ON OUTER AUTOMORPHISM GROUPS OF COXETER GROUPS

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Abstract. It is shown that the outer automorphism group of a Coxeter group W of finite rank is finite if the Coxeter graph contains no infinite bonds. A key step in the proof is to show that if the group is irreducible and Π_1 and Π_2 any two bases of the root system of W, then $\Pi_2 = \pm w\Pi_1$ for some $w \in W$. The proof of this latter fact employs some properties of the dominance order on the root system introduced by Brink and Howlett.

1. Introduction

The main aim of this paper is to prove the following result.

Theorem 1.1. *If W is a Coxeter group of finite rank whose diagram has no infinite bonds, then* $\mathrm{Out}(W)$ *is a finite group.*

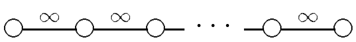
We recall that a Coxeter group is a group with a presentation of the form

$$W = \mathrm{gp}\langle \{ r_a \mid a \in \Pi \} \mid (r_a r_b)^{m_{ab}} = 1 \text{ for all } a,\, b \in \Pi \rangle \tag{1.1}$$

where Π is some indexing set, whose cardinality is called the *rank* of W, and (m_{ab}) is a *Coxeter matrix*, that is, the parameters m_{ab} satisfy $m_{ab} = m_{ba} \in \{ m \in \mathbb{Z} \mid m \geq 1 \} \cup \{\infty\}$ and $m_{ab} = 1$ if and only if $a = b$. (A "relation" $(r_a r_b)^\infty = 1$ is to be interpreted as vacuous.)

The diagram of W is a graph with vertex set Π in which an edge (or bond) labelled m_{ab} joins a, $b \in \Pi$ whenever $m_{ab} \geq 3$. Conventionally, we omit the label when $m_{ab} = 3$. The diagram provides a compact encoding of the presentation (1.1). Every Coxeter group can be realized geometrically—possibly in several inequivalent ways—as a group of linear transformations of a real vector space V. We shall deal only with geometric realizations which embed W in the orthogonal group of some symmetric bilinear form F on V, as described in more detail in §2 below. In this situation we call V a *reflection module* for W.

For Coxeter groups whose diagram is of the form



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the automorphism group is completely described by James [7]. Tits [11] extended this to cover all finite rank Coxeter groups whose diagram has no triangles and only infinite bonds. An important ingredient in Tits' work is the result that every finite subgroup of a Coxeter group is contained in a finite parabolic subgroup. This also plays a vital role in our work. Specifically, it allows us to prove that $[\mathrm{Aut}(W) : \mathrm{Aut}_V(W)] < \infty$, where $\mathrm{Aut}_V(W)$ consists of automorphisms of W induced by orthogonal transformations of a reflection module V. Our second main theorem, which we consider to be of independent interest, provides information about $\mathrm{Aut}_V(W)$, and thereby enables us to complete the proof of Theorem 1.1.

Theorem 1.2. *Let W be an irreducible Coxeter group of finite rank and Φ its root system in a reflection module V. Let Π be a root basis in Φ. If $\varphi : V \to V$ is an orthogonal transformation permuting Φ, then $\varphi\Pi = \pm w\Pi$, for some $w \in W$.*

We understand that J. Y. Hée has an unpublished proof of Theorem 1.2. See also Kac's book [8], Proposition 5.9.

Our strategy for proving Theorem 1.2 is to show that (after replacing φ by $-\varphi$ if necessary) the set

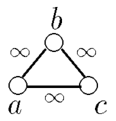
$$\{ a \in \Phi^+ \mid \varphi(a) \in \Phi^- \} \tag{1.2}$$

is finite. Here Φ^+ and Φ^- denote, respectively, the positive and negative roots of Φ. This is equivalent to showing that φ preserves the Tits cone of W. (For the definition of the Tits cone see §3.) For affine and hyperbolic Coxeter groups we use a characterization of the Tits cone in terms of the bilinear form to achieve this. In particular, this proves Theorem 1.2 for Coxeter groups of rank at most 3. To pass to the general case we employ the dominance partial order on Φ which was first introduced in [2]. In particular, we show that φ or $-\varphi$ preserves dominance, and then the finiteness of the set

$$\mathcal{E} = \{ a \in \Phi^+ \mid a \text{ does not dominate any } b \in \Phi^+ \setminus \{a\} \}$$

yields that the set in (1.2) is finite.

We close this section with an example which shows that the assumption of no infinite bonds in Theorem 1.1 cannot be omitted. Let W be the Coxeter group whose diagram is



Put $x = r_a$, $y = r_b$, $z = r_c$ and $t = xy$. Let $\alpha : W \to W$ be the homomorphism for which

$$\alpha(x) = x, \quad \alpha(y) = y \quad \text{and} \quad \alpha(z) = tzt^{-1}.$$

It is easy to check that, in fact, $\alpha \in \mathrm{Aut}(W)$. Suppose that $\alpha^i \in \mathrm{Inn}(W)$ for some $i \in \mathbb{Z}$. Then $\alpha^i(x) = x = wxw^{-1}$, $\alpha^i(y) = y = wyw^{-1}$ and $\alpha^i(z) = t^i z t^{-i} = wzw^{-1}$ for some $w \in W$. Hence

$$w \in C_W(x) \cap C_W(y) = \langle\, x \,\rangle \cap \langle\, y \,\rangle = 1,$$

and so $t^i \in C_W(z) = \langle\, z \,\rangle$. Since t has infinite order, this implies that $i = 0$. Consequently the image of α in $\mathrm{Out}(W)$ has infinite order.

2. Notation and Preliminary Results

Assume that V is a vector space over $\mathbb{R}$ equipped with a symmetric bilinear form $F : V \times V \to \mathbb{R}$ and assume that Π is a finite subset of V which spans V and is a root basis in the sense of the next definition.

Definition 2.1. *A subset* $\Pi \subseteq V$ *is a* root basis *(relative to the bilinear form F) if*

(i) for all a, $b \in \Pi$,

$$F(a,b) = -\cos\frac{\pi}{m_{ab}} \qquad \text{if } m_{ab} \neq \infty,$$
$$F(a,b) \leq -1 \qquad \text{if } m_{ab} = \infty,$$

where (m_{ab}) is a Coxeter matrix, and

(ii) the zero vector is not contained in the set

$$X = \{\, \sum_{a\in\Pi} \lambda_a a \mid \lambda_a \geq 0 \text{ for all } a, \text{ and } \lambda_a \neq 0 \text{ for some } a \,\}.$$

For each $x \in V$ with $F(x,x) = 1$ we define $r_x : V \to V$ (the *reflection* corresponding to x) by

$$r_x v = v - 2F(x,v)x$$

for all $v \in V$. It is easily checked that r_x is linear and preserves F.

It can be shown that (1.1) is a presentation for the subgroup of $O(V,F)$ (the orthogonal group with respect to F) generated by $\{\, r_a \mid a \in \Pi \,\}$, identifying the generators in (1.1) with the reflections as the notation suggests. We refer the reader to Bourbaki [1] for the proof of this and all other assertions about Coxeter groups whose proofs we omit (unless we give an alternative reference). Note, however, that unlike Bourbaki we do not assume that Π is linearly independent, as the weaker assumption 2.1(ii) suffices for the proofs. If Π is linearly independent we call it a *free* root basis, and W a *standard geometric realization* of the Coxeter group (1.1). Note that, given any presentation of the form (1.1), we can construct a standard geometric realization by defining V to be the vector space which has the indexing set Π as a linear basis, and then defining the form F so that 2.1(i) is satisfied.

The *root system* of W in V is the set

$$\Phi = \{\, wa \mid w \in W \text{ and } a \in \Pi \,\},$$

and the elements of $\Phi^+ = \Phi \cap X$ and $\Phi^- = \Phi \cap (-X)$ are called *positive* and *negative* roots respectively. The elements of Π are called *simple* roots.

By Definition 2.1(ii) it is clear that $\Phi^+ \cap \Phi^- = \emptyset$, and it can be shown that $\Phi = \Phi^+ \cup \Phi^-$. For each $w \in W$ define

$$N(w) = \{\, a \in \Phi^+ \mid wa \in \Phi^- \,\},$$

and note that $N(r_a) = \{a\}$ whenever $a \in \Pi$. It follows that for each $w \in W$ and $a \in \Pi$,

$$N(wr_a) = \begin{cases} r_a N(w) \cup \{a\} & \text{if } wa \in \Phi^+ \\ r_a(N(w) \setminus \{a\}) & \text{if } wa \in \Phi^-, \end{cases} \tag{2.1}$$

and $|N(w)| = \ell(w)$, the length of a shortest possible expression for w as a product of the generators $\{\, r_a \mid a \in \Pi \,\}$.

We say that W (or the root system Φ) is *irreducible* if it is impossible to express Φ as the union of two non-empty subsets which are orthogonal relative to F. It is easily seen that Φ is irreducible if and only if Π cannot be expressed as the union of two non-empty orthogonal subsets and this in turn is equivalent to connectedness of the associated diagram. Clearly every root system is uniquely expressible as a union of irreducible root subsystems.

The group W is finite if and only if the form F is positive definite. In consequence, if W is finite then the root basis Π is necessarily free. If Φ is infinite and irreducible and F is positive semidefinite, then W is a so-called "affine Weyl group", corresponding to one of the diagrams on page 34, §2.5 of [6]. In each case the radical of F is one-dimensional.

It is easily checked that if $g \in O(V,F)$ and if $x \in V$ satisfies $F(x,x) = 1$, then $gr_xg^{-1} = r_{gx}$. Applying this with $x = a \in \Pi$ and $g \in W$ it follows that $r_b \in W$ for all $b \in \Phi$. In the converse direction we have the following result

Lemma 2.2. *Let $b \in V$ such that $F(b,b) = 1$ and $r_b \in W$. Then $b \in \Phi$.*

Proof. We use induction on $\ell(r_b)$. If $\ell(r_b) = 1$, then $r_b = r_a$ for some $a \in \Pi$, and hence $b = \pm a \in \Phi$. Now suppose that $\ell(r_b) > 1$, and choose $a \in \Pi$ such that $\ell(r_a r_b) = \ell(r_b) - 1$. By (2.1), $r_b a \in \Phi^-$. Note that $\ell(r_a r_b) = \ell(r_b r_a)$ since these elements are inverses of each other. So by (2.1) again,

$$\ell(r_a r_b r_a) = \begin{cases} \ell(r_b) & \text{if } r_a(r_b a) \in \Phi^+, \\ \ell(r_b) - 2 & \text{if } r_a(r_b a) \in \Phi^-. \end{cases}$$

However, if $r_a(r_b a) \in \Phi^+$, then $-r_b a = a$ because $-r_b a \in \Phi^+$ and $N(r_a) = \{a\}$. But then $b = \pm a \in \Phi$, and so we may assume that $r_a(r_b a) \in \Phi^-$, and hence $\ell(r_a r_b r_a) = \ell(r_b) - 2$. But since $r_a r_b r_a = r_{r_a(b)}$ it follows by the inductive hypothesis that $r_a b \in \Phi$, and hence $b \in \Phi$, as required. □

For each $\Gamma \subseteq \Phi$ the subgroup W_Γ generated by $\{ r_a \mid a \in \Gamma \}$ is called a *reflection subgroup* of W. The set $\Phi_\Gamma = \{ a \in \Phi \mid r_a \in W_\Gamma \}$ is called the *(root) subsystem* generated by Γ. Let $\Phi_\Gamma^+ = \Phi_\Gamma \cap \Phi^+$ and $\Phi_\Gamma^- = \Phi_\Gamma \cap \Phi^-$, and define

$$\Pi_\Gamma = \{ a \in \Phi_\Gamma^+ \bigm| |N(r_a) \cap \Phi_\Gamma| = 1 \}.$$

The main theorem of Deodhar [4] (in particular, steps 2 and 3 of its proof) and also Theorem (3.3) of Dyer [5] yield the following result.

Theorem 2.3. *If $\Gamma \subseteq \Phi$, then the subgroup W_Γ is a Coxeter group on the generating set $\{ r_a \mid a \in \Pi_\Gamma \}$.*

The following is Theorem (4.4) of Dyer [5].

Theorem 2.4. *A subset $\Pi' \subseteq \Phi^+$ equals Π_Γ for some $\Gamma \subseteq \Phi$ if and only if the following condition holds: for all a, $b \in \Pi'$ either $F(a,b) = -\cos\frac{\pi}{m}$ for some integer m, or else $F(a,b) \leq -1$.*

It is clear from Definition 2.1(*ii*) that if $\Pi' \subseteq \Phi^+$ then the zero vector cannot be nontrivially expressed in the form $\sum_{a \in \Pi'} \lambda_a a$ with all $\lambda_a \geq 0$. Hence Theorem 2.4 shows that Π_Γ is a root basis in all cases. Thus Φ_Γ is finite if and only if the restriction of F to $\mathrm{Span}(\Gamma)$ is positive definite. We remark that for an arbitrary Γ it is not necessarily the case that Π_Γ is finite, despite our assumption that Π is.

If $J \subseteq \Pi$ then it is easily shown that $\Pi_J = J$. In this case the subgroup W_J is called a *standard parabolic subgroup* of W, and Φ_J a *standard parabolic subsystem* of Φ. A *parabolic subgroup* of W is any subgroup of the form wW_Jw^{-1}, where $J \subseteq \Pi$ and $w \in W$. Similarly, a *parabolic subsystem* is a root subsystem of the form $w\Phi_J$ for some $w \in W$ and $J \subseteq \Pi$.

For a standard geometric realization, if Ψ is a parabolic subsystem of Φ then Ψ contains all roots expressible as linear combinations of Ψ; that is, $\Psi = \Phi \cap \operatorname{Span} \Psi$. In the converse direction, if Φ is finite and Ψ is a root subsystem of Φ, then $\Phi \cap \operatorname{Span} \Psi$ is a parabolic subsystem of rank equal to $\dim(\operatorname{Span} \Psi)$.

As mentioned in §1, we require the following result, which appears in [1], exercise 2d, p. 130.

Theorem 2.5 (Tits). *If H is a finite subgroup of W, then H is contained in some finite parabolic subgroup of W.*

Since a Coxeter group acts faithfully on its root system, the reflection subgroup corresponding to a finite root system will be finite, and so it follows from Theorem 2.5 that every such subsystem is contained in a finite parabolic subsystem. Thus we obtain the following result, which is essentially contained in [9], Lemma 3.2.3.

Proposition 2.6. *If Ψ is a finite subsystem of Φ then $\Phi \cap \operatorname{Span} \Psi$ is a finite parabolic subsystem of Φ.*

Proof. Since $|\Psi| < \infty$ the subspace $\operatorname{Span} \Psi$ is Euclidean, and so the subsystem $\Lambda = \Phi \cap \operatorname{Span} \Psi$ is finite. By Theorem 2.5 there exist $w \in W$ and $J \subseteq \Pi$ such that $W_\Lambda \leq w^{-1} W_J w$, and $|W_J| < \infty$. Replacing Ψ by $w\Psi$ permits us to assume that $W_\Lambda \leq W_J$, and hence that $\Lambda = \{ a \in \Phi \mid r_a \in W_\Lambda \} \subseteq \Phi_J$. Since $\Lambda = \Phi_J \cap \operatorname{Span} \Lambda$, we conclude that Λ is a parabolic subsystem of Φ_J, and hence a parabolic subsystem of Φ. $\square$

The following result can be found in Deodhar [3], Proof of Proposition 4.2.

Proposition 2.7. *If Φ is infinite and irreducible then $\Phi \setminus \Phi_J$ is infinite whenever J is a proper subset of Π.*

We also need the following technical lemma.

Lemma 2.8. *Suppose that $\Gamma \subseteq \Phi$ has the property that $W_\Gamma = W$. Then Γ spans V.*

Proof. An easy induction shows that if $a_1, a_2, \ldots, a_\ell \in \Gamma$, then the vector $r_{a_1} r_{a_2} \ldots r_{a_\ell} v - v$ is a linear combination of $a_1, a_2 \ldots, a_\ell$, for all $v \in V$. Since the simple reflections r_a can be expressed as products $r_{a_1} r_{a_2} \ldots r_{a_\ell}$, it follows that $2a = a - r_a a \in \operatorname{Span} \Gamma$ for all $a \in \Pi$. Because Π spans V, so too does Γ. $\square$

The following was proved in a preliminary version of [2].

Proposition 2.9. *If $\Gamma \subseteq \Phi$, then $|\Pi_\Gamma| \leq |\Gamma|$.*

Proof. Let $\widetilde{V}$ be a real vector space with basis $\widetilde{\Pi}$ in bijective correspondence with Π_Γ, and define a bilinear form $\widetilde{F}$ on $\widetilde{V}$ so that the linear map $\psi : \widetilde{V} \to V$ induced by the bijection $\widetilde{\Pi} \to \Pi_\Gamma$ satisfies $F(\psi u, \psi v) = \widetilde{F}(u, v)$ for all u, $v \in \widetilde{V}$.

Then $\widetilde{W} = \langle \{ r_a \mid a \in \widetilde{\Pi} \} \rangle$ is a standard geometric realization of the Coxeter group W_Γ, and there is an isomorphism $\sigma : \widetilde{W} \to W_\Gamma$ such that $\sigma r_a = r_{\psi a}$. It is easily checked that $\psi(r_a v) = r_{\psi a}(\psi v)$ for all $a \in \widetilde{\Pi}$ and $v \in \widetilde{V}$. By induction on $\ell(w)$ it follows that

$$\psi(wv) = (\sigma w)(\psi v) \tag{2.2}$$

for all $w \in \widetilde{W}$ and $v \in \widetilde{V}$. Hence ψ maps the root system $\widetilde{\Phi}$ of $\widetilde{W}$ in $\widetilde{V}$ to Φ_Γ, and since $\psi\widetilde{\Pi} = \Pi_\Gamma$ it follows that $\psi(\widetilde{\Phi}^+) = \Phi_\Gamma^+$.

Since σ maps the reflections in $\widetilde{W}$ bijectively to the reflections in W_Γ, the restriction of ψ to $\widetilde{\Phi}^+$ is a bijection $\widetilde{\Phi}^+ \to \Phi_\Gamma^+$. Hence $a \mapsto \psi a$ is a bijection from $\widetilde{\Phi}$ to Φ_Γ.

The required conclusion is now an easy consequence of Lemma 2.8. Let $\widetilde{\Gamma} = \{ a \in \widetilde{\Phi} \mid \psi a \in \Gamma \}$. Since $\{ r_a \mid a \in \Gamma \}$ generates W_Γ it follows that $\{ \sigma^{-1}(r_a) \mid a \in \Gamma \} = \{ r_b \mid b \in \widetilde{\Gamma} \}$ generates $\widetilde{W}$, and hence $\widetilde{\Gamma}$ spans $\widetilde{V}$. Since $\widetilde{\Pi}$ is a basis for $\widetilde{V}$ it follows that $|\widetilde{\Pi}| \le |\widetilde{\Gamma}|$, whence $|\Pi_\Gamma| \le |\Gamma|$ as claimed. □

We conclude this section with another elementary technical lemma which will be needed in Section 4 below.

Lemma 2.10. *If Φ is irreducible and a, $b \in \Phi$ then there exists $c \in \Phi$ with $F(a, c) \neq 0$ and $F(b, c) \neq 0$.*

Proof. If $F(a, b) \neq 0$ we may take $c = a$. So assume that $F(a, b) = 0$. Now $\Phi_{\{a,b\}} = \{\pm a, \pm b\}$ is finite, and so, by Proposition 2.6, $\mathrm{Span}\{a, b\} \cap \Phi$ is a parabolic subsystem of rank 2. If this system contains $c \notin \{\pm a, \pm b\}$ then clearly $F(a, c)$ and $F(b, c)$ are both nonzero, as required. So we may assume that $\mathrm{Span}\{a, b\} \cap \Phi = \{\pm a, \pm b\}$ is a parabolic subsystem, and replacing a and b by wa and wb (if necessary) for some $w \in W$, we may assume that a, $b \in \Pi$.

Since Φ is irreducible, we may choose a_0, a_1, $\ldots$, $a_k \in \Pi$ with $a = a_0$, $b = a_k$ and $F(a_{i-1}, a_i) \neq 0$ for $i = 1, 2, \ldots , k$. Do this so as to minimize k. Then $d = r_{a_{k-1}} r_{a_{k-2}} \ldots r_{a_2}(a_1)$ has the required properties (since $d = \sum_{i=1}^{k-1} \mu_i a_i$ for some positive coefficients μ_i). □

3. Cones, Hyperbolic Coxeter Groups and Dominance

A *cone* in a finite dimensional real vector space V is a subset of V which is closed under addition and multiplication by positive scalars.

If V^* is the dual space and S is a cone in V then the polar cone of S is the set

$$\{ f \in V^* \mid fv \leq 0 \text{ for all } v \in S \}.$$

It is more convenient for us to work with the negative of this, and so we define

$$S^* = \{ f \in V^* \mid fv \geq 0 \text{ for all } v \in S \}.$$

It is clear that S^* is topologically closed (relative to the usual topology for real vector spaces) and also that $S \subseteq S^{**}$.

The following is well-known and not hard to prove.

Proposition 3.1. *If S is any cone in V, then $S^{**} = \overline{S}$, the topological closure of S.*

Note that the set X appearing in Definition 2.1(*ii*) is a cone. It is also well known that the assumption that $0 \notin X$ is equivalent to the statement that there exists an element $f \in V^*$ such that $fa > 0$ for all $a \in \Pi$. That is, the set

$$C = \{ f \in V^* \mid fv > 0 \text{ for all } v \in X. \}$$

is non-empty. Note that $\overline{C} = X^*$.

For $w \in W$ and $f \in V^*$ we define $wf \in V^*$ by

$$(wf)v = f(w^{-1}v) \qquad \text{for all } v \in V.$$

The *Tits cone* is defined to be the set

$$U = \bigcup_{w \in W} w\overline{C}.$$

See Tits [12] (or Bourbaki [1]) for the proof that U is in fact a cone.

If Φ is reducible then the Tits cone for Φ is the sum of the Tits cones for the irreducible components. Note that we continue to assume that the space V is spanned by Φ. For finite root systems the Tits cone equals V^*. For infinite irreducible systems we have the following result, which appears in [9], Theorem 2.1.6; see also Vinberg [13], Lemma 15, p. 1112.

Proposition 3.2. *If Φ is infinite and irreducible, then $U \cap (-U) = \{0\}$.*

Proof. Since $U \cap (-U) = \cup_{w_1,w_2 \in W}(w_1\overline{C} \cap w_2(-\overline{C}))$ it follows that if $U \cap (-U) \neq \{0\}$ then there exists a nonzero $f \in \overline{C}$ such that $-wf \in \overline{C}$ for some $w \in W$. Let $J = \{ a \in \Pi \mid fa = 0 \}$ so that $fa > 0$ for all $a \in \Pi \setminus J$. If $b \in \Phi^+ \setminus \Phi_J$ then $fb > 0$, whence $(-wf)(wb) = -fb < 0$ and so $wb \in \Phi^-$ (since $-wf \in \overline{C}$). Hence $\Phi^+ \setminus \Phi_J$ is contained in the finite set $N(w)$. By Proposition 2.7 it follows that $J = \Pi$, contradicting $f \neq 0$. ∎

Observe that

$$\begin{aligned} U^* &= \{\, v \in V \mid fv \geq 0 \text{ for all } v \in U \,\} \\ &= \{\, v \in V \mid (wf)v \geq 0 \text{ for all } f \in \overline{C} \text{ and } w \in W \,\} \\ &= \{\, v \in V \mid f(w^{-1}v) \geq 0 \text{ for all } f \in \overline{C} \text{ and all } w \in W \,\} \\ &= \{\, v \in V \mid w^{-1}v \in \overline{X} \text{ for all } w \in W \,\} \end{aligned}$$

since $\overline{X} = X^{**} = \overline{C}^*$. Note also that $\overline{X} = X \cup \{0\}$ is the set of all non-negative linear combinations of Π. Thus U^* consists of all $v \in V$ such that wv is a non-negative linear combination of Π for all $w \in W$. Although we make no subsequent use of it, the following fact may be of some interest.

Proposition 3.3. *Suppose that $v \in \overline{X}$ and $F(a, v) \leq 0$ for all $a \in \Pi$. Then $v \in U^*$.*

Proof. We use induction on $\ell(w)$ to show that $wv \in \overline{X}$ for all $w \in W$. This holds by hypothesis when $\ell(w) = 0$. Assume then that $\ell(w) > 0$, so that $N(w) \neq \emptyset$ and hence $N(w) \cap \Pi \neq \emptyset$, and choose $a \in N(w) \cap \Pi$. By (2.1) we have that $w = w'r_a$ where $\ell(w') = \ell(w) - 1$, and so the inductive hypothesis yields $w'v \in \overline{X}$. Now

$$\begin{aligned} wv &= (w'r_a)v \\ &= w'(v - 2F(a, v)a) \\ &= w'v - 2F(a, v)w'a \in \overline{X} \end{aligned}$$

since $w'a = -wa \in \Phi^+ \subseteq \overline{X}$ and $-2F(a, v) \geq 0$. □

We remark that it is not necessarily the case that every element of U^* has the form wv for some $w \in W$ and some v satisfying $F(a, v) \leq 0$ for all $a \in \Pi$. For example, if $\Pi = \{a, b\}$ with $F(a, b) = -2$, then $a + (2 + \sqrt{3})b$ is in U^* but cannot be expressed in the stated form.

The next result was proved by Maxwell [10] for the case of a free root basis. The proof for an arbitrary root basis is somewhat more complicated.

Proposition 3.4. *If $v \in U^*$ then $F(v, v) \leq 0$.*

Proof. To prove this we first choose a free root basis $\widetilde{\Pi}$ in a space $\widetilde{V}$, relative to a form $\widetilde{F}$, such that there is a linear map $\eta : \widetilde{V} \to V$ which restricts to a bijection $\widetilde{\Pi} \to \Pi$ and carries $\widetilde{F}$ to F. Then η is a W-module homomorphism and $Z = \ker \eta$ is contained in the radical of $\widetilde{F}$.

Let

$$Y = \{ \sum_{a\in\widetilde{\Pi}} \lambda_a a \mid \lambda_a \geq 0 \text{ for all } a \in \widetilde{\Pi} \},$$

and for $v = \sum_{a\in\widetilde{\Pi}} \lambda_a a \in \widetilde{V}$ define $S(v) = \sum_{a\in\widetilde{\Pi}} \lambda_a$. Observe that $\eta^{-1}(U^*) = \bigcap_{w\in W}(wY + Z)$, and note also by Definition 2.1(ii) that $Y \cap Z = \{0\}$.

Lemma 3.5. *For each $v \in \widetilde{V}$ the set $(v + Z) \cap Y$ is bounded.*

Proof. If not, then there exist elements $v + z_n \in (v + Z) \cap Y$ such that $S(v + z_n) \to \infty$ as $n \to \infty$. Now the elements $S(v + z_n)^{-1}(v + z_n)$ lie in the compact set $Y_1 = \{ y \in Y \mid S(y) = 1 \}$. A convergent subsequence gives a limit in $Z \cap Y_1$, contradicting $Z \cap Y = \{0\}$. □

For each $v \in Z + Y$ define $Z(v) = \{ z \in Z \mid v + z \in Y \}$ and $M_v = \sup\{ S(v + z) \mid z \in Z(v) \}$ (which is well-defined in view of Lemma 3.5).

Now suppose, for a contradiction, that $\eta^{-1}(U^*)$ contains elements v with $\widetilde{F}(v, v) > 0$, and choose $v_0 \in \eta^{-1}(U^*)$ with $\widetilde{F}(v_0, v_0) = 1$. Put $M = M_{v_0}$ and $\varepsilon = {}^2\!/nM$, where $n = |\widetilde{\Pi}|$. Clearly v_0 is contained in

$$\mathcal{U} = \{ v \in \eta^{-1}(U^*) \mid \widetilde{F}(v, v) = 1 \text{ and } M_v \leq M \}.$$

We show that for any given $v \in \mathcal{U}$ there exists $v' \in \mathcal{U}$ with $M_{v'} \leq M_v - \varepsilon$. A finite number of iterations of this process yields a $v \in \mathcal{U}$ with $M_v < 0$, which is absurd.

Given $v \in \mathcal{U}$, since $v \in \eta^{-1}(U^*) \subseteq Z + Y$, we may choose $z \in Z(v)$, and write $v + z = \sum_{a\in\widetilde{\Pi}} \lambda_a a$. Then

$$1 = \widetilde{F}(v + z, v + z) = \sum_{a\in\widetilde{\Pi}} \lambda_a \widetilde{F}(a, v),$$

and so there exists an $a \in \widetilde{\Pi}$ with $\lambda_a \widetilde{F}(a, v) \geq {}^1\!/n$. Since $\lambda_a \leq S(v + z) \leq M_v \leq M$ it follows that $\widetilde{F}(a, v) \geq {}^1\!/nM = {}^\varepsilon\!/2$.

Put $v' = r_a v$. Then $v' \in \eta^{-1}(U^*)$ (since $\eta^{-1}(U^*)$ is W-invariant) and $\widetilde{F}(v', v') = \widetilde{F}(v, v) = 1$; so to show that $v' \in \mathcal{U}$ it remains to verify that $M_{v'} \leq M$.

Since $v' = r_a v = v - 2\widetilde{F}(a, v)a$ we see that

$$S(v') = S(v) - 2\widetilde{F}(a, v) \leq S(v) - \varepsilon.$$

Hence if $z' \in Z(v')$ then

$$v + z' = v' + z' + 2\widetilde{F}(a, v)a \in Y,$$

and so $z' \in Z(v)$. Hence,

$$\begin{aligned} S(v'+z') &= S(v') + S(z') \\ &\leq S(v) - \varepsilon + S(z') \\ &\leq M_v - \varepsilon, \end{aligned}$$

and consequently $M_{v'} \leq M_v - \varepsilon$. Since $M_v \leq M$ this yields $M_{v'} \leq M$, so that v' is an element of $\mathcal{U}$ satisfying $M_{v'} \leq M_v - \varepsilon$, as required to complete the proof of the proposition. □

Corollary 3.6. *If W is an irreducible affine Weyl group then $U^* = \{\, \lambda\rho \mid \lambda \geq 0 \,\}$ where ρ spans the radical of F.*

Proof. Since W is infinite we have $U \cap (-U) = \{0\}$ by Proposition 3.2, and so $U^* \neq \{0\}$ (since otherwise $\overline{U} = U^{**}$ would equal V^*). By Proposition 3.4 there exists nonzero $\rho \in U^*$ such that $F(\rho,\rho) \leq 0$, whence $F(\rho,\rho) = 0$ and ρ spans the radical. Since $U^* \cap (-U^*) \subseteq \overline{X} \cap (-\overline{X}) = \{0\}$ it follows that U^* consists only of the nonnegative multiples of ρ. □

For our next two results we assume that W is a hyperbolic Coxeter group, by which we mean that V has a basis $x_1, x_2, \ldots, x_n$ such that $F(x_i, x_j) = 0$ for $i \neq j$ and

$$F(x_i, x_i) = \begin{cases} 1 & 1 \leq i \leq n-1, \\ -1 & i = n. \end{cases}$$

Let $H = \{\, \sum \lambda_i x_i \mid \lambda_n \geq \sqrt{\sum_{i=1}^{n-1} \lambda_i^2} \,\}$, and observe that

$$\{\, v \in V \mid F(v,v) \leq 0 \,\} = H \cup (-H).$$

Furthermore, it can be checked (using the Cauchy-Schwarz inequality) that H is a cone.

Since $U^* \neq \{0\}$ (as in the proof of Corollary 3.6) it follows from Proposition 3.4 that $U^* \cap (H \cup (-H)) \neq \{0\}$. Replacing x_n by $-x_n$ if necessary, we may suppose that $U^* \cap H \neq \{0\}$. Define

$$L = \{\, f \in V^* \mid f(x) > 0 \text{ for all nonzero } x \in H \,\}.$$

Proposition 3.7 (Maxwell [10]). *(i) $U^* \subseteq H$.*
(ii) $L \subseteq U$.

Proof. Let $0 \neq v \in H \cap U^*$ and suppose, for a contradiction, that there exists a nonzero vector $v' \in U^* \cap (-H)$. Write $v = \sum \lambda_i x_i$ and $v' = \sum \mu_i x_i$ and put $v'' = \lambda_n v' - \mu_n v$. Since v'' is a linear combination of $x_1, \ldots, x_{n-1}$ we have that $F(v'', v'') \geq 0$. But $v'' \in U^*$, since v, $v' \in U^*$ and the coefficients λ_n and $-\mu_n$ are both positive. So

$F(v'', v'') \le 0$, by Proposition 3.4, whence $v'' = 0$ since F is positive definite on $\mathrm{Span}\{x_1, \dots, x_{n-1}\}$. So v' is a positive multiple of $-v$, which shows that v and $-v$ both lie in $U^* \subseteq \overline{X} = X \cup \{0\}$. Since X is a cone and $0 \notin X$, this forces $v = 0$, a contradiction. Hence $U^* \cap (-H) = \emptyset$ and so (i) holds.

From $U^* \subseteq H$ we obtain $H^* \subseteq U^{**} = \overline{U}$. Since U is convex, U contains the interior of $\overline{U}$. As $L \subseteq \mathrm{int}(H^*) \subseteq \mathrm{int}\,\overline{U}$, the assertion of (ii) follows. □

Corollary 3.8. *$C \cap L \neq \emptyset$.*

Proof. Clearly W preserves $L \cup (-L)$, but since it also preserves U, and as $U \cap (-U) = \{0\}$, it follows from Proposition 3.7 that W preserves L. Since $U = \bigcup_{w \in W} w\overline{C}$ this implies that $\overline{C} \cap L \neq \emptyset$, and hence, since L is open, $C \cap L \neq \emptyset$. □

We conclude this section with some results concerning the dominance partial order.

Definition 3.9. *Let a, $b \in \Phi$. If*

$$\{\, w \in W \mid wa \in \Phi^- \,\} \subseteq \{\, w \in W \mid wb \in \Phi^- \,\}$$

we say that a dominates *b, and write $a \succcurlyeq b$ (and $a \succ b$ if $a \neq b$).*

Although in [2] dominance was defined only on Φ^+, it is clear that defining it on all of Φ, as we have done, yields a partial order on Φ. Furthermore, this partial order is W-invariant. Clearly, also, if $a \succcurlyeq b$ then $-b \succcurlyeq -a$.

Lemma 2.3 of [2] yields the following criterion for dominance.

Proposition 3.10. *If a, $b \in \Phi$, then $F(a, b) \ge 1$ if and only if either $a \succcurlyeq b$ or $b \succcurlyeq a$.*

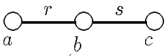
Suppose that Ψ is a subsystem of Φ and (temporarily) let $\succeq$ denote dominance on Ψ defined with respect to W_Ψ. Using Proposition 3.10 it is not hard to show, for all a, $b \in \Psi$ that $a \succcurlyeq b$ if and only if $a \succeq b$.

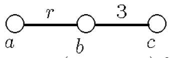
A positive root a is said to be *elementary* if $\{\, b \in \Phi^+ \mid a \succ b \,\}$ is empty. The main result of [2] (Theorem 2.8) says that the set $\mathcal{E} = \{\, a \in \Phi^+ \mid a \text{ is elementary} \,\}$ is finite.

Lemma 3.11. *Suppose that $\Pi = \{a, b, c\}$ and that Φ is infinite and irreducible. Then there exists α, β, $\gamma \in \Phi$ with $\alpha \succ a$, $\beta \succ b$ and $\gamma \succ c$. Suitable values for α, β and γ for the various cases are as follows.*

(i)

, where r, s, $t \ge 3$: $\alpha = (r_a r_b)c$, $\beta = (r_b r_c)a$, $\gamma = (r_c r_a)b$.

(*ii*) , *where* $r,\ s \geq 4$*:* $\alpha = (r_a r_b r_c r_b)a$, $\beta = (r_b r_a r_c)b$, $\gamma = (r_c r_b r_a r_b)c$.

(*iii*) , *where* $r \geq 6$*:* $\alpha = (r_a r_b r_c r_a r_b)a$, $\beta = (r_b r_a r_c)b$, $\gamma = (r_c r_b r_a)b$.

Proof. To prove that $\alpha \succ a$ it suffices to show that $\alpha \in \Phi^+ \setminus \{a\}$ and $F(\alpha, a) \geq 1$ since $r_a(a) \in \Phi^-$ and $r_a(\alpha) \in \Phi^+$ shows that a does not dominate α.

So, writing $\alpha' = r_a\alpha$, it suffices to show that $\alpha' \in \Phi^+$ and $F(\alpha', a) \leq -1$. Similarly to prove that $\beta \succ b$ and $\gamma \succ c$ it suffices to show that $\beta' = r_b\beta$ and $\gamma' = r_c\gamma$ are positive roots satisfying $F(\beta', b) \leq -1$ and $F(\gamma', c) \leq -1$.

In case (i), $\alpha' = c + 2(\cos\dfrac{\pi}{s})b$ and so

$$F(\alpha', a) = -\cos\frac{\pi}{t} - 2(\cos\frac{\pi}{s})(\cos\frac{\pi}{r}) \leq -1,$$

since the cosines are all at least $1/2$. Symmetrical arguments apply for β' and γ'.

In case (ii), $\alpha' = a + 8\cos\dfrac{\pi}{r}\cos^2\dfrac{\pi}{s}\,b + 4\cos\dfrac{\pi}{r}\cos\dfrac{\pi}{s}\,c$, and so

$$F(\alpha', a) = 1 - 8\cos^2\frac{\pi}{r}\cos^2\frac{\pi}{s} \leq -1,$$

since the cosines are both at least $1/\sqrt{2}$. A symmetrical argument applies for γ'. We leave the calculation for β' to the reader.

In case (iii) similar calculations show that

$$\alpha' = (4\lambda^2 - 1)a + 8(\lambda^3 - 2\lambda)b + 2\lambda c$$

and $F(\alpha', a) = -8\lambda^4 + 6\lambda^2 - 1$, where $\lambda = \cos\dfrac{\pi}{r}$. Elementary calculus, using the fact that $\lambda \geq \sqrt{3}/2$, yields $F(\alpha', a) \leq -1$. The calculations for β' and γ' are again left to the reader. $\square$

Proposition 3.12. *In every infinite irreducible root system* Φ*, for each* $a \in \Phi$ *there exists* $b \in \Phi$ *with* $b \succ a$*.*

Proof. Since dominance is preserved by the W-action it suffices to prove that for each $a \in \Pi$ there exists $b \in \Phi$ with $b \succ a$. Since $\mathcal{E}$ is finite and Φ^+ is infinite there certainly exist $c,\ d \in \Phi^+$ with $c \succ d$, and by W-invariance of dominance we may choose $d \in \Pi$.

By irreducibility of Π it now suffices to show that if $e \in \Pi$ with $F(d, e) \neq 0$, there exists $f \in \Phi$ with $f \succ e$. If $F(d, e) \leq -1$ we may take $f = r_e(d)$; so we may assume that the parabolic subsystem generated by d and e is finite. Let $\Gamma = \{c, d, e\}$, and note that the subsystem Φ_Γ generated by Γ is infinite since $F(c-d,c-d) \le 0$. Because Π_Γ contains d and e (as is immediate from the definition of Π_Γ) it follows from Proposition 2.9 that Φ_Γ has rank 3 and is irreducible. By Lemma 3.11 there is an $f \in \Phi_\Gamma$ with the required property. □

Our final result for this section is a key ingredient in the proof of Theorem 1.2.

Proposition 3.13. *Suppose that $\varphi : \Phi \to \Phi$ is a bijection that preserves dominance. Then*

$$\mathcal{A} = \{\, a \in \Phi^+ \mid \varphi a \in \Phi^- \,\}$$

is a finite set.

Proof. Let

$$\mathcal{B} = \{\, b \in \Phi^+ \mid -\varphi e \succcurlyeq b \text{ for some } e \in \mathcal{A} \cap \mathcal{E} \,\}.$$

Note that if $e \in \mathcal{A} \cap \mathcal{E}$ then $f = -\varphi e \in \Phi^+$, and so if $f \succcurlyeq b$ then $b \in N(r_f)$ (since $f \in N(r_f)$). Hence $\mathcal{B}$ is a subset of the finite set $\bigcup_{f \in -\varphi\mathcal{E}} N(r_f)$.

Let $a \in \mathcal{A}$ and choose $e \in \mathcal{E}$ such that $a \succcurlyeq e$. Then $\varphi a \succcurlyeq \varphi e$ since φ preserves dominance. Since $\varphi a \in \Phi^-$ it follows that $\varphi e \in \Phi^-$. So $e \in \mathcal{A}$. Also $\varphi a \succcurlyeq \varphi e$ gives $-\varphi e \succcurlyeq -\varphi a$, and so $-\varphi a \in \mathcal{B}$. Hence $\mathcal{A} \subseteq \varphi^{-1}(-\mathcal{B})$, a finite set. □

4. Proof of Theorem 1.2

In this section we shall often be dealing with two Coxeter groups simultaneously. We retain the notation of §3, using subscripts 1 and 2 to distinguish the groups.

Theorem 4.1. *Let Φ_1 and Φ_2 be irreducible root systems spanning the spaces V_1 and V_2 with root bases Π_1 and Π_2. Suppose that $\varphi : V_1 \to V_2$ maps Φ_1 bijectively to Φ_2 and takes the bilinear form on V_1 to that on V_2. Then $\Pi_2 = \pm\varphi w \Pi_1$ for some $w \in W_1$.*

Proof. If the root systems are finite this is well-known—indeed, $\Pi_2 = \varphi w \Pi_1$ for some w in this case—and so we assume that they are infinite. Note that $x \mapsto \varphi x \varphi^{-1}$ is an isomorphism $O(V_1, F_1) \to O(V_2, F_2)$ which takes W_1 to W_2.

Assume first that W_1 is an irreducible affine Weyl group. Then so is W_2 (since if F_1 is positive semidefinite and Φ_1 cannot be expressed as a union of two nonempty orthogonal subsets, then the same is true of

F_2 and Φ_2), and by what was shown in §3 the duals of the Tits cones are

$$U_1^* = \{\, \lambda\rho_1 \mid \lambda \geq 0 \,\}$$
$$U_2^* = \{\, \lambda\rho_2 \mid \lambda \geq 0 \,\}$$

where ρ_1 and ρ_2 span the radicals of V_1 and V_2 respectively. Since φ maps the radical of V_1 to the radical of V_2, it follows that $\varphi U_1^* = \pm U_2^*$, and, replacing φ by $-\varphi$ if necessary, we may assume that $\varphi U_1^* = U_2^*$.

Denoting by φ^* the map $V_2^* \to V_1^*$ induced by $\varphi : V_1 \to V_2$, we have

$$\varphi^*\overline{U}_2 = \varphi^*(U_2^{**}) = U_1^{**} = \overline{U}_1.$$

Hence $\varphi^*(\operatorname{int}(\overline{U}_2)) = \operatorname{int}(\overline{U}_1) = \operatorname{int} U_1$, and in particular, choosing $f \in C_2$ arbitrarily, we have $\varphi^* f \in \operatorname{int} U_1$. We conclude that, in the case of affine Weyl groups, there exists $f \in C_2$ with $\varphi^* f \in \operatorname{int} U_1$ (φ having possibly been replaced by $-\varphi$).

Assume alternatively that the Coxeter groups W_1 and W_2 are hyperbolic. Choose $f \in C_2 \cap L_2$. Then $\varphi^* f \in \varphi^* L_2$, and after possibly replacing φ by $-\varphi$ so that $\varphi H_1 = H_2$ we have

$$\begin{aligned}
\varphi^* L_2 &= \{\, \varphi^* f \mid f \in V_2^* \text{ and } f(x) > 0 \text{ for all nonzero } x \in H_2 \,\}\\
&= \{\, \varphi^* f \mid f \in V_2^* \text{ and } (\varphi^* f)(\varphi^{-1}x) > 0 \text{ for all nonzero } x \in H_2 \,\}\\
&= \{\, f \in V_1^* \mid f(\varphi^{-1}x) > 0 \text{ for all nonzero } x \in H_2 \,\}\\
&= \{\, f \in V_1^* \mid f(x) > 0 \text{ for all nonzero } x \in H_1 \,\}\\
&= L_1 \subseteq \operatorname{int} U_1
\end{aligned}$$

So in the hyperbolic case, as well as in the affine, there exists $f \in C_2$ with $\varphi^* f \in \operatorname{int} U_1$. For such an f, we have that $\varphi^* f \in w\overline{C}_1$ for some $w \in W_1$. But since $f(a) \neq 0$ for all $a \in \Phi_2$, and since φ takes Φ_1 to Φ_2, it follows that

$$(w^{-1}\varphi^* f)(a) = f(\varphi(wa)) \neq 0 \quad \text{for all } a \in \Phi_1.$$

So $w^{-1}\varphi^* f \in C_1$. Now

$$\begin{aligned}
\Phi_2^+ &= \{\, a \in \Phi_2 \mid f(a) > 0 \,\}\\
&= \varphi w(\{\, a \in \Phi_1 \mid f(\varphi(wa)) > 0 \,\})\\
&= \varphi w(\{\, a \in \Phi_1 \mid (w^{-1}\varphi^* f)(a) > 0 \,\})\\
&= \varphi w(\Phi_1^+) \quad (\text{since } w^{-1}\varphi^* f \in C_1).
\end{aligned}$$

So $\Pi_2 = \varphi w(\Pi_1)$, and so the result holds for affine and hyperbolic Coxeter groups (as well as finite ones).

Consider now the general case. If Ψ_1 is a subsystem of Φ_1 of rank 3 or less, corresponding to a reflection subgroup $T_1 \leq W_1$, then the restriction of φ yields a bijection from Ψ_1 to $\Psi_2 = \varphi(\Psi_1)$ satisfying the hypotheses of the theorem. Since root systems of rank at most 3 are either finite, affine or hyperbolic, we conclude by the first part of the proof that there exists $w \in T_1$ such that either $\Psi_2^+ = \varphi w \Psi_1^+$ or $\Psi_2^+ = -\varphi w \Psi_1^+$. We show that the dominance partial order on Ψ_1 is preserved by φ in the former case, reversed by φ in the latter.

Write T_2 for the reflection subgroup of W_2 corresponding to the subsystem Ψ_2, noting that the isomorphism $x \mapsto \varphi x \varphi^{-1}$ takes T_1 to T_2. Assuming the former of our two cases, if $a, b \in \Psi_1$ with $a \succcurlyeq b$, then

$$\begin{aligned}
\{\, x \in T_2 \mid x\varphi a \in \Psi_2^- \,\} &= \varphi\{\, y \in T_1 \mid \varphi y a \in \Psi_2^- \,\}\varphi^{-1} \\
&= \varphi\{\, y \in T_1 \mid \varphi y a \in \varphi w \Psi_1^- \,\}\varphi^{-1} \\
&= \varphi\{\, y \in T_1 \mid w^{-1} y a \in \Psi_1^- \,\}\varphi^{-1} \\
&\subseteq \varphi\{\, y \in T_1 \mid w^{-1} y b \in \Psi_1^- \,\}\varphi^{-1} \\
&= \varphi\{\, y \in T_1 \mid \varphi y b \in \varphi w \Psi_1^- \,\}\varphi^{-1} \\
&= \varphi\{\, y \in T_1 \mid \varphi y b \in \Psi_2^- \,\}\varphi^{-1} \\
&= \{\, x \in T_2 \mid x\varphi b \in \Psi_2^- \,\},
\end{aligned}$$

and hence $\varphi a \succcurlyeq \varphi b$. So φ preserves dominance on Ψ_1, as claimed. Since multiplication by -1 reverses dominance it is clear that dominance is reversed in the other case.

We now show that φ either preserves dominance or reverses it on the entire root system. Let $a, b \in \Phi_1$ with $a \succ b$. If $\varphi b \succ \varphi a$ replace φ by $-\varphi$, so that $\varphi a \succ \varphi b$. We show that now whenever $c, d \in \Phi_1$ with $c \succ d$ then $\varphi c \succ \varphi d$.

Let $c, d \in \Phi_1$ with $c \succ d$. If $F(b, c) \neq 0$ then $\Phi_{\{a,b,c\}}$ is an infinite irreducible root subsystem of rank at most 3, and since $a, b \in \Phi_{\{a,b,c\}}$ the restriction of φ to $\Phi_{\{a,b,c\}}$ must preserve dominance rather than reverse it. Furthermore, there exists $f \in \Phi_{\{a,b,c\}}$ with $f \succ c$. Now $\varphi f \succ \varphi c$ and hence φ preserves dominance on $\Phi_{\{f,c,d\}}$. Thus $\varphi c \succ \varphi d$.

We may suppose that $F(b, c) = 0$. By Lemma 2.10 there exists $f \in \Phi$ with $F(f, b) \neq 0$ and $F(f, c) \neq 0$. Since $\Phi_{\{a,b,f\}}$ is infinite and irreducible, there exists $g \in \Phi_{\{a,b,f\}}$ with $g \succ f$. Furthermore φ preserves dominance on $\Phi_{\{a,b,f\}}$ and so $\varphi g \succ \varphi f$. Now $\Phi_{\{g,f,c\}}$ is infinite and irreducible and so there exists $h \in \Phi_{\{g,f,c\}}$ with $h \succ c$. Moreover, φ preserves dominance on $\Phi_{\{g,f,c\}}$ (since $\varphi g \succ \varphi f$) and so $\varphi h \succ \varphi c$. Now applying the same argument to $\Phi_{\{h,c,d\}}$ we conclude that $\varphi c \succ \varphi d$, as required.

Since φ preserves dominance, the set $\{\, a \in \Phi_1^+ \mid \varphi a \in \Phi_2^- \,\}$ is finite, by Proposition 3.13. Now choose $w \in W$ so as to minimize the size of the set $S = \{\, a \in \Phi_1^+ \mid \varphi w a \in \Phi_2^- \,\}$. If $S \neq \emptyset$ then S must contain a simple root b, and then it is clear that

$$\{\, a \in \Phi_1^+ \mid (\varphi w r_b)a \in \Phi_2^- \,\} = r_b(S \setminus \{b\})$$

has fewer elements than S, contradicting the choice of w. So $(\varphi w)\Phi_1^+ = \Phi_2^+$, and hence $(\varphi w)\Pi_1 = \Pi_2$, as required. □

Of course, Theorem 1.2 is simply the case $\Phi_1 = \Phi_2$ of Theorem 4.1.

5. Proof of Theorem 1.1

Recall that our aim is to prove that if W is a Coxeter group such that $|\Pi| < \infty$ and $m_{ab} < \infty$ for all a and $b \in \Pi$, then $\mathrm{Out}(W)$ is a finite group.

Proof. Choose a standard geometric realization of W, so that Π is a basis for a real vector space V on which W acts in the manner described in § 2 above. Since $|\Pi| < \infty$ clearly $\mathcal{J} = \{\, J \subseteq \Pi \bigm| |W_J| < \infty \,\}$ is a finite set. Let I be the set of involutions of W and $\mathcal{P} = \{\, (r, s) \mid r, s \in I \text{ and } |rs| \text{ is finite} \,\}$. The group W acts on $\mathcal{P}$ by conjugation and $\mathrm{Aut}\, W$ permutes the orbits. By Theorem 2.5 if $(r, s) \in \mathcal{P}$ then there exists $w \in W$ and a $J \in \mathcal{J}$ such that $w\langle\, r, s \,\rangle w^{-1} \subseteq W_J$. Since $\mathcal{J}$ is finite and for each $J \in \mathcal{J}$ there are only finitely many pairs of involutions in W_J, it follows that W has only finitely many orbits on $\mathcal{P}$.

Let K be the kernel of the action of $\mathrm{Aut}\, W$ on the set of W-orbits on $\mathcal{P}$. Since the number of orbits is finite, the index of K in $\mathrm{Aut}(W)$ is finite. If a, $b \in \Pi$ then $(r_a, r_b) \in I$, since $m_{ab} < \infty$, and if $\sigma \in K$ then $(\sigma(r_a), \sigma(r_b))$ is in the same W orbit of I as (r_a, r_b). Hence for each $\sigma \in K$ there exists $w = w(\sigma, a, b) \in W$ such that

$$\sigma(r_a) = w r_a w^{-1}, \quad \sigma(r_b) = w r_b w^{-1}.$$

We may choose these elements so that $w(\sigma, a, b) = w(\sigma, b, a)$ for each two element subset $\{a, b\}$ of Π.

Given $\sigma \in K$ and $a \in \Pi$, define $\hat{\sigma}(a) = w(a)$, where $w = w(\sigma, a, a)$. Note that if $v = w(\sigma, a, b)$, where $b \in \Pi$, then

$$r_{w(a)} = w r_a w^{-1} = \sigma(r_a) = v r_a v^{-1} = r_{v(a)}$$

and so $v(a) = \pm w(a)$. It follows that $\hat{\sigma}(a) = \pm w(\sigma, a, b)(a)$ for all $b \in \Pi$.

Let $a, b \in \Pi$. Then for each $\sigma \in K$ there exist signs $\varepsilon(\sigma, a, b) = \pm 1$ such that

$$\begin{aligned} F(\hat{\sigma}(a), \hat{\sigma}(b)) &= \varepsilon(\sigma, a, b) F\big(w(\sigma, a, b)(a), w(\sigma, a, b)(b)\big) \\ &= \varepsilon(\sigma, a, b) F(a, b). \end{aligned}$$

We show that if $\sigma, \tau \in K$ and $\varepsilon(\sigma, a, b) = \varepsilon(\tau, a, b)$ for all $a, b \in \Pi$, then $\tau \in (\mathrm{Aut}_V(W) \cap K)\sigma$.

By Lemma 2.8 $\{\, \hat\sigma(a) \mid a \in \Pi \,\}$ and $\{\, \hat\tau(a) \mid a \in \Pi \,\}$ both span V, and hence are both bases of V. Furthermore, $F(\hat\sigma(a), \hat\sigma(b)) = F(\hat\tau(a), \hat\tau(b))$ for all $a, b \in \Pi$. Hence there exists an orthogonal transformation $T\colon V \to V$ such that $T\hat\sigma(a) = \hat\tau(a)$ for all $a \in \Pi$. Now

$$T\sigma(r_a)T^{-1} = Tr_{\hat\sigma(a)}T^{-1} = r_{T\hat\sigma(a)} = r_{\hat\tau(a)} = \tau(r_a) = (\tau\sigma^{-1})(\sigma(r_a))$$

and since the elements $\sigma(r_a)$ generate W it follows that $TwT^{-1} = (\tau\sigma^{-1})(w)$ for all $w \in W$. Hence $\tau\sigma^{-1} \in \mathrm{Aut}_V(W)$; so $\tau \in (\mathrm{Aut}_V(W) \cap K)\sigma$, as claimed.

As there only finitely many possibilities for the signs $\varepsilon(\sigma, a, b)$ for each σ, it follows that the index of $\mathrm{Aut}_V(W) \cap K$ in K is finite. Since the index of K in $\mathrm{Aut}(W)$ is finite, the index of $\mathrm{Aut}_V(W)$ in $\mathrm{Aut}(W)$ is also finite. So it remains to show that the index of $\mathrm{Inn}(W)$ in $\mathrm{Aut}_V(W)$ is finite. But if $\sigma \in \mathrm{Aut}_V(W)$ then there exists $\varphi \in O(V, F)$ such that $\sigma(x) = \varphi x \varphi^{-1}$ for all $x \in W$, and by Theorem 1.2 there exists a permutation π of Π, a sign $\varepsilon = \pm 1$ and a $w \in W$ such that $\varphi a = w(\varepsilon\pi a)$ for all $a \in \Pi$. It follows that $F(\varepsilon\pi a, \varepsilon\pi b) = F(a, b)$ for all a, $b \in \Pi$, and so there exists a $T \in O(V, F)$ satisfying $Ta = \varepsilon\pi a$ for all $a \in \Pi$. Now $\varphi = wT$, and, furthermore, since T preserves Φ there exists $\tau \in \mathrm{Aut}_V(W)$ given by $\tau(x) = TxT^{-1}$ for all $x \in W$. Thus

$$\sigma(x) = \varphi x \varphi^{-1} = wTxT^{-1}w^{-1} = w\tau(x)w^{-1}$$

for all $x \in W$, whence $\sigma \in \mathrm{Inn}(W)\tau$. As there are only finitely many possibilities for the permutation π and the sign ε, there are only finitely many possibilities for τ, and so the index of $\mathrm{Inn}(W)$ in $\mathrm{Aut}_V(W)$ is finite, as required.

$\square$

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