ON THE REGULARITY OF THE FINITE REFLECTION GROUPS

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Abstract. We show that an irreducible finite reflection group G of rank greater than 1 is regular if and only if the associated root system is perfect. This holds if and only if G is a Coxeter group. Hence we get a characterization of Coxeter groups among all the irreducible finite reflection groups of rank greater than 1.

The irreducible finite unitary reflection groups were classified by Shephard and Todd ([16]) and by Cohen ([4]). They have been studied extensively by many people since then (see [2; 3; 10; 11; 12; 13; 14; 15; 17]). Finite Coxeter groups are a special family of reflection groups, whose properties are relatively well known. It is interesting to ask what properties of Coxeter groups are shared also by the other reflection groups, and what are not. In the present paper, we shall consider a property, which we call *regularity*, of a reflection group, defined by the existence of a *basic section* of the associated root system R (see 3.1). We shall show that an irreducible finite reflection group G of rank > 1 is regular if and only if all the root line circles in R are perfect (see 2.3 for the definitions). Then we further show that this holds if

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and only if G is a Coxeter group. Thus we get a characterization of Coxeter groups among all the finite reflection groups.

Let G be a finite reflection group which acts irreducibly on a hermitian space V, and let S be a minimal set of reflections which generates G. Our strategy for proving our main result is as follows. We show in Lemma 1.7 that the order $|Z(G)|$ of the center $Z(G)$ of G divides the cardinality of any root line in R. We further show in Lemma 4.2 (2) that if G is regular then $|Z(G)|$ divides the orders $o(s)$ of all $s \in S$. By the classification of the irreducible finite complex reflection groups, there are only a few irreducible finite complex reflection groups G satisfying this property (see Proposition 4.3). On the other hand, in each of these exceptional irreducible complex reflection groups G, we can find at least one root line circle which is imperfect (see 4.5-4.11). This proves our main result.

The contents of the paper are organized as follows. In Section 1 we collect some definitions and results concerning irreducible finite reflection groups G which are either well known or easily proved. Then in Section 2 we introduce the concept of perfectness for a root line circle, a root system and a reflection group. Lemma 2.6, concerned with perfectness, is crucial in the proof of our main result. Regularity is introduced in Section 3, where we establish the equivalence of regularity of G and perfectness of its associated root system R (see Theorem 3.4). Our main result, Theorem 4.12, is proved in Section 4.

§1. Roots and reflections.

We collect some definitions and results concerning irreducible finite reflection groups; many of them follow from Cohen's paper [4].

1.1. Let V be a complex vector space of dimension n. A reflection in V is a linear transformation of V of finite order with exactly $n-1$ eigenvalues equal to 1. A reflection group G in V is a finite group generated by reflections in V. The group G is reducible if it is a direct product of two proper reflection subgroups and irreducible otherwise. The action of G on V is said irreducible if V has no nonzero proper G-invariant subspace. In the present paper we shall always assume that G is irreducible and acts irreducibly on V. A reflection group G in V is called a real group or a Coxeter group if there is a G-invariant $\mathbb{R}$-subspace V_0 of V such that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} V_0 \to V$ is bijective. If this is not the case, G will be called complex. (Note that, according to this definition, a real reflection group is not complex.)

There exists a unitary inner product $(\ ,\ )$ on V invariant under G; hence we may assume that G is a subgroup of $U(V)$, the group of all unitary transformations with respect to the unitary inner product. From now on we will make this assumption.

1.2. A root of a reflection in V is an eigenvector corresponding to the unique nontrivial eigenvalue of the reflection. A root of G is a root of a reflection in G.

Let s be a reflection in V of order $d > 1$. There is a vector $a \in V$ of length 1 and a primitive d-th root ζ of unity such that $s = s_{a,\zeta}$, where $s_{a,\zeta}$ is defined by

$$s_{a,\zeta}(v) = v + (\zeta - 1)(v, a)a \tag{1.2.1}$$

for all $v \in V$. We also write $s_{a,d}$ for $s_{a,\zeta}$ if $\zeta = e^{2\pi i/d}$.

We use the notation $|x|$ for the cardinality of x if x is a set, and for the absolute value of x if x is a complex number. The meaning will always be clear from the context.

For each $v \in V$ define $o_G(v)$ to be the order of the (necessarily cyclic) group consisting of the identity and the reflections in G which have v as a root. (This group is $G_W = \{\, g \in G \mid gu = u \text{ for all } u \in W \,\}$, where $W = v^{\perp}$.)

Thus $o_G(v) > 1$ if and only if v is a root of G. If a is a root of G, then the number $o_G(a)$ will be called the order of a (with respect to G). We shall denote $o_G(a)$ simply by $o(a)$ when G is clear from the context.

Note that we shall also use the notation $o(\zeta)$ for the order of ζ, where ζ could be either a root of unity, or a group element. This should cause no confusion.

Lemma 1.3. *$o(gv) = o(v) = o(cv)$ for all $v \in V$, $g \in G$ and $c \in \mathbb{C}^*$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.*

Proof. Let $W = v^\perp$. Then we have

$$u \in W \iff (u, v) = 0 \iff (gu, gv) = 0 \iff gu \in (gv)^\perp.$$

This implies $gW = (gv)^\perp$. Now we have

$$\begin{aligned} h \in G_{gW} &\iff h(gu) = gu \quad \text{ for all } u \in W, \\ &\iff (g^{-1}hg)u = u \quad \text{ for all } u \in W, \\ &\iff g^{-1}hg \in G_W, \\ &\iff h \in {}^gG_W. \end{aligned}$$

So we get $G_{gW} = {}^gG_W$ and hence $o(gv) = |G_{gW}| = |{}^gG_W| = |G_W| = o(v)$. The equation $o(v) = o(cv)$ follows from the fact that $v^\perp = (cv)^\perp$. $\square$

1.4. A pair (R, f) is called a root system in V, if

(i) R is a finite set of vectors of V of length 1;

(ii) $f : R \to \mathbb{N}\backslash\{1\}$ is a map such that $s_{a,f(a)}R = R$ and $f(s_{a,f(a)}(b)) = f(b)$ for all a, $b \in R$;

(iii) the group G generated by $\{\, s_{a,f(a)} \mid a \in R \,\}$ is a (finite) reflection group, and for all $a \in R$ and $c \in \mathbb{C}$,

$$ca \in R \iff ca \in Ga.$$

The group G is called the reflection group associated with the root system (R, f). We have $o_G(a) = f(a)$ for any $a \in R$.

We shall denote a root system (R, f) simply by R when f is clear from the context.

1.5. A system of simple roots is a pair (B, w), where B is a finite set of vectors of V and w is a map from B to $\mathbb{N} \setminus \{1\}$, satisfying the following conditions:

(i) for all $a, b \in B$, we have $|(a, b)| = 1 \iff a = b$;

(ii) the group G generated by $S = \{\, s_{a,w(a)} \mid a \in B \,\}$ is finite;

(iii) there is a root system (R, f) with $R = GB$ and $f(a) = w(a)$ for all $a \in B$;

(iv) for any $B' \subsetneqq B$, the group G' generated by $\{\, s_{a,w(a)} \mid a \in B' \,\}$ is a proper subgroup of G.

We call the elements of S simple reflections, and the subgroups of G generated by the subsets of S standard parabolic subgroups of G (relative to S). We also call (R, f) the root system of G generated by B, and B a simple system for R.

Note that we do not require B to be linearly independent. If B is linearly independent, then condition (iv) holds automatically.

The above definition of simple system is considerably weaker the usual definition for Coxeter groups; in particular, it is not true that whenever B_1 and B_2 are simple systems for the same root system R then there is an element $g \in G$ with $gB_1 = B_2$.

It is easily seen that G acts irreducibly on V if and only if the root system R (resp. the simple root system B) spans V and cannot be decomposed into a disjoint union of two proper subsets R_1 and R_2 (resp. B_1 and B_2) with $R_1 \perp R_2$ (resp. $B_1 \perp B_2$).

By Lemma 1.3 we see that if $\alpha \in B$ and $\beta = \zeta\alpha \in R$ for some $\zeta \in \mathbb{C}^*$, then $B' = (B \setminus \{\alpha\}) \cup \{\beta\}$ also forms a simple root system for R.

1.6. Let R be a root system with G the associated reflection group. For any $\alpha \in R$, let $R_\alpha = \{\, \gamma \in R \mid \gamma \in \mathbb{C}\alpha \,\}$, called a root line in R. By Property (iii) of a root system, we see that a root line of R is contained in a single G-orbit, and that the action of G on R induces a permutation on $\overline{R}$, the set of all root lines of R. It is easily seen that if $\alpha, \beta \in R$ then $s_\beta(R_\alpha) \neq R_\alpha$ unless $\beta \in R_\alpha$ or $(\beta, \alpha) = 0$.

We have the following result concerning the root lines.

Lemma 1.7. *Let R be a root system with G the associated reflection group. Then the order of the center $Z(G)$ of G divides $|R_\alpha|$ for any $\alpha \in R$.*

Proof. Since G acts irreducibly on V, it follows by Schur's Lemma that elements of $Z(G)$ act as scalar multipliers. Hence R_α is a union of some $Z(G)$-orbits, each of which has the same cardinality $|Z(G)|$. So the result follows. □

1.8. In the subsequent sections, we always assume that G is an irreducible finite reflection group of rank greater than 1. Let R be the root system associated with G and B a simple system for R. Let $S = \{\, s_\alpha \mid \alpha \in B \,\}$ be the set of the simple reflections associated with the simple system B; that is

$$s_\alpha(v) = v + (e^{2\pi i/d_\alpha} - 1)(v, \alpha)\alpha, \quad (\text{for all } v \in V),$$

where d_α is the order of α (and also of s_α).

Let $\overline{R}$ be the set of all root lines of R.

§2. Perfectness.

2.1. A sequence of root lines $\xi : l_1, l_2, \ldots, l_r$ $(r > 1)$ in $\overline{R}$ is called a circle if the following conditions hold.

(i) $l_i \neq l_{i+1}$ for all $i = 1, 2, \ldots, r$, with the convention that $l_{r+1} = l_1$.

(ii) For any h with $1 \leqslant h \leqslant r$, there exists some $s_h \in S$ satisfying $s_h(l_h) = l_{h+1}$.

2.2. Given two distinct root lines l and l' such that $l' = s(l)$ for some $s \in S$, it is possible that there exists an $s' \in S$ such that $s'(l) = l'$ and $s' \neq s$. (This happens, for example, for the group G_7 of Shephard and Todd [16]). Thus the sequence of simple reflections $s_1, s_2, \ldots, s_r$ associated with a root line circle ξ need not be unique.

2.3. We say that a root line circle $\xi : l_1, l_2, \ldots, l_r$ in $\overline{R}$ is perfect (with respect to S), if for any sequence of simple reflections $s_1, \ldots, s_r$ satisfying $s_h(l_h) = l_{h+1}$ for all h, we have $s_r s_{r-1} \cdots s_1(\alpha) = \alpha$ for each $\alpha \in l_1$. Otherwise we say that the root line circle is imperfect.

Remark 2.4. (1) If one root α in l_1 satisfies the required condition that $s_r s_{r-1} \cdots s_1(\alpha) = \alpha$, then so do all others; so when checking perfectness we need only consider a fixed $\alpha \in l_1$. However, we do need to consider every possible sequence of simple reflections $s_1, s_2, \ldots, s_r$ associated with the root line circle, since, as explained in 2.2 above, it need not be unique.

(2) Perfectness is defined relative to a set S of simple reflections for G. In the sequel, S is always fixed, and so we shall not mention the dependence on S explicitly.

2.5. Let $R = \bigcup_{i \in I} R_i$ be the decomposition of R into a disjoint union of G-orbits R_i, and $\overline{R} = \bigcup_{i \in I} \overline{R}_i$ the corresponding decomposition of $\overline{R}$. We say that R and $\overline{R}$ (resp. R_i and $\overline{R}_i$) are perfect if all the root line circles in $\overline{R}$ (resp. $\overline{R}_i$) are perfect (relative to S). We say that G is perfect if its associated root system R is perfect.

Lemma 2.6. *Let $\overline{R}_i$ be a perfect G-orbit in $\overline{R}$. Suppose that we have two root line sequences $\xi : l_1, \ldots, l_t$ and $\eta : l'_1, \ldots, l'_r$ in $\overline{R}_i$ with $l_1 = l'_1$ and*

$l_t = l'_r$, and two simple reflection sequences $s_1, \ldots, s_{t-1}$ and $s'_1, \ldots, s'_{r-1}$ such that $s_h(l_h) = l_{h+1} \neq l_h$ and $s'_k(l'_k) = l'_{k+1} \neq l'_k$ for all h and k (with $1 \leqslant h < t$ and $1 \leqslant k < r$). Take any root $\alpha \in l_1$. Then the equation

$$s_{t-1}s_{t-2}\cdots s_2 s_1(\alpha) = s'_{r-1}s'_{r-2}\cdots s'_2 s'_1(\alpha)$$

is satisfied.

Proof. Let $\delta = s_{t-1}s_{t-2}\cdots s_2 s_1(\alpha)$ and $\delta' = s'_{r-1}s'_{r-2}\cdots s'_2 s'_1(\alpha)$, and suppose that the result false. Then $\delta' = c\delta$ for some complex number $c \neq 1$. For all h and k with $1 \leqslant h \leqslant t$ and $1 \leqslant k \leqslant r$, put $\alpha_h = s_{h-1}s_{h-2}\cdots s_1(\alpha)$ and $\beta_k = s'_{k-1}s'_{k-2}\cdots s'_1(\alpha)$. Thus $\alpha_t = \delta$ and $\beta_r = \delta'$, and $\alpha_1 = \beta_1 = \alpha$.

Define a new sequence of roots $\zeta' : \mu_1 = \delta, \mu_2, \ldots, \mu_p = \alpha$ from the sequence $\zeta : \alpha_1 = \alpha, \alpha_2, \ldots, \alpha_t = \delta$ as follows. Start by reversing the order of ζ to get the sequence $\zeta^- : \alpha_t, \ldots, \alpha_2, \alpha_1$; then, for each h $(1 \leqslant h < t)$ such that the order m_h of s_h is greater than 2, insert the sequence of roots

$$s_h(\alpha_{h+1}),\ s_h^2\alpha_{h+1}),\ \ldots,\ s_h^{m_h-2}(\alpha_{h+1})$$

between α_{h+1} and α_h in ζ^-. This gives the desired sequence ζ'. Now let τ be the root line sequence $l_1, l_2, \ldots, l_{p+r-1}$ such that l_h contains μ_h for $1 \leqslant h \leqslant p$, and l_{p+k} contains β_k for $1 \leqslant k < r$. Then τ is an imperfect circle, contradicting our assumption. $\square$

§3. Basic sections of root systems.

As before, let B be a simple system for a root system R, let G be the associated reflection group and $S = \{s_\alpha \mid \alpha \in B\}$ the set of simple reflections.

3.1. A subset $R' \subset R$ is called a section of R if it contains a unique root in each root line $l \in \overline{R}$. A section R' of R is basic if, for every $\beta \in B$, the simple reflection s_β permutes the roots in the set $R' \setminus R_\beta$. Equivalently, the section R' is basic if $s(\gamma) \in R'$ whenever $\gamma \in R'$ and s is a simple reflection such that $s(R_\gamma) \neq R_\gamma$.

Basicness, like perfectness, is defined relative to the set S, which we shall consider to be fixed.

A positive root system of a Coxeter group forms a basic section of its root system. Also, any root of a reflection group of rank 1 forms a basic section.

3.2. In general, a root system always contains a section, but need not contain a basic section. A finite reflection group G is called regular, relative to a set S of simple reflections, if its root system R contains a basic section.

The main goal of the present paper is to show that an irreducible finite reflection group G is regular if and only if it is either a Coxeter group or a group of rank 1.

By Lemma 2.6, we see that if all the root line circles of R are perfect then we can define a section R' of R in the following way. Let $R = \bigcup_{i\in I} R_i$ be the decomposition of R into a disjoint union of G-orbits R_i, and $\overline{R} = \bigcup_{i\in I} \overline{R}_i$ the corresponding decomposition of the root line set $\overline{R}$. For any $i \in I$, fix a root α in R_i, and let $l = R_\alpha \in \overline{R}_i$ be the root line containing α. For any root line $l' \in \overline{R}_i$ there exists a sequence $\xi : l_1 = l, l_2, \dots , l_r = l'$ in $\overline{R}_i$ such that $l_h \neq l_{h+1}$ and $s_h(l_h) = l_{h+1}$ for $1 \leqslant h < r$ and some $s_h \in S$. Define a root $\gamma \in l'$ by $\gamma = s_{r-1}s_{r-2}\cdots s_1(\alpha)$. By Lemma 2.6, γ depends only on α and not on the choice of ξ; so each l' contains a unique such γ. Let $R^{(i)}$ be the set of all roots γ obtained from α in this way for $l' \in \overline{R}_i$. Then the set $R' = \bigcup_{i\in I} R^{(i)}$ is a section of R.

Lemma 3.3. *The section R' defined above is basic.*

Proof. Let $\beta \in B$ and $\gamma \in R' \setminus R_\beta$. It suffices to prove that $s_\beta(\gamma) \in R'$. We may assume that $(\beta, \gamma) \neq 0$, since otherwise $s_\beta(\gamma) = \gamma \in R'$.

Let R_i be the G-orbit of R in which γ lies, and let α be the fixed root in R_i used in defining R'. Then by the definition of R' there exist root lines $l_1 = R_\alpha,\ l_2,\ \ldots,\ l_r = R_\gamma$ and simple reflections $s_1,\ \ldots,\ s_{r-1}$ such that $l_h \neq l_{h+1}$ and $s_h(l_h) = l_{h+1}$ for $1 \leqslant h < r$, and $s_{r-1}\cdots s_1(\alpha) = \gamma$. Now as $\beta \notin \mathbb{C}\gamma$ and $(\beta, \gamma) \neq 0$, it follows that $s_\beta(l_r) = s_\beta(R_\gamma) \neq R_\gamma$, and so if we define $l_{r+1} = s_\beta(l_r)$ then the conditions $l_h \neq l_{h+1}$ and $s_h(l_h) = l_{h+1}$ hold for $1 \leqslant h < r+1$. Hence $s_\beta(\gamma) = s_r s_{r-1}\cdots s_1(\alpha)$ is in R', as required. $\square$

Lemma 3.3 immediately yields the "if" part of the following Theorem.

Theorem 3.4. *An irreducible finite reflection group G is regular if and only if all the root line circles in $\overline{R}$ are perfect.*

Proof. By Lemma 3.3, it suffices to show the implication $\Longrightarrow$. Let R' be a basic section of R. Let $\xi : l_1,\ \ldots,\ l_r$ be a root line circle of $\overline{R}$ and $s_1,\ \ldots,\ s_r$ simple reflections with $s_h(l_h) = l_{h+1}$ for $1 \leqslant h \leqslant r$. Choose a root $\alpha_1 \in l_1$. By Remark 2.4 (1) we can assume without loss of generality that $\alpha_1 \in R'$. Proceeding inductively, define $\alpha_{j+1} = s_j(\alpha_j)$ for $1 \leqslant j \leqslant r$. Since $s_j(l_j) \neq l_j$ it follows from the definition of a basic section that $\alpha_j \in R'$ for all j. Since $\alpha_{r+1} \in l_1 \cap R'$ and since α_1 is the unique root in $l_1 \cap R'$, this implies that $\alpha_{r+1} = \alpha_1$, and so the root line circle ξ is perfect. $\square$

§4. The main result.

In this section, we assume that G is an irreducible finite regular reflection group of rank > 1 with S a set of simple reflections. Let (R, o_G) be the root system associated with G and let B be a simple root system for R contained in a basic section R' of R. Let $\overline{R}$ be the root line set of R.

Lemma 4.1. *For any $\alpha \in R$, the root line R_α has cardinality $o_G(\alpha)$.*

Proof. Without loss of generality, we may assume that $\alpha \in R'$. Since s_α

acts on R_α by multiplication by the scalar $\zeta = e^{2\pi i/o(\alpha)}$, it follows that R_α is a union of $\langle s_\alpha\rangle$-orbits, each orbit having the same cardinality $o(\alpha)$. We shall show that R_α actually consists of a single $\langle s_\alpha\rangle$-orbit, and hence has cardinality $o(\alpha)$.

Let Z be the subgroup of $\mathbb{C}^*$ generated by ζ, and let $\gamma \in R_\alpha$ be arbitrary. Since $R_\alpha = \mathbb{C}\alpha \cap G\alpha$, there exists an element $y \in G$ with $\gamma = y(\alpha)$, and we may choose $s_1, s_2, \ldots, s_r \in S$ such that $y = s_r s_{r-1} \cdots s_1$. Put $\alpha_0 = \alpha$, and for $1 \leqslant k \leqslant r$, put $\alpha_k = s_k s_{k-1} \cdots s_1(\alpha)$ and let l_k be the root line containing α_k. We use induction on k to show that for each k, there exists an $\alpha'_k \in R'$ such that $\alpha_k \in Z\alpha'_k$.

The case $k = 0$ is trivial since $\alpha_0 = \alpha \in R'$. So assume that $k > 1$ and $\alpha_{k-1} = \zeta^m \alpha'_{k-1}$ for some $\alpha'_{k-1} \in R'$ and $m \in \mathbb{Z}$. If $s_k(l_{k-1}) \neq l_{k-1}$ then by the definition of a basic section the root $s_k(\alpha_{k-1})$ is in R'; so defining $\alpha'_k = s_k(\alpha'_{k-1})$, we conclude that $\alpha_k = s_k(\alpha_{k-1}) = \zeta^m \alpha'_k$, as required. So we may assume that $s_k(l_{k-1}) = l_{k-1}$, and hence either $\beta \in l_{k-1}$ or else $(\beta, \alpha_{k-1}) = 0$, where β is the simple root such that $s_\beta = s_k$. In the latter case $\alpha_k = s_\beta(\alpha_{k-1}) = \alpha_{k-1}$, and defining $\alpha'_k = \alpha'_{k-1} \in R'$ gives $\alpha_k \in Z\alpha'_k$ as required. On the other hand, if $\beta \in l_{k-1}$, then by Lemma 1.3, we have $o(\beta) = o_G(\alpha_{k-1}) = o_G(\alpha)$, since α_{k-1} and α are in the same G-orbit. Hence s_β acts on l_{k-1} by multiplication by ζ; so $\alpha_k = s_\beta(\alpha_{k-1}) = \zeta\alpha_{k-1} = \zeta^{m+1}\alpha'_k$, where $\alpha'_k = \alpha'_{k-1} \in R'$, and this completes the induction.

Since $\gamma = \alpha_r$, we conclude from the above that $\gamma \in Z\alpha'_r$, where $\alpha'_r \in R' \cap l_r$. But $l_r = R_\gamma = R_\alpha$, and $\alpha \in R'$ by hypothesis. Since R' is a section, it follows that $\alpha'_r = \alpha$, and thus $\gamma \in Z\alpha$. But γ was chosen as an arbitrary element of $R\alpha$, and so we conclude that $R\alpha = Z\alpha$, as required. $\square$

Lemma 4.2. *(1) For any $\alpha \in R$, we have $|R_\alpha| = o(s)$ for some $s \in S$;*

(2) $|Z(G)|$ is a divisor of $o(s)$ for every $s \in S$.

Proof. Since $R = GB$, there is an element $\beta \in B$ with $\alpha = g(\beta)$ for some $g \in G$. By Lemmas 4.1 and 1.3, we have $|R_\alpha| = o_G(\alpha) = o_G(\beta) = o(s_\beta)$. So (1) follows. Then (2) follows from (1), Lemma 1.7 and $R = GB$. $\square$

Proposition 4.3. *Let G be an irreducible finite complex reflection group of rank > 1 with R the associated root system. If $|Z(G)|$ is a divisor of $o(s)$ for every $s \in S$, then G must be one of the following groups (in the notation of Shephard and Todd, see* [4; 16]*):*

(1) the primitive complex reflection groups G_8, G_{12}, G_{24}, G_{25}, G_{33};

(2) the imprimitive complex reflection groups $G(m, m, n)$ (with $m > 2$ and $\gcd(m, n) \in \{1, 2\}$) and $G(2p, p, n)$ (with $p > 1$ and $\gcd(p, n) = 1$).

Proof. This follows from [3, Tables 1-4], using the fact that the order of $Z(G)$ is the greatest common divisor of the degrees of G. $\square$

4.4. We shall show that none of the groups listed in Proposition 4.3 is regular. By Theorem 3.4, it suffices to find in each case a sequence of roots $\gamma_0, \gamma_1, \ldots, \gamma_t$ in the root system R associated with G such that the root lines $R_{\gamma_0}, R_{\gamma_1}, \ldots, R_{\gamma_t}$ form an imperfect circle.

Let $\epsilon_1, \epsilon_2, \ldots$ be an orthonormal basis in the hermitian space V. We shall give a simple root system $B = \{\alpha_1, \alpha_2, \ldots\}$ associated with G, expressing the α_i's in terms of the ϵ_i's. The order $o_G(\alpha_i)$ of α_i will be 2 unless otherwise specified. We specify roots of G by giving their coordinates relative to B if B is linearly independent, or relative to some maximal linearly independent subset of B if B is linearly dependent. We use the notation $i = \sqrt{-1} \in \mathbb{C}$.

4.5. $G = G_8$: Let $B = \{\alpha_1, \alpha_2\}$ with $\alpha_1 = \epsilon_1$ and $\alpha_2 = \frac{1+i}{2}(\epsilon_2 - \epsilon_1)$, and $o_G(\alpha_1) = o_G(\alpha_2) = 4$. Then $\gamma_0 = (0, 1)$, $\gamma_1 = (1, 1)$, $\gamma_2 = (1, 0)$ and $\gamma_3 = (1, -i)$ are four roots of G. We have

$$\gamma_0 \xmapsto{s_1} \gamma_1 \xmapsto{s_2} \gamma_2 \xmapsto{s_2} \gamma_3 \xmapsto{s_1} -i\gamma_0,$$

where the notation $a \overset{s}{\longmapsto} b$ means $s(a) = b$.

4.6. $G = G_{12}$: Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ with $\alpha_1 = \epsilon_1$, $\alpha_2 = \frac{-1+i\sqrt{2}}{2}\epsilon_1 + \frac{\sqrt{3}-i\sqrt{6}}{6}\epsilon_2$ and $\alpha_3 = -\frac{i\sqrt{2}}{2}\epsilon_1 + \frac{2\sqrt{3}+i\sqrt{6}}{6}\epsilon_2 = \alpha_1 + i\sqrt{2}\alpha_2$. Using coordinates relative to α_1 and α_2, the four roots given by $\gamma_0 = (0,1)$, $\gamma_1 = (1+i\sqrt{2}, -1+i\sqrt{2})$, $\gamma_2 = (i\sqrt{2}, -1+i\sqrt{2})$ and $\gamma_3 = (-1+i\sqrt{2}, -1)$ satisfy

$$\gamma_0 \overset{s_3}{\longmapsto} \gamma_1 \overset{s_1}{\longmapsto} \gamma_2 \overset{s_3}{\longmapsto} \gamma_3 \overset{s_1}{\longmapsto} -\gamma_0.$$

4.7. $G = G_{24}$: Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ with $\alpha_1 = \epsilon_2$, $\alpha_2 = \frac{\overline{\lambda}}{2}(\epsilon_2 + \epsilon_3)$ and $\alpha_3 = -\frac{1}{2}(\epsilon_1 + \epsilon_2 - \lambda\epsilon_3)$, where $\lambda = \frac{1}{2}(1 + i\sqrt{7})$. Then

$$\begin{array}{lll}
\gamma_0 = (1,0,0) & \gamma_1 = (1,0,1) & \gamma_2 = (0,0,1) \\
\gamma_3 = (0,1,1) & \gamma_4 = (0,1,0) & \gamma_5 = (-\overline{\lambda},1,0) \\
\gamma_6 = (-\overline{\lambda},1,\lambda) & \gamma_7 = (-\overline{\lambda},1+\lambda,\lambda) & \gamma_8 = (-1-\overline{\lambda},1+\lambda,\lambda) \\
\gamma_9 = (-1-\overline{\lambda},1+\lambda,-\overline{\lambda}) & \gamma_{10} = (-1-\overline{\lambda},\lambda,-\overline{\lambda}) & \gamma_{11} = (-1,\lambda,-\overline{\lambda}) \\
\gamma_{12} = (-1,\lambda,0) & &
\end{array}$$

are thirteen roots of G. We have

$$\gamma_0 \overset{s_3}{\longmapsto} \gamma_1 \overset{s_1}{\longmapsto} \gamma_2 \overset{s_2}{\longmapsto} \gamma_3 \overset{s_3}{\longmapsto} \gamma_4 \overset{s_1}{\longmapsto} \gamma_5 \overset{s_3}{\longmapsto} \gamma_6 \overset{s_2}{\longmapsto} \gamma_7$$
$$\overset{s_1}{\longmapsto} \gamma_8 \overset{s_3}{\longmapsto} \gamma_9 \overset{s_2}{\longmapsto} \gamma_{10} \overset{s_1}{\longmapsto} \gamma_{11} \overset{s_3}{\longmapsto} \gamma_{12} \overset{s_2}{\longmapsto} -\gamma_0.$$

4.8. $G = G_{25}$: Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ with $\alpha_1 = \overline{\omega}\epsilon_3$, $\alpha_2 = (i/\sqrt{3})(\epsilon_1 + \epsilon_2 + \epsilon_3)$, $\alpha_3 = \overline{\omega}\epsilon_2$, and $o_G(\alpha_j) = 3$ for each $j \in \{1,2,3\}$, where ω is a primitive cube root of unity. Then $\gamma_0 = (1,1,0)$, $\gamma_1 = (-\overline{\omega},1,0)$ are two roots of G, and

$$\gamma_0 \overset{s_1}{\longmapsto} \gamma_1 \overset{s_2}{\longmapsto} -\overline{\omega}\gamma_0.$$

4.9. $\mathbf{G} = \mathbf{G}_{33}$: Let $B = \{\alpha_j \mid 1 \leqslant j \leqslant 5\}$ with $\alpha_1 = 2^{-1/2}\omega(\epsilon_5 + \epsilon_6)$, $\alpha_2 = -2^{-3/2}\omega(-\epsilon_1 + (1+2\omega)\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6)$, $\alpha_3 = 2^{-1/2}(\epsilon_1 - \epsilon_2)$, $\alpha_4 = 2^{-1/2}(\epsilon_2 - \epsilon_3)$ and $\alpha_5 = 2^{-1/2}(\epsilon_3 - \epsilon_4)$. Then $\gamma_0 = (0,0,0,1,0)$,

$\gamma_1 = (0,0,1,1,0)$, $\gamma_2 = (0,0,1,0,0)$, $\gamma_3 = (0,1,1,0,0)$, $\gamma_4 = (0,1,0,0,0)$ and $\gamma_5 = (0,1,0,\overline{\omega},0)$ are six roots of G such that

$$\gamma_0 \xmapsto{s_3} \gamma_1 \xmapsto{s_4} \gamma_2 \xmapsto{s_2} \gamma_3 \xmapsto{s_3} \gamma_4 \xmapsto{s_4} \gamma_5 \xmapsto{s_2} \overline{\omega}\gamma_0.$$

We see that in each of the above five groups, the corresponding root lines $R_{\gamma_0}, R_{\gamma_1}, \ldots$ form an imperfect circle.

4.10. It remains to find imperfect root line circles for the imprimitive complex reflection groups listed in Proposition 4.3 (2). We shall actually do this for all the groups $G(m,m,n)$ with $m > 2$ and $G(2p,p,n)$ with $p > 1$, without the restrictions $\gcd(m,n) \in \{1,2\}$ and $\gcd(p,n) = 1$.

For each $m > 2$, let $\zeta_m = e^{2\pi i/m}$. The group $G(m,m,n)$ has a simple root system $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ given by $\alpha_i = \frac{1}{\sqrt{2}}(\epsilon_i - \epsilon_{i+1})$ for $1 \leqslant i < n$ and $\alpha_n = \frac{1}{\sqrt{2}}(\epsilon_1 - \zeta_m \epsilon_2)$. The root system R generated by B consists of all the vectors $\pm(\zeta_m^k \epsilon_i - \zeta_m^l \epsilon_j)$, where i, j are distinct elements of $\{1,2,\ldots,n\}$ and k, l are integers. Then $\gamma_0 = \frac{1}{\sqrt{2}}(\epsilon_2 - \epsilon_3)$, $\gamma_1 = \frac{1}{\sqrt{2}}(\overline{\zeta}_m \epsilon_1 - \epsilon_3)$, $\gamma_2 = \frac{1}{\sqrt{2}}(\overline{\zeta}_m \epsilon_2 - \epsilon_3)$, $\gamma_3 = \frac{1}{\sqrt{2}}(\overline{\zeta}_m \epsilon_3 - \epsilon_2)$ and $\gamma_4 = \frac{\overline{\zeta}_m}{\sqrt{2}}(\epsilon_3 - \epsilon_1)$ are five roots of $G(m,m,n)$. Since $\zeta_m \neq \pm 1$, they belong to five different root lines. Now

$$\gamma_0 \xmapsto{s_n} \gamma_1 \xmapsto{s_1} \gamma_2 \xmapsto{s_2} \gamma_3 \xmapsto{s_n} \gamma_4 \xmapsto{s_1} -\overline{\zeta}_m \gamma_0,$$

whence R_{γ_0}, R_{γ_1}, R_{γ_2}, R_{γ_3}, R_{γ_4} constitute an imperfect root line circle.

4.11. The group $G(2p,p,n)$ has a simple root system $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha'_n\}$ with α_j as in 4.10 for $1 \leqslant j < n$, while $\alpha_n = \frac{1}{\sqrt{2}}(\epsilon_1 - \zeta_{2p}\epsilon_2)$ and $\alpha'_n = \epsilon_1$. The root system generated by B is $R(2p,p,n) = \{\pm\epsilon_k \mid 1 \leqslant k \leqslant n\} \cup R(2p,2p,n)$, where $R(2p,2p,n)$ is the root system of $G(2p,2p,n)$ as described in 4.10. In particular, $G(2p,p,n)$ contains $G(2p,2p,n)$ as a standard parabolic subgroup (see 1.5); so the existence of an imperfect root line circle in $R(2p,p,n)$ follows from the existence of one in $R(2p,2p,n)$, as established in 4.10.

We are now ready to prove our main result.

Theorem 4.12. *An irreducible finite reflection group G is regular if and only if G is either a Coxeter group or has rank 1.*

Proof. The implication $\Leftarrow$ is well known (see 3.1); so it suffices to show that if G has rank > 1 then G is not regular. By Lemma 4.2 (2) and Proposition 4.3, if G is regular, then it must be one of the groups listed in Proposition 4.3. The results in 4.5–4.11 show that each of these has at least one imperfect root line circle in its root system, and so is not regular by Theorem 3.4. □

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