^2 \ , \tag{12}$$

it is the Laplace operator $\Delta(x, x)$.

Lemma 4. *The algebra of trace polynomials $\mathrm{t}_{k+1}[\mathbf{O}]$ is closed under Δ.*

Proof. By lemma 1, $\mathrm{t}_{k+1}[\mathbf{O}]$ is spanned by 1 and $f = \mathrm{t}(u_1) \dots \mathrm{t}(u_m)$, where $\mathrm{t}(u_p)$ are polynomials (a1)-(a2).

Since $\mathrm{D}_x(e_i)$ is a derivation, we have

$$f \cdot \mathrm{D}_x^2(e_i) = \sum_{p,q=1;\ p \neq q}^{m} \mathrm{t}(u_1) \dots (\mathrm{t}(u_p) \cdot \mathrm{D}_x(e_i)) \dots (\mathrm{t}(u_q) \cdot \mathrm{D}_x(e_i)) \dots \mathrm{t}(u_m) +$$

$$+ \sum_{p=1}^{m} \mathrm{t}(u_1) \dots (\mathrm{t}(u_p) \cdot \mathrm{D}_x^2(e_i)) \dots \mathrm{t}(u_m) \ .$$

The lemma will be proven, if we show that

$$\sum_{i=1}^{8} (\mathrm{t}(u_p)\cdot \mathrm{D}_x(e_i))\,(\mathrm{t}(u_q)\cdot \mathrm{D}_x(e_i))$$

and

$$\sum_{i=1}^{8} \mathrm{t}(u_p)\cdot \mathrm{D}_x^2(e_i)$$

belong to $\mathrm{t}_{k+1}[\mathbf{O}]$.

Consider the first sum; we may assume that $u_p = v_p x$, where $\deg_x(v_p) = 0$ or $v_p = x$. Anyway, by (11) the first sum lies in $\mathrm{t}_{k+1}[\mathbf{O}]$.

The latter sum is not equal to zero only if $u_p = x^2$, in this case it is a constant. □

Proof of theorem 2.

Pick $f \in F[k\mathbf{O}]^G \subseteq F[(k+1)\mathbf{O}]^G$, by proposition 1 $f\cdot R(x)^m \in \mathrm{t}_{k+1}[\mathbf{O}]$ for some m.

If $m = 0$, then $f \in \mathrm{t}[k\mathbf{O}]$; otherwise we shall show that $f\cdot R(x)^{m-1} \in \mathrm{t}_{k+1}[\mathbf{O}]$.

By lemma 4, $(f\cdot R(x)^m)\cdot\Delta \in \mathrm{t}_{k+1}[\mathbf{O}]$. On the other hand, since f does not depend on x,

$$(f\cdot R(x)^m)\cdot\Delta = f\cdot(R(x)^m\cdot\Delta)\ .$$

Denote $\mathrm{D} = \mathrm{D}_x(e_i)$, $a = \mathrm{t}(x)$ and $b = \mathrm{t}(x^2)$, then $R = R(x) = 2b - a^2$ and

$$R^m\mathrm{D}^2 = m(m-1)R^{m-2}(R\mathrm{D})^2 + mR^{m-1}(R\mathrm{D}^2)$$

$$R\mathrm{D} = 2b\mathrm{D} - 2a(a\mathrm{D})\ ;$$

$$R\mathrm{D}^2 = 2b\mathrm{D}^2 - 2(a\mathrm{D})^2 - 2a(a\mathrm{D}^2) = 2b\mathrm{D}^2 - 2(a\mathrm{D})^2,$$

because $a\mathrm{D}^2 = 0$. Hence, by (11)

$$\sum_{i=1}^{8} R\cdot\mathrm{D}_x(e_i)^2 = 4\sum_{i=1}^{8}\mathrm{t}(e_i^2) - 2\sum_{i=1}^{8}\mathrm{t}(e_i)^2 = 4\times 8 - 2\sum_{i=1}^{8}\mathrm{t}(e_i\cdot 1)\mathrm{tr}(e_i\cdot 1) = 32 - 2\mathrm{t}(1) = 28\ .$$

Also

$$(R\cdot\mathrm{D})^2 = 4a^2(a\mathrm{D})^2 - 8a(a\mathrm{D})(b\mathrm{D}) + 4(b\mathrm{D})^2$$

and by (11)

$$\sum_{i=1}^{8}(R\cdot\mathrm{D}_x(e_i))^2 = 4\sum_{i=1}^{8}\mathrm{t}(x)^2\mathrm{t}(e_i)^2 - 8\sum_{i=1}^{8}\mathrm{t}(x)\mathrm{t}(e_i)\cdot 2\mathrm{t}(xe_i) + 4\sum_{i=1}^{8}(2\mathrm{t}(xe_i))^2 =$$

$$= 4\mathrm{t}(x)^2\mathrm{t}(1) - 16\cdot\mathrm{t}(x)^2 + 16\mathrm{t}(x^2) = 8\mathrm{t}(x)^2 - 16\mathrm{t}(x)^2 + 16\mathrm{t}(x^2) = 8R\ .$$

Thus,

$$R^m\cdot\Delta = \sum_{i=1}^{8} R^m\mathrm{D}_x(e_i)^2 = m(m-1)R^{m-2}\sum_{i=1}^{8}(R\mathrm{D}_x(e_i))^2 + mR^{m-1}\sum_{i=1}^{8}(R\mathrm{D}_x(e_i)^2)$$

$$= m(m-1)R^{m-2}8R + 28mR^{m-1} = m(8m+20)R^{m-1}\ .$$

Theorem 2 is proven. □

3. Slices of $k\mathbf{A} : F_4$.

Recall the definition of Albert algebra [9, 14]. Let $\mathbf{O}_3 = \mathbf{O} \otimes_F F_3$ be the algebra of 3×3-matrices over $\mathbf{O}$; a linear mapping $a \otimes b \to \bar{a} \otimes b^\top$, where $b \to b^\top$ is the matrix transposition, is an involution of $\mathbf{O}_3$; the subspace $H(\mathbf{O}_3)$ of symmetric elements with the multiplication $a \circ b = 1/2(ab + ba)$ is a simple Jordan algebra $\mathbf{A}$ of dimension 27 called an Albert algebra.

Let $e_i = e_{ii}, e_{ij}$ $(i, j \in \{1, 2, 3\},\ i \neq j)$ denote the elementary matrices in $\mathbf{O}_3$ (we identify $1 \otimes F_3$ with F_3), then an arbitrary element in $\mathbf{A}$ can be written in the form

$$\sum_{i=1}^{3} \alpha_i e_i + \sum_{(ijk)} [\mathrm{c}_i]_{jk}, \tag{13}$$

where $\alpha_i \in F$, $\mathrm{c}_i \in \mathbf{O}$ and $[\mathrm{c}_i]_{jk}$ means $\mathrm{c}_i \cdot e_{jk} + \bar{\mathrm{c}}_i \cdot e_{kj}$, and (ijk) runs over the subgroup A_3 of even permutations. Note that $[\mathrm{c}]_{kj} = [\bar{\mathrm{c}}]_{jk}$.

The next formulae will be used very often

$$[a]_{ij} \circ [b]_{jk} = [\overline{ab}]_{ki}\ ; \tag{14}$$

$$[a]_{ij} \circ [b]_{ij} = \frac{1}{2}\mathrm{t}(a\bar{b})(e_i + e_j)\ . \tag{15}$$

If $\mathrm{c} = \begin{pmatrix} \alpha & v \\ w & \beta \end{pmatrix}$, then $\begin{bmatrix} \alpha & v \\ w & \beta \end{bmatrix}_{jk}$ denotes $[\mathrm{c}]_{jk}$. In fact, the subspace of such elements is the Pierce component $\mathbf{A}_{ij} = \mathbf{A}U_{e_i,e_j}$, where $U_{x,y} \in \mathrm{End}_F\mathbf{A}$, $zU_{x,y} = 1/2((zx)y + (zy)x) - z(xy))$; also $Fe_i = \mathbf{A}_{ii} = \mathbf{A}U_{e_i,e_i}$ (cf. [9, p.118]).

The subspace of elements (13), where

$$\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3 \in \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \subseteq \mathbf{O}\ ,$$

is a subalgebra of $\mathbf{A}$ isomorphic to the Jordan algebra $F_3^{(+)}$ (the associative algebra F_3 with the symmetrized multiplication $a \circ b$). For all $j, k \in \{1, 2, 3\}$, $j \neq k$, put

$$\epsilon_{jk} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{jk}\ , \tag{16}$$

then e_i, ϵ_{jk} form a basis of $F_3^{(+)}$. In particular, $[\mathrm{c}_i]_{jk} = c_{jk}\epsilon_{jk} + c_{kj}\epsilon_{kj}$ for $\mathrm{c}_i \in F^2$.

Also, denote by F^3 the subalgebra of diagonal matrices $Fe_1 + Fe_2 + Fe_3 \subseteq F_3^{(+)} \subseteq \mathbf{A}$.

It is known that the automorphism group $G = Aut(\mathbf{A})$ is a simple linear algebraic group of type F_4 (e.g.[8]). We need certain subgroups of G:

(c1) the symmetric group S_3 whose action is induced by permutation of the idempotents e_1, e_2, e_3 [9, p.389,ex.3];

(c2) the subgroup $\mathbf{Z}_2$ generated by an automorphism of the second order (cf. [7, sect.2]):

$$\sigma : e_i \to e_i\ \forall i \in \{1, 2, 3\}\ ;$$

$$\sigma : \begin{bmatrix} \alpha & v \\ w & \beta \end{bmatrix}_{ij} \to \begin{bmatrix} \beta & -w \\ -v & \alpha \end{bmatrix}_{ij}$$

for all $(ijk) \in A_3$;

(c3) the subgroup $T_2 = \{\xi = (\xi_1, \xi_2, \xi_3) \in F^* \times F^* \times F^* |\ \xi_1\xi_2\xi_3 = 1\}$ acting in the following way:

$$\xi : e_i \to e_i\ \forall i \in \{1, 2, 3\}\ ;$$

$$\xi : \begin{bmatrix} \alpha & v \\ w & \beta \end{bmatrix}_{ij} \to \begin{bmatrix} \xi_i\xi_j^{-1}\alpha & \xi_k^{-1}v \\ \xi_k w & \xi_i^{-1}\xi_j\beta \end{bmatrix}_{ij}$$

for all $(ijk) \in A_3$;

(c4) by [9, p.386, th.8], the stabilizer M of $F_3^{(+)}$ is isomorphic to the special linear group SL_3. More precisely, if $Q \in SL_3$, then we can define an automorphism $\hat{Q} \in M$ in following way (it suffices to consider only elements in the complement of $F_3^{(+)}$ to $\mathbf{A}$):

$$\hat{\psi} : \begin{bmatrix} 0 & v \\ w & 0 \end{bmatrix}_{ij} \to \begin{bmatrix} 0 & vQ \\ w(Q^\top)^{-1} & 0 \end{bmatrix}_{ij}$$

for all $(ijk) \in A_3$. Then $Q \to \hat{Q}$ is an isomorphism of SL_3 onto M;

(c5) the stabilizer of F^3 is isomorphic to a spinor group $Spin_7$ [9, p.388,ex.2]; note that $SL_3, T_2, \mathbf{Z}_2 \subseteq Spin_7$.

Put $W = k\mathbf{A} \oplus F_3^{(+)} \oplus F^3 \subseteq (k+2)\mathbf{A}$; obviously, subgroups (c1)-(c4) are contained in $N_G(W)$. Also, it is easy to check that $\mathbf{Z}_2$ normalizes T_2 and $\mathbf{Z}_2 \cap T_2 = \{1\}$, therefore, $H_0 = T_2\mathbf{Z}_2$ is a semidirect product $T_2 \rightthreetimes \mathbf{Z}_2$. Besides, H_0 normalizes SL_3 and $H_0 \cap SL_3 = \{1\}$, hence, $H = SL_3 \rightthreetimes H_0 \leq N_G(W)$. Finally, S_3 normalizes H and the semidirect product $H \rightthreetimes S_3$ is isomorphic to a subgroup $H \cdot S_3$ of $N_G(W)$. Moreover, one can check that they coincide and W is a slice (e.g. [1, lemma 4.5],[6, lemma 5]).

Similarly, let $W_0 = (k+1)\mathbf{A} \oplus F^3 \subset (k+2)\mathbf{A}$, then it is also a slice and $N_G(W_0)$ is a semidirect product of S_3 and the stabilizer of e_1, e_2, e_3, i.e., $Spin_7$.

Denote by θ (θ_0) the algebra homomorphism from $F[(k+2)\mathbf{A}]$ to $F[W]$ (respectively, $F[W_0]$) which takes f to the restriction $f|_W$ ($f|_{W_0}$). The corresponding mapping from $\mathrm{Pol}[(k+2)\mathbf{A}, \mathbf{A}]$ to $\mathrm{Pol}[W, \mathbf{A}]$ ($\mathrm{Pol}[W_0, \mathbf{A}]$) will be denoted by the same symbol.

Recall that W (W_0) is a slice, in particular, θ (θ_0) restricts to a monomorphism from $F[(k+2)A]^G$ to $F[W]$ ($F[W_0]$).

Denote $\alpha_i = \alpha_i^{(k+1)}$, $\beta_i = \alpha_i^{(k+2)}$, $x_{ij} = x_{ij}^{(k+1)}$ for all i, j. Notice that $\theta(x^{(r)}) = \theta_0(x^{(r)}) = x^{(r)}$ for all $r \leq k$; also $\theta_0(x^{(k+1)}) = x^{(k+1)}$. Next, put $x = \theta(x^{(k+1)}) = \sum_{i=1}^3 \alpha_i e_i + \sum_{(ijk)} [\mathbf{x}_i]_{jk}$, where

$$[\mathbf{x}_i]_{jk} = \begin{bmatrix} x_{jk} & 0 \\ 0 & x_{kj} \end{bmatrix}_{jk} .$$

Finally, put $y = \theta(x^{(k+2)}) = \theta_0(x^{(k+2)}) = \sum_{i=1}^3 \beta_i e_i$,

By the classical result on invariants of SL_n the algebra $K_1 = F[W]^{SL_3}$ is generated by

(d1) $\alpha_i^{(r)}, \alpha_i, \beta_i$;

(d2) $x_{ij}^{(r)}$, x_{ij}, $i \neq j$;

(d3) $(v_i^{(p)}, w_j^{(q)})$;

(d4) $\det(v_i^{(p)}, v_j^{(q)}, v_k^{(r)})$, $\det(w_i^{(p)}, w_j^{(q)}, w_k^{(r)})$, where $r,p,q \in \{1,\dots,k\}$, $i,j,k \in \{1,2,3\}$.

Let B denote the Jordan subalgebra generated by $x^{(r)}$, $r \le k$ and x,y in $\mathrm{Pol}[W,\mathbf{A}]$, i.e., $B=\theta(\mathcal{F}_{k+2}(\mathbf{A}))$ (in fact, θ is an isomorphism from $\mathcal{F}_{k+2}(\mathbf{A})$ onto B [7, lemma 1]).

Also, denote by $\tilde{B}$ the Jordan subalgebra generated by B and e_i, ϵ_{ij} and let P be the subalgebra of $F[W]$ generated by $\mathrm{tr}(u)$, where $u \in \tilde{B}$.

Proposition 2. *P contains K_1.*

Proof. It suffices to show that the generators (d1)-(d4) belong to P.

First of all, the elements $\alpha_i^{(r)}e_i = x^{(r)}U_{e_i,e_i}$, $[\mathbf{x}_k^{(r)}]_{ij} = x^{(r)}U_{e_i,e_j}$, $\alpha_i e_i$, $\beta_i e_i$, $[\mathbf{x}_k]_{ij}$ lie in $\tilde{B}$. Hence, $\alpha_i^{(r)} = \mathrm{tr}(\alpha_i^{(r)}e_i) \in P$ and similarly, $\alpha_i, \beta_i \in P$.

Besides, $x_{ij} = \mathrm{tr}(\epsilon_{ij} \circ [\mathbf{x}_k]_{ij}) \in P$ for all i,j. Hence,

$$\begin{bmatrix} 0 & v_k^{(r)} \\ w_k^{(r)} & 0 \end{bmatrix}_{ij} = [\mathbf{x}_k]_{ij} - x_{ij}\epsilon_{ij} - x_{ji}\epsilon_{ji} \in \tilde{B}$$

and

$$\begin{bmatrix} 0 & v_k^{(r)} \\ 0 & 0 \end{bmatrix}_{ij} = (\begin{bmatrix} 0 & v_k^{(r)} \\ w_k^{(r)} & 0 \end{bmatrix}_{ij} \circ \epsilon_{kj}) \circ \epsilon_{jk} \in \tilde{B}\ .$$

Multiplying these elements by ϵ_{ij}, we get

$$\begin{bmatrix} 0 & v_k^{(r)} \\ 0 & 0 \end{bmatrix}_{st}, \begin{bmatrix} 0 & 0 \\ w_k^{(r)} & 0 \end{bmatrix}_{st} \in \tilde{B}$$

for all k,r,s,t. Therefore,

$$(v_i^{(p)}, w_j^{(q)}) = \mathrm{tr}\left(\begin{bmatrix} 0 & v_i^{(p)} \\ 0 & 0 \end{bmatrix}_{st} \circ \begin{bmatrix} 0 & 0 \\ w_j^{(q)} & 0 \end{bmatrix}_{st} \right) \in P$$

and

$$\det(v_i^{(p)}, v_j^{(q)}, v_k^{(r)}) = \mathrm{tr}\left(\left(\begin{bmatrix} 0 & v_i^{(p)} \\ 0 & 0 \end{bmatrix}_{12} \circ \begin{bmatrix} 0 & v_j^{(q)} \\ 0 & 0 \end{bmatrix}_{23}) \right) \circ \begin{bmatrix} 0 & v_k^{(r)} \\ 0 & 0 \end{bmatrix}_{31} \right) \in P \quad ;$$

similarly, $\det(w_i^{(p)}, w_j^{(q)}, w_k^{(r)}) \in P$. So, all generators of K_1 belong to P. $\square$

The image $B_0 = \theta_0(\mathcal{F}_{k+2}(\mathbf{A})) \subseteq \mathrm{Pol}[W_0,\mathbf{A}]$ is generated by $x^{(r)}$, $r \le k+1$, and y; consider the Jordan algebra $\tilde{B}_0$ generated by B_0 and e_1,e_2,e_3. Let P_0 be the subalgebra of $F[W_0]$ generated by $\mathrm{tr}(u)$, where u runs over $\tilde{B}_0$. Since $Spin_7$ leaves fixed e_i and y, we have $P_0 \subseteq F[W_0]^{Spin_7}$.

Put $t = \mathrm{t}(\mathbf{x}_1^{(k+1)}\mathbf{x}_2^{(k+1)} \cdot \mathbf{x}_3^{(k+1)})$, $t_i = \mathrm{t}(\mathbf{x}_i^{(k+1)}\bar{\mathbf{x}}_i^{(k+1)})$ and

$$\omega = t^2 - \frac{1}{2}t_1t_2t_3\ ; \tag{17}$$

since $[\mathbf{x}_i^r]_{jk} = x^{(r)}U_{e_i,e_j} \in \tilde{B}_0$, by (15) $t, t_i, \omega \in P_0$.

Proposition 3. *The algebra $F[W_0]^{Spin_7}$ is contained in the algebra*

$$P_0\omega^{-1} = \{f/\omega^i |\ f \in P_0; \quad i = 0,1,\dots\}$$

Proof. Consider a homomorphism θ_1 from $\mathrm{Pol}[W_0, \mathbf{A}]$ ($F[W_0]$) to $\mathrm{Pol}[W, \mathbf{A}]$ ($F[W]$) which takes f to $f|_W$.

Notice that θ_1 restricts to a monomorphism from $F[W_0]^{Spin_7}$ to $F[W]$. Actually, there is a dense subset U of $W_0 = (k+1)\mathbf{A} \oplus F^3$ such that for all $a \in U$ there is an automorphism $\sigma \in G$ which takes a to W (cf.[7, lemma 1]). Using a suitable element from S_3, we can choose such σ leaving fixed all e_i, i.e., $\sigma \in Spin_7$. This means that the ordit $W \cdot Spin_7$ is a dense subset of W_0, hence, θ_1 is a monomorphism.

Since $P_0 \subseteq F[W_0]^{Spin_7}$, θ_1 extends to an isomorphism from $P_0\omega^{-1}$ onto $\theta_1(P_0)\theta_1(\omega)^{-1}$. Therefore, it suffices to show that

$$\theta_1(F[W_0]^{Spin_7}) \subseteq \theta_1(P_0)\theta(\omega)^{-1} .$$

Pick $f \in \theta_1(F[W_0]^{Spin_7})$, then it belongs to $(F[W]^{SL_3})^{T_2 \leftthreetimes \mathbf{Z}_2} \subseteq P^{T_2 \leftthreetimes \mathbf{Z}_2}$. Let $\tilde{B}'$ be the Jordan subalgebra of $\mathrm{Pol}[W, \mathbf{A}]$ generated by B and e_1, e_2, e_3; obviously, $\theta_1(\tilde{B}_0) = \tilde{B}' \subseteq \tilde{B}$ and $\theta_1(P_0)$ is generated by $\mathrm{tr}(u)$, where $u \in \tilde{B}'$.

Put $\mu = x_{12}x_{23}x_{31}$, $\bar{\mu} = x_{21}x_{13}x_{32}$, then $\theta_1(t_i) = 2x_{jk}x_{kj}$, $\theta_1(t) = \mu\bar{\mu}$ and therefore

$$\theta_1(\omega) = (\mu - \bar{\mu})^2 ;$$

also, one can check that

$$\epsilon_{ij} = \frac{1}{\mu - \bar{\mu}}([\mathbf{x}_k]_{ij}x_{jk}x_{ki} - [\mathbf{x}_i]_{jk} \circ [\mathbf{x}_j]_{ki}x_{ij})$$

hence, for all $f \in F[W]^{SL_3} \subseteq P$ replacing all occurrences of ϵ_{ij} $(i, j \in \{1, 2, 3\},\ i \neq j)$ in f, we get f to belong to the algebra generated by $\theta_1(P_0)$, x_{ij}, $i, j = 1, 2, 3;\ i \neq j$ and $\theta_1(\omega)^{-1}$. Hence,

$$f = \theta_1(\omega)^{-d} \sum_q f_q h_q$$

where $f_q \in \theta_1(P_0)$ and $h_q \in F[x_{ij}]$ (all polynomials in x_{ij}). Moreover, $T_2 \leftthreetimes \mathbf{Z}_2$ is a linearly reductive nilpotent group, hence h_p is equal to a sum of semi-invariants of $T_2 \leftthreetimes \mathbf{Z}_2$. Since $f_q \in \theta_1(P_0) \subseteq K_1^{T_2 \leftthreetimes \mathbf{Z}_2}$, we may assume that $h_p \in F[x_{ij}]^{T_2 \leftthreetimes \mathbf{Z}_2}$

Lemma 5. *$F[x_{ij}]^{T_2 \leftthreetimes \mathbf{Z}_2}$ is generated by $\theta_1(\omega)$, $\theta_1(t_i)$, $\theta_1(t)$.*

Proof. Recall that $(\xi_1, \xi_2, \xi_3) \in T_2$ takes x_{ij} to $\xi_i^{-1}\xi_j x_{ij}$; using this fact one can easily check that $F[x_{ij}]^{T_2}$ is generated by $\theta_1(t_i) = x_{jk}x_{kj}$, μ and $\bar{\mu}$. In addition, $\mathbf{Z}_2$ swops μ and $\bar{\mu}$ and leaves fixed $\theta_1(t_i)$, hence, $F[x_{ij}]^{T_2 \leftthreetimes \mathbf{Z}_2} = (F[x_{ij}]^{T_2})^{\mathbf{Z}_2}$ is generated by $\theta_1(t_i)$, $\mu + \bar{\mu} = \theta_1(t)$ and $\theta_1(\omega) = (\mu - \bar{\mu})^2$. □

Obviously, lemma 5 completes the proof of proposition 3. □

Recall that W_0 is a slice of $(k + 2)\mathbf{A} : G$, hence, θ_0 restricts to a monomorphism from $F[(k + 2)\mathbf{A}]^G$ and this restriction extends to a field isomorphism from $F((k + 2)\mathbf{A})^G$ onto $F(W_0)^{N_G(W_0)}$, where $N_G(W_0) = Spin_7 \leftthreetimes S_3$. Hence, $P_0^{S_3} \subseteq F(W_0)^{N_G(W_0)}$; denote by Q_0 the pre-image $\theta_0^{-1}(P_0^{S_3})$ in $F((k + 2)\mathbf{A})^G$; we shall describe it explicitly.

Let $R(x)$ denote the descriminant of the polynomial (2), then $R(x)$ is a polynomial in $\mathrm{tr}(x^i)$, $i = 1, 2, 3$.

Proposition 4. *The algebra Q_0 is contained in*

$$\mathrm{tr}_{k+2}[\mathbf{A}] \cdot R^{-1}$$

where $R = R(x^{(k+2)})$.

Proof. In $\mathrm{Pol}[W_0, \mathbf{A}]$ consider a system of linear equations

$$\sum_{i=1}^{3} \beta_i^p e_i = \theta_0(x^{(k+2)})^p = y^p \ ,$$

where $p = 0, 1, 2$ (we assume that $(x^{(k+2)})^0 = 1$). The square of the determinant of this system $\det(\beta_i^p)$ is equal to $\theta_0(R)$, hence, e_i is equal to a linear combination of powers of $\theta_0(x^{(k+2)})$ with coefficients in $F[\beta_i]\theta_0(R)^{-1}$.

Now pick $f \in P_0$, like in the proof of proposition 3 we replace all occurrences of e_i with the corresponding linear combination of powers of $\theta_0(x^{(k+2)})$ and get

$$f = \theta_0(R)^{-d} \sum_q f_q h_q \ ,$$

where $d \in \mathbf{N}$, $f_q \in \theta_0(\mathrm{tr}_{k+2}[\mathbf{A}])$ and $h_q = h_q(\beta_1, \beta_2, \beta_3)$. Since S_3 is a finite group, we may assume that h_q is an S_3-invariant, i.e., it is a symmetric polynomial. It is well-known that h_q is a polynomial in $\mathrm{tr}(y^i) = \theta_0(\mathrm{tr}((x^{(k+2)})^i))$, where $i = 1, 2, 3$. Thus, $f = \theta_0(R)^{-d}\theta(h)$ for some $h \in \mathrm{tr}_{k+2}[\mathbf{A}]$. Recall that θ_0 is monic on $F((k+2)\mathbf{A})^G$, thus this yields proposition 4 □

Recall that $\omega \in P_0$, denote by $\Omega \in Q_0$ the pre-image of ω.

Corollary 3. *For all $f \in F[(k+2)\mathbf{A}]^G$ there exist $p, q \geq 0$ such that $f\Omega^p R^q \in \mathrm{tr}_{k+2}[\mathbf{A}]$.*

Proof. The mapping θ_0 takes $F[(k+2)\mathbf{A}]^G$ to $F[W_0]^{N_G(W_0)} = (F[W_0]^{Spin_7})^{S_3}$ which lies in $(P_0\omega^{-1})^{S_3} = P_0^{S_3}\omega^{-1}$ by proposition 3.

Since θ_0 is monic on $F((k+2)\mathbf{A})^G$, the pre-image of $F[W_0]^{N_G(W_0)}$ is contained in $Q_0\Omega^{-1}$, hence, $F[(k+2)\mathbf{A}]^G \subseteq Q_0\Omega^{-1}$; so, the statement follows from proposition 4. □

4. Differentiation

In the case of G_2 the algebra of trace polynomials $\mathrm{t}[\mathbf{O}]$ is closed under the action of Laplace operators; it happens because $\mathrm{t}[\mathbf{O}]$ is generated by polynomials of small degree (≤ 4). The situation becomes more complicated in the case of F_4.

Lemma 6. *The algebra $\mathrm{tr}[\mathbf{A}]$ is generated by homogeneous invariants of degree ≤ 85.*

Proof. Recall that $\mathbf{A}$ is the direct sum of submodules $F \cdot 1$ and $\mathbf{A}'$, $\dim_F \mathbf{A}' = 26$. Using a standard argument based on representation of the symmetric group (or that of the general linear group) one can show that $\mathrm{tr}[\mathbf{A}]$ is generated by linearization (i.e., polarization) of generators of $\mathrm{tr}_{26}[\mathbf{A}]$.

Thanks to the cubic identity (2) it suffices to show that a Jordan nil algebra A of nil-index 3 generated by 26 variables is nilpotent of index ≤ 84.

The linearization of the identity $x^3 = 0$ in a commutative algebra is the Jacobi identity

$$(x_1x_2)x_3 + (x_2x_3)x_1 + (x_3x_1)x_2 = 0;$$

hence, the product of two ideals of A is also an ideal and the power J^k of an ideal $J \triangleleft A$ is equal to JR_J^{k-1} (like in a Lie algebra). Let $I = (A^2)^k/(A^2)^{k+1}$, then I is an A-module and $IR_A^{27} = 0$. By [3, lemma 3] $A^2 R_{A^2}^3 R_A = 0$, hence, $A^2 R_A^{82} = 0$, that is $A^{84} = 0$. $\square$

The next statements says that Laplace operators might give new invariants. Probably, it is a well-known fact, because the action of a Laplace operator on a multilinear polynomial can be viewed as a tenzor contraction.

Lemma 7. *Let V be a vector space of dimension n with a nondegenerate symmetric bilinear form $\langle x, y\rangle$. Then the definition (3) of Laplace operator $\Delta(x, y)$ does not depend on the choice of orthonormal basis.*

Besides, if $\langle a, b\rangle$ is G-invariant then

$$F[kV]^G \Delta(x, y) \subseteq F[kV]^G \ .$$

Proof. Let $f \in F[kV]$, we shall show that $f\Delta(x, y)$ does not depend on the choice of orthonormal basis and if f is G-invariant, then $f\Delta(x, y) \in F[\vec{V}]^G$ (therefore, $f\Delta(x, y) \in F[kV]^G = F[\vec{V}]^G \cap F[kV]$). We may assume that f is homogeneous in each generic element and $\Delta(x, y)$ acts nontrivially (hence, f depends on x, y as well).

Pick different generic elements $x', y' \notin \{x^{(1)}, \dots, x^{(k)}\}$, then for all a $f\mathrm{D}_x(a) = f\mathrm{D}_x(x')\mathrm{D}_{x'}(a)$, hence

$$f\Delta(x, y) = f\mathrm{D}_x(x')\mathrm{D}_y(y')\Delta(x', y'). \tag{18}$$

Observe that $f\mathrm{D}_x(x')\mathrm{D}_y(y')$ is linear in x' and y' and belongs to $F[\vec{V}]^G$, if $f \in F[kV]^G$. So, we may assume that $f = f(x, y, x^{(1)}, \dots, x^{(k)})$ is linear in x, y.

Since $\langle a, b\rangle$ is a nondegenerate symmetric bilinear form, there exists

$$\Phi = \Phi(x^{(1)}, \dots, x^{(k)}) \in \mathrm{Pol}[\vec{V}, \mathrm{End}_F V]$$

such that

$$f = \langle y \cdot \Phi,\ x\rangle \ .$$

Hence, the trace of Φ is equal to

$$\sum_{i=1}^{n} \langle E_i \cdot \Phi\ ,\ E_i\rangle =$$

$$= \sum_{i=1}^{n} f(E_i, E_i, x^{(1)}, \dots, x^{(k)}) = f \cdot \Delta(x, y) \ ,$$

thus $f \cdot \Delta(x, y)$ does not depend on the choice of $E_1, \dots, E_n$.

Put $\Phi\sigma = \Phi(x^{(1)} \cdot \sigma^{-1}, \dots, x^{(k)} \cdot \sigma^{-1})$, then for all $\sigma \in GL(V)$ we have

$$\langle y \cdot \Phi,\ x\rangle = f = f(x \cdot \sigma^{-1}, y \cdot \sigma^{-1}, x^{(1)} \cdot \sigma^{-1}, \dots, x^{(k)} \cdot \sigma^{-1}) =$$

$$= \langle y \cdot \sigma^{-1}(\Phi\sigma)\ ,\ (x \cdot \sigma^{-1})\rangle = \langle y \cdot \sigma^{-1}(\Phi\sigma)(\sigma^{-1})^\top\ ,\ x\rangle \ .$$

where $\sigma \to \sigma^\top$ is the transposition with respect to $\langle a, b\rangle$. Hence,

$$\Phi\sigma = \sigma\Phi\sigma^\top \ .$$

If $\sigma \in G$, then $\sigma^\top = \sigma^{-1}$, hence, $f\Delta(x, y) = \mathrm{tr}(\Phi)$ is a G-invariant. $\square$

Let $\mathrm{tr}[\mathbf{A}]^\circ$ denote the minimal subalgebra which contains $\mathrm{tr}[\mathbf{A}]$ and is closed under Laplace operators.

Proposition 5. *The algebra* $\mathrm{tr}[\mathbf{A}]^\circ$ *is generated by elements (4).*

Proof. Let A be the subalgebra of $F[\vec{\mathbf{A}}]$ generated by (4); then A contains $\mathrm{tr}[\mathbf{A}]$ and $A \subseteq \mathrm{tr}[\mathbf{A}]^\circ$. To prove the reciprocal inclusion it suffices to show that A is closed under Laplace operators, i.e., $f\Delta(x,y) \in A$ for all $f \in A$.

We may assume that f is a product of elements (4), i.e., $f = \prod_{i=1}^q f_i$, where $f_i = g_i\Delta_1^{(i)} \dots \Delta_{t_i}^{(i)}$ and $g_i \in S$.

Using (18) and a sufficient number of different generic elements, we can replace all $\Delta_j^{(i)}$ with another ones $\bar{\Delta}_j^{(i)}$ depending on different generic elements (which differ x, y as well); at the same time, each g_i is replaced with some linearization (it lies in $\mathrm{tr}[\mathbf{A}]$). Moreover, we choose the generic elements so that $\bar{\Delta}_j^{(i)} = \Delta(x^{(r)}, x^{(l)})$ and all g_k, $k \neq i$ do not depend on $x^{(r)}, x^{(l)}$, therefore, $\mathrm{D}_{x^{(r)}}(a), \mathrm{D}_{x^{(l)}}(a)$ kill g_k. Hence, f can be written in the form $g\Delta_1 \dots \Delta_t$, where $\Delta_1 \dots \Delta_t$ is the product of all $\bar{\Delta}_j^{(i)}$ and $g \in \mathrm{tr}[\mathbf{A}]$ is multilinear in all generic elements whose the operators depend on. Recall that $\Delta(x,y)$ commutes with all Δ_i, hence, by (18) $f\Delta(x,y)$ can be written in the same form (we linearize g and attach $\Delta(x', y')$). Finally, since S generates $\mathrm{tr}[\mathbf{A}]$, we can replace g with a monomial in the generators, hence it suffices to show that the following element belong to A:

$$(\prod_{i=1}^q h_i)\Delta_1 \dots \Delta_d \tag{19}$$

where $h_i \in S$, $\Delta_i = \Delta(x^{(i)}, y^{(i)})$, all $x^{(i)}, y^{(j)}$ ($i \in \{1, \dots, d\}$) are different and $\prod_{i=1}^q h_i$ is linear in every $x^{(i)}, y^{(i)}$.

We carry out the induction on d; the case $d = 0$ is trivial. Now, suppose that there are $i \in \{1, \dots, d\}$, $s, p \in \{1, \dots, q\}$ such that and h_s (h_p) depends on $x^{(i)}$ (respectively, $y^{(i)}$). Since Δ-s commute, we may assume $i = 1$; besides, by [9, p.227, cor. 4], $\mathrm{tr}((ab)c - (ba)c)) = 0$, hence, $h_s = \mathrm{tr}(w_s x^{(1)})$, $h_p = \mathrm{tr}(w_p y^{(1)})$ for some

$$w_s, w_p \in \{\sum_j u_j t_j |\ u_j \in \mathcal{F}(\mathbf{A}) \cup \{1\};\ t_j \in \mathrm{tr}[\mathbf{A}]\}$$

and using (10), we reduce d. If such i, s, p do not exist, Δ-s are distributed on multiples h_i, thus (19) becomes a product of elements (4). $\square$

Proof of theorem 1.

By lemma 7, $S \subseteq F[\vec{\mathbf{A}}]^G$; hence, by lemma 5 we need to show only $F[\vec{\mathbf{A}}]^G \subseteq \mathrm{tr}[\mathbf{A}]^\circ$.

Pick a homogeneous $f \in F[\vec{\mathbf{A}}]^G$ depending on k vector variables, i.e., $f \in F[k\mathbf{A}]^G$; by corollary 3 there are $p, q \in \mathbf{N}$ such that $f\Omega^p R^q \in \mathrm{tr}[\mathbf{A}]$. Put $\Delta_1 = \Delta(x^{(k+1)}, x^{(k+1)})$, $\Delta_2 = \Delta(x^{(k+2)}, x^{(k+2)})$; recall that Ω is equal to a fraction g/h, where $h = R^n$ depends on $x^{(k+2)}$, hence, $\Omega^p\Delta_1$ is defined (it is equal to $(g^p\Delta_1)/h^p$). Moreover, since the mapping θ_0 does not change the first $k+1$ generic elements of $(k+2)\mathbf{A}$, Δ_1 is defined on $F[W_0]$ and commutes with θ_0.

We may assume that p, q are even; in the next section, proposition 6, we show that $\omega^p\Delta_1^{3p}$ is a non-zero constant c. Hence,

$$f\Omega^p R^q \Delta_1^{3p} = f(\Omega^p\Delta_1^{3p})R^q = f\theta_0(\omega^p\Delta_1^{3p})R^q = c(fR^q)\ ,$$

therefore, $fR^q \in \mathrm{tr}[A]^\circ$. A similar fact about R and Δ_2 from proposition 7 enables us to reduce R^q and prove theorem 1.

5. Cancellation of ω and R

Recall that E_p, $p = 1, \dots, 27$, is an orthonormal basis of $\mathbf{A}$ with respect to the bilinear form $\mathrm{tr}(xy)$. Moreover, by (14), the Pierce components are pairwise orthogonal, hence, we may find such E_p in $\mathbf{A}_{ij}$; in particular, $E_p = e_p$ for $p = 1, 2, 3$.

Given $a \in \mathbf{A}$ and $(ijk) \in A_3$, denote by $(a)_k$ the element c $\in \mathbf{O}$ such that $[c]_{ij} = aU_{e_i,e_j}$; it is a function from $\mathbf{A}$ to $\mathbf{O}$ and in these notations,

$$(x^{(r)})_k = \mathbf{x}_k^{(r)} \tag{20}$$

for all r.

By (15) for all $a, b \in \mathbf{O}$

$$\mathrm{tr}([a]_{ij} \circ [b]_{ij}) = \mathrm{tr}(\frac{1}{2}\mathrm{t}(a\bar{b})(e_i + e_j)) = \mathrm{t}(a\bar{b}) \ . \tag{21}$$

Let x, y be generic elements of $\mathbf{A}$; by (21) and (10) for all (ijk):

$$\sum_{p=1}^{27} \mathrm{t}(\bar{\mathbf{x}}_k(E_p)_k)\mathrm{t}(\bar{\mathbf{y}}_k(E_p)_k) = \sum_{p=1}^{27} \mathrm{tr}([\mathbf{x}_k]_{ij} \circ E_p)\mathrm{tr}([\mathbf{y}_k]_{ij} \circ E_p) = \mathrm{tr}([\mathbf{x}_k]_{ij} \circ [\mathbf{y}_k]_{ij}) = \mathrm{t}(\bar{\mathbf{x}}_k\mathbf{y}_k) \ . \tag{22}$$

Proposition 6. *For all $s \in \mathbf{N}$ $\omega^{2s} \cdot \Delta_1^{6s} \neq 0$.*

Proof. Recall that $x = \theta_0(x^{(k+1)})$ and $t_i = \mathrm{t}(\mathbf{x}_i\bar{\mathbf{x}}_i)$, $t = \mathrm{t}(\mathbf{x}_1\mathbf{x}_2 \cdot \mathbf{x}_3)$.

Put $\mathrm{D}_p = \mathrm{D}_x(E_p)$; recall that $(E_p)_q \neq 0$ for some q and $(E_p)_i = 0$ for all $i \neq q$. Fix $q \in \{1, 2, 3\}$, since $\mathbf{x}_j\mathbf{x}_k = \overline{([x_k]_{ij} \circ [x_j]_{ki})_i}$, we have ($\sum_p$ denotes $\sum_{p=1}^{27}$)

$$\sum_p (t\mathrm{D}_p)(t_q\mathrm{D}_p) = \sum_p \sum_{(ijk)} \mathrm{t}((E_p)_i \cdot \mathbf{x}_j\mathbf{x}_k)2\mathrm{t}((E_p)_q\bar{\mathbf{x}}_q) = \tag{23}$$

$$= \sum_{(ijk),\ i=q} \sum_p \mathrm{t}((E_p)_i \cdot \mathbf{x}_j\mathbf{x}_k)2\mathrm{t}((E_p)_q\bar{\mathbf{x}}_q) = 2t \ ;$$

Similarly,

$$\sum_p (t_i\mathrm{D}_p)^2 = 4t_i \ ; \tag{24}$$

also, by (22) and $\mathrm{t}(\bar{a}a) = 2\bar{a}a$

$$\sum_p (t\mathrm{D}_p)^2 = \sum_p (\sum_{(ijk)} \mathrm{t}((E_p)_i \cdot \mathbf{x}_j\mathbf{x}_k))^2 = \tag{25}$$

$$= \sum_p \sum_{(ijk)} \mathrm{t}((E_p)_i \cdot \mathbf{x}_j\mathbf{x}_k)^2 = \sum_{(ijk)} \mathrm{t}(\overline{\mathbf{x}_j\mathbf{x}_k} \cdot \mathbf{x}_j\mathbf{x}_k) = \sum_{(ijk)} 1/2 \cdot t_it_j \ ;$$

$$\sum_p (t_i\mathrm{D}_p^2) = \sum_p 2\mathrm{t}((E_p)_i\overline{(E_p)_i}) = 2 \sum_{(E_p)_i \neq 0} \mathrm{t}((E_p)_i\overline{(E_p)_i}) = 16 \ . \tag{26}$$

For all $i \neq j$

$$t\mathrm{D}_p^2 = (t_i\mathrm{D}_p)(t_j\mathrm{D}_p) = 0 \ . \tag{27}$$

Hence, by (27)

$$t^m t_1^{n_1} t_2^{n_2} t_3^{n_3} \mathrm{D}_p^2 = m(m-1) t^{m-2} t_1^{n_1} t_2^{n_2} t_3^{n_3} \cdot (t\mathrm{D}_p)^2 + \tag{28}$$

$$+ \sum_{(ijk)} 2mn_i \cdot t^{m-1} t_i^{n_i-1} t_j^{n_j} t_k^{n_k} (t\mathrm{D}_p)(t_i \mathrm{D}_p) + \sum_{(ijk)} n_i \cdot t^m t_i^{n_i-1} t_j^{n_j} t_k^{n_k} (t_i \mathrm{D}_p^2) +$$

$$+ \sum_{(ijk)} n_i(n_i-1) \cdot t^m t_i^{n_i-2} t_j^{n_j} t_k^{n_k} (t_i \mathrm{D}_p)^2 \ .$$

Next, by (23)-(26)

$$t^m t_1^{n_1} t_2^{n_2} t_3^{n_3} \Delta_1 = \frac{1}{2} m(m-1) t^{m-2} \sum_{(ijk)} t_i^{n_i+1} t_j^{n_j+1} t_k^{n_k} + \tag{29}$$

$$+ \sum_{(ijk)} (4mn_i + 16n_i + 4n_i(n_i-1)) \cdot t^{m-1} t_i^{n_i-1} t_j^{n_j} t_k^{n_k} \ .$$

For $m = 2r$ we have

$$t^{2r} t_1^{n_1} t_2^{n_2} t_3^{n_3} \Delta_1 = \tag{30}$$

$$= r(2r-1) \sum_{(ijk)} t^{2(r-1)} t_i^{n_i+1} t_j^{n_j+1} t_k^{n_k} +$$

$$+ \sum_{(ijk)} 4n_i(n_i + 2r + 3) \cdot t^{2r} t_i^{n_i-1} t_j^{n_j} t_k^{n_k} \ .$$

Put

$$\gamma_r = \prod_{q=1}^{r} \frac{4(q+3)}{2q-1} \tag{31}$$

for all $r \geq 1$ and $\gamma_0 = 1$.

Lemma 8. *For all $r, n_1, n_2, n_3 \geq 0$*

$$(t_1^{n_1+r} t_2^{n_2+r} t_3^{n_3+r} - \gamma_r t^{2r} t_1^{n_1} t_2^{n_2} t_3^{n_3}) \Delta_1^{3k+n_1+n_2+n_3} = 0$$

Proof. We shall prove it by induction on r. If $r = 0$, then it is trivial; suppose that for all $s < r$ and for all $n_1, n_2, n_3 \geq 0$ it is proven , we carry out the second induction on $n = n_1 + n_2 + n_3$.

Let $n = 0$, then $n_1 = n_2 = n_3 = 0$; by (30)

$$t_1^r t_2^r t_3^r \Delta_1 = \sum_{(ijk)} 4r(r+3) t_i^{r-1} t_j^r t_k^r$$

and

$$t^{2r} \Delta_1 = \sum_{(ijk)} r(2r-1) t^{2(r-1)} t_i t_j$$

Denote by $V(d)$ the subspace of polynomials which vanish under Δ_1^d; by the induction hypothesis on r modulo $V(3r-1)$ the latter sum is equal to

$$\sum_{(ijk)} \frac{r(2r-1)}{\gamma_{r-1}} t_i^r t_j^r t_k^{r-1} = \sum_{(ijk)} \frac{4r(r+3)}{\gamma_r} t_i^{r-1} t_j^r t_k^r \ ,$$

therefore,

$$t^{2r} \Delta_1 - \gamma_r t_1^r t_2^r t_3^r \Delta_1 \in V(3r-1) \ .$$

and the statement holds for $n = 0$. Now let us show it for $n > 0$.

By (30)

$$t_1^{n_1+r}t_2^{n_2+r}t_3^{n_3+r}\Delta_1 = \sum_{(ijk)} 4(n_i+r)((n_i+r)+3)t_i^{n_i+r-1}t_j^{n_j+r}t_k^{n_k+r}$$

on the other hand, modulo $V(3r+n_1+n_2+n_3)$

$$t^{2r}t_1^{n_1}t_2^{n_2}t_3^{n_3}\Delta_1 \equiv \frac{r(2r-1)}{\gamma_{r-1}} \sum_{(ijk)} t_i^{n_i+1+(r-1)}t_j^{n_j+1+(r-1)}t_k^{n_k+(r-1)} +$$
$$+ \sum_{(ijk)} \frac{4n_i(n_i+2r+3)}{\gamma_r} t_i^{n_i-1+r}t_j^{n_j+r}t_k^{n_k+r} =$$
$$= \frac{r(2r-1)}{\gamma_{r-1}} \sum_{(ijk)} t_i^{n_i+r}t_j^{n_j+r}t_k^{n_k+r-1} +$$
$$+ \frac{1}{\gamma_r} \sum_{(ijk)} 4n_i(n_i+2r+3)t_i^{n_i+r-1}t_j^{n_j+r}t_k^{n_k+r}$$

Hence, by (31)

$$(t_1^{n_1+r}t_2^{n_2+r}t_3^{n_3+r} - \gamma_r t^{2r}t_1^{n_1}t_2^{n_2}t_3^{n_3})\Delta_1 \equiv$$
$$\equiv \sum_{(ijk)} (4r(r+3) - \frac{\gamma_r r(2r-1)}{\gamma_{r-1}})t_i^{n_i+r-1}t_j^{n_j+r}t_k^{n_k+r} = 0$$

□.

Recall that $\omega = t^2 - 1/2t_1t_2t_3$, hence,

$$\omega^{2s} = \sum_{r=0}^{2s} C_{2s}^r(-2)^{-2s+r}t^{2r}t_1^{2s-r}t_2^{2s-r}t_3^{2s-r}$$

By lemma 8, modulo $V(6s)$ it is equal to

$$2^{-2s}t_1^{2s}t_2^{2s}t_3^{2s}\sum_{r=0}^{2s} C_{2s}^r 2^r \frac{1}{\gamma_r}(-1)^{2s-r} \ ,$$

hence, by (30) it follows that $t_1^{2s}t_2^{2s}t_3^{2s}\Delta_1^{6s}$ is a positive integer, hence, to prove the proposition it suffices to show that

$$T = \sum_{r=0}^{2s} C_{2s}^r 2^r \frac{1}{\gamma_r}(-1)^{2s-r} \neq 0$$

Lemma 9.

$$T = \frac{3}{2\pi}\int_0^1\int_0^{x_3}\int_0^{x_2}\int_0^{\pi/2}(x_1 Sin^2 x - 1)^{2s}\mathrm{d}x\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \ .$$

Obviously, this yields $T > 0$.

Proof. We make use of the following well-known formulae

$$\int_0^1\int_0^{x_3}\int_0^{x_2} x_1^r\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = \frac{1}{(r+3)(r+2)(r+1)} \ ;$$
$$\int_0^{\pi/2} Sin^{2r}(x)\mathrm{d}x = \frac{\pi}{2}\prod_{q=1}^{r}\frac{2q-1}{2q} \ .$$

This gives

$$T = \sum_{r=0}^{2s} C_{2s}^r \frac{2^r}{\gamma_r}(-1)^{2s-r} =$$

$$= \sum_{r=0}^{2s} C_{2s}^r 2^r \prod_{q=1}^{r} \frac{2q-1}{4(q+3)}(-1)^{2s-r} = \sum_{r=0}^{2s} C_{2s}^r \prod_{q=1}^{r} \frac{2q-1}{2(q+3)}(-1)^{2s-r}$$

$$= \sum_{r=0}^{2s} C_{2s}^r \frac{3!}{2^3(r+3)(r+2)(r+1)} \prod_{q=1}^{r} \frac{2q-1}{2q}(-1)^{2s-r} =$$

$$= \sum_{r=0}^{2s} C_{2s}^r (-1)^{2s-r} \frac{3}{2\pi} \int_0^1 \int_0^{x_3} \int_0^{x_2} \int_0^{\pi/2} x_1^r Sin^{2r}(x) \mathrm{d}x \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 =$$

$$= \frac{3}{2\pi} \int_0^1 \int_0^{x_3} \int_0^{x_2} \int_0^{\pi/2} \left(\sum_{r=0}^{2s} C_{2s}^r (-1)^{2s-r} x_1^r Sin^{2r}(x) \right) \mathrm{d}x \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 =$$

$$= \frac{3}{2\pi} \int_0^1 \int_0^{x_3} \int_0^{x_2} \int_0^{\pi/2} (x_1 Sin^2 x - 1)^{2s} \mathrm{d}x \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \ .$$

□.

Now let us prove an analog of proposition 6 for R.

Proposition 7. *For all* $s \in \mathbf{N}$ $R^{2s} \cdot \Delta_2^{6s} \neq 0$.

Denote $y = x^{(k+2)}$, then $R(y)\mathrm{D}_y(1) = 0$. Indeed, if y is diagonal matrix $\sum_{i=1} \beta_i e_i$, then $R(y) = \det(\beta_i^j|_{i=1,j=0}^{3,2})$ and one may check it easily. On the other hand, $R(y)\mathrm{D}_y(1)$ is a G-invariant, hence it is equal to 0 for all $y \in \mathbf{A}$.

Put $z = y - 1/3\mathrm{tr}(y)$, then $\mathrm{tr}(z) = 0$ and $R = R(y) = R(z) = \alpha a^3 + \beta b^2$, where $a = \mathrm{tr}(z^2)$, $b = \mathrm{tr}(z^3)$, because $R(y)$ is a polynomial in $\mathrm{tr}(y), \mathrm{tr}(y^2), \mathrm{tr}(y^3)$ of degree 6 with respect to y. Substituting $z = 2e_2 - e_1 - e_3$ and $z = e_1 - e_2$ we get

$$R(z) = 1/2a^3 - 3b^2 \ .$$

We fix an orthonormal basis G_p, $p = 1, \dots, 27$, of $\mathbf{A}$ where $G_1 = 1/\sqrt{3}$ and $\mathrm{tr}(G_p) = 0$ for all $p \geq 2$.

By (5) for any polynomial $f = f(z)$

$$f(z)\Delta_2 = f(z)\Delta(z,z) \ .$$

Now $\sum_p$ denotes $\sum_{p=2}^{27}$; for all $u, v \in \mathbf{A}'$ $\mathrm{tr}(uG_1) = \mathrm{tr}(vG_1) = 0$, hence, by (10) we have

$$\sum_p \mathrm{tr}(uG_p)\mathrm{tr}(vG_p) = \sum_{p=1}^{27} \mathrm{tr}(uG_p)\mathrm{tr}(vG_p) = \mathrm{tr}(uv) \ . \tag{32}$$

Put $\mathrm{D}_p = D_y(G_p)$, then

$$\sum_p (a\mathrm{D}_p)^2 = 4\sum_p \mathrm{tr}(zG_p)^2 = 4\mathrm{tr}(z^2) = 4a \ . \tag{33}$$

Next,

$$\sum_p a\mathrm{D}_p^2 = 2\sum_p \mathrm{tr}(G_p^2) = 52 \ . \tag{34}$$

Also,

$$\sum_p (a\mathrm{D}_p)(b\mathrm{D}_p) = \sum_p 2\mathrm{tr}(zG_p)\cdot 3\mathrm{tr}(z^2G_p) = 6b \tag{35}$$

and by (2)

$$\sum_p (b\mathrm{D}_p)^2 = 9\sum_p \mathrm{tr}(z^2G_p)^2 = 9\mathrm{tr}(z^4) = \tag{36}$$

$$= 9\mathrm{tr}(\frac{1}{2}\mathrm{tr}(z^2)z^2 + \frac{1}{3}\mathrm{tr}(z^3)z) = 9/2a^2\ .$$

Besides, since $\mathrm{tr}(z) = 0$ and $G_1^2 = 1/3$, we have (cf. lemma 7)

$$\sum_p b\mathrm{D}_p^2 = 6\sum_p \mathrm{tr}(zG_p^2) = 6\sum_{p=1}^{27}\mathrm{tr}(zG_p^2) = 6\mathrm{Tr}_{\mathbf{A}}(R_z) = 0 \tag{37}$$

because $\mathrm{Tr}_{\mathbf{A}}(R_x) = 9\mathrm{tr}(x)$ for all $x \in \mathbf{A}$.

Using the previous formulae, we get

$$a^m b^n \mathrm{D}_p^2 = ma^{m-1}b^n(a\mathrm{D}_p^2) + m(m-1)a^{m-2}b^n(a\mathrm{D}_p)^2 +$$

$$+2mna^{m-1}b^{n-1}(a\mathrm{D}_p)(b\mathrm{D}_p) + na^m b^{n-1}(b\mathrm{D}_p^2) + n(n-1)a^m b^{n-2}(b\mathrm{D}_p)^2$$

hence,

$$a^m b^n \Delta_2 = 52ma^{m-1}b^n + 4m(m-1)a^{m-1}b^n + \tag{38}$$

$$+12mna^{m-1}b^n + 0 + 9/2n(n-1)a^{m+2}b^{n-2} =$$

$$a^m b^n \Delta_2 = 4m(m+3n+12)a^{m-1}b^n + 9/2n(n-1)a^{m+2}b^{n-2}\ .$$

Put

$$\gamma_r = \prod_{q=1}^{r}\frac{4(q+4)}{2q-1}$$

for all $r \geq 1$ and $\gamma_0 = 1$; using the induction on r, one may check that for all $r, m \geq 0$

$$(a^{m+3r} - \gamma_r a^m b^{2r})\Delta^{2m+3r} = 0\ .$$

This gives

$$R^{2s} = 2^{-2s}\sum_{r=0}^{2s} C_{2s}^r 6^r b^{2r} a^{3(2s-r)}(-1)^{2s-r} \equiv$$

$$2^{-2s}(\sum_{r=0}^{2s} C_{2s}^r \frac{6^r}{\gamma_r}(-1)^{2s-r})a^{6s}$$

modulo $V(6s) = \{f|\ f\Delta_2^{6s} = 0\}$. By (38), $a^{6s} \notin V(6s)$, hence, the next lemma (the proof is similar to lemma 9 and we leave it as an exercise) yields proposition 7 and, therefore, completes the proof of theorem 1.

Lemma 10.

$$(\sum_{r=0}^{2s} C_{2s}^r \frac{6^r}{\gamma_r}(-1)^{2s-r}) =$$

$$= \frac{3}{4\pi}\int_0^1\int_0^{x_4}\int_0^{x_3}\int_0^{x_2}\int_0^{\pi/2}(3x_1 Sin^2 x - 1)^{2s}\mathrm{d}x\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\ .$$

6. Example

In this section we show that Laplace operators do extend the algebra $\mathrm{tr}[\mathbf{A}]$.

Proposition 8. $\mathrm{tr}[\mathbf{A}] \neq \mathrm{tr}[\mathbf{A}]^{\circ}$.

Proof. First of all, observe that $\mathrm{tr}[\mathbf{A}]$ has no multilinear polynomials of degree 5 which are skew-symmetric in 4 variables.

Actually, let $u = u(x_1, \dots, x_n) \in \mathcal{F}(\mathbf{A})$ be a multilinear monomial; since $\mathbf{A}$ is commutative, the alternation of u in $x_1, \dots, x_{n-1}$ might be nonzero only if (up to the enumeration of the first $n-1$ variables)

$$u = x_n R_{x_1} \dots R_{x_{n-1}} \ .$$

Suppose that $n \geq 3$, put $v = x_n R_{x_1} \dots R_{x_{n-3}}$, then $\mathrm{tr}(v x_{n-2} \cdot x_{n-1}) - \mathrm{tr}(v x_{n-1} \cdot x_{n-2}) = 0$ in $\mathbf{A}$ (cf. [9, p.229]). Hence, the alternation of $\mathrm{tr}(u)$ in any $n-1$ variables makes it equal to 0.

Next, any polynomial from $\mathrm{tr}[\mathbf{A}]$ of degree 5 is a linear combination of polynomials

$$f = \mathrm{tr}(u_1) \dots \mathrm{tr}(u_t)$$

where $t \leq 5$. Consider the alternation of any 4 variables in f; if $t \geq 4$, then 3 of u-s are variables, hence f vanishes. Otherwise, there is u_i with $\deg(u_i) \geq 3$, hence f vanishes too, or there are u_i, u_j with $deg(u_i), deg(u_j) \geq 2$, hence, f also becomes zero under the alternation.

Now we shall exhibit a multilinear polynomial $f \in \mathrm{tr}[\mathbf{A}]^{\circ}$ of degree 5 which is skew-symmetric in 4 variables.

Let $\mathrm{Tr}_{\mathbf{A}}(\Phi)$ denote the trace of a linear operator $\Phi \in \mathrm{End}_F \mathbf{A}$. Put

$$Q = \sum_{\sigma \in S_4} \mathrm{sgn}(\sigma) R_{x_{\sigma(1)}} R_{x_{\sigma(2)}} R_{x_{\sigma(3)}} R_{x_{\sigma(4)}} R_{x_5} \ ,$$

then $\mathrm{Tr}_{\mathbf{A}}(Q) = f\Delta(x, y)$, where $f = \mathrm{tr}(yQ \cdot x)$. Thus, we only need to show that $\mathrm{Tr}_{\mathbf{A}}(Q) \neq 0$. Note that invariants of this type are not new (cf.[4, 11]).

Put

$$x_1 = e_{12} \ , x_2 = e_{23}, \ x_3 = e_{31}, \ x_4 = e_1 = e_{11}, \ x_5 = e_2 = e_{22} \ ; \tag{39}$$

all these elements belong to $F_3^{(+)}$. Consider a basis of $\mathbf{A}$ consisting of e_i, ϵ_{ij} (a basis of $F_3^{(+)}$) and $\begin{bmatrix} 0 & s_r \\ 0 & 0 \end{bmatrix}_{jk}$, $\begin{bmatrix} 0 & 0 \\ s_r & 0 \end{bmatrix}_{jk}$, where s_1, s_2, s_3 is a basis of the 3-dimensional vector space. Then for any element a of the basis, the subspace $F_3^{(+)} + Fa$ lies in an associative subalgebra of $\mathbf{O}_3 = F_3 \otimes_F \mathbf{O}$; it follows by the definition of multiplication (in fact, this subalgebra is isomorphic to $\mathbf{Q}_3$, where $\mathbf{Q}$ is the algebra of quaternions).

Let λ_x (ρ_x) denote the operator of left (right) multiplication in $\mathbf{O}_3$, then the operator $R_x = 1/2(\rho_x + \lambda_x)$ and we get the following (as usual, (ijk) runs over A_3 and $[a,b] = ab - ba$):

$$Q = 2 \sum_{(ijk)} [R_{x_i}, R_{x_j}] \circ [R_{x_k}, R_{x_4}] R_{x_5} =$$

$$= 1/16 \sum_{(ijk)} [\rho_{x_i} + \lambda_{x_i}, \rho_{x_j} + \lambda_{x_j}] \circ [\rho_{x_k} + \lambda_{x_k}, \rho_{x_4} + \lambda_{x_4}](\rho_{x_5} + \lambda_{x_5}) \ .$$

The associativity law can be written in the operator form as $\rho_x\rho_y = \rho_{xy}$, $\lambda_x\lambda_y = \lambda_{yx}$, $[\rho_x, \lambda_y] = 0$, hence

$$16 \cdot Q = \sum_{(ijk)} ([\rho_{x_i}, \rho_{x_j}] + [\lambda_{x_i}, \lambda_{x_j}]) \circ ([\rho_{x_k}, \rho_{x_4}] + [\lambda_{x_k}, \lambda_{x_4}])(\rho_{x_5} + \lambda_{x_5})$$

$$= \sum_{(ijk)} \rho_{[x_i,x_j]\circ[x_k,x_4]x_5} + \rho_{[x_i,x_j]\circ[x_k,x_4]}\lambda_{x_5} + \lambda_{x_5[x_i,x_j]\circ[x_k,x_4]} + \lambda_{[x_i,x_j]\circ[x_k,x_4]}\rho_{x_5} +$$

$$-\rho_{[x_i,x_j]}\lambda_{x_5[x_k,x_4]} - \rho_{[x_i,x_j]x_5}\lambda_{[x_k,x_4]} - \lambda_{[x_i,x_j]}\rho_{[x_k,x_4]x_5} - \lambda_{x_5[x_i,x_j]}\rho_{[x_k,x_4]} .$$

Using 39) and the multiplication table for matrix units, we get

$$16Q = \frac{1}{2}(-\rho_{e_2} - \lambda_{e_2} + \rho_{e_3}\lambda_{e_2} + \lambda_{e_3}\rho_{e_2} - 2\lambda_{e_2}\rho_{e_2} + 2\rho_{e_{12}}\lambda_{e_{21}}) .$$

Observe that the vector space $\mathbf{A}$ is closed under those operators and $\mathrm{Tr}_{\mathbf{A}}(\rho_{e_i}) = \mathrm{Tr}_{\mathbf{A}}(\lambda_{e_i}) = 9$, $\mathrm{Tr}_{\mathbf{A}}(\rho_{e_i}\lambda_{e_j}) = 4$, $\mathrm{Tr}_{\mathbf{A}}(\rho_{e_i}\lambda_{e_i}) = 1$ for all $i \neq j$ and $\mathrm{Tr}_{\mathbf{A}}(\rho_{e_{ij}}\lambda_{e_{pq}}) = 0$ if $i \neq j$ or $p \neq q$. Therefore,

$$\mathrm{Tr}_{\mathbf{A}}(16Q) = -6.$$

The proposition is proven. □

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