ORDERS IN POWER SEMIGROUPS

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Abstract

We consider the Fountain-Petrich notion of a semigroup of quotients. If Q is a semigroup of quotients of S we also say that S is an *order* in Q. We concentrate here on orders in the *restricted power semigroup* $\mathcal{P}(Q)$ of a semigroup Q, where $\mathcal{P}(Q)$ consists of all non-empty finite subsets of Q under the natural multiplication. Our first result shows that if S is a commutative cancellative semigroup with group of quotients G, then $\mathcal{P}(S)$ is an order in $\mathcal{P}(G)$. In the latter part of the paper we give necessary and sufficient conditions for $\mathcal{P}(S)$ to be an order in $\mathcal{P}(Q)$, where Q is a semilattice of torsion-free commutative groups and S is an order in Q.

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1 Introduction

In this paper we consider examples of orders in restricted power semigroups, where for any semigroup S the *restricted power semigroup* $\mathcal{P}(S)$ is given by $\mathcal{P}(S) = \{X \subseteq S : 1 \leq |X| < \aleph_0\}$ with multiplication $XY = \{xy : x \in X, y \in Y\}$ for all $X, Y \in \mathcal{P}(S)$. We use the notion of order introduced by Fountain and Petrich in [2] which first appears in the form used here in [3]. If S is a subsemigroup of Q then S is an *order* in Q and Q is a *semigroup of quotients* of S if any $q \in Q$ can be written as $q = a^*b = cd^*$ where $a, b, c, d \in S$ and $a^*(d^*)$ is the inverse of $a(d)$ in a subgroup of Q, and in addition, all elements of S satisfying a weak cancellability condition called *square-cancellability* lie in a subgroup of Q.

It is clear that the concept of a semigroup of quotients extends that of the group of quotients G of a commutative cancellative semigroup S. Our first result shows that for such an S and G, the restricted power semigroup $\mathcal{P}(S)$ is an order in $\mathcal{P}(G)$.

In the latter part of the paper we turn our attention to orders in a semigroup Q which is a semilattice Y of commutative groups $G_\alpha, \alpha \in Y$. To handle the idempotents of $\mathcal{P}(Q)$ we make the further assumption that the groups $G_\alpha, \alpha \in Y$, are torsion-free. We find a necessary and sufficient condition for an order S in such a semigroup Q to have the property that $\mathcal{P}(S)$ is an order in $\mathcal{P}(Q)$. In fact we prove a slightly stronger result. We say that a subsemigroup S of Q is a *weak order* in Q if any $q \in Q$ can be written as $q = a^*b = cd^*$ where $a, b, c, d \in S$ and $a^*(d^*)$ is the inverse of $a(d)$ in a subgroup of Q. Proposition 4.1 gives a necessary and sufficient condition on S such that $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$, where S is an order in Q and Q is a semilattice of torsion-free commutative groups. We then show that for such an S and Q, if $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$ then necessarily $\mathcal{P}(S)$ is an order in $\mathcal{P}(Q)$.

Section 2 consists of some preliminary definitions and results concerning orders and Green's relations in certain restricted power semigroups. Section 3 considers restricted power semigroups of orders in commutative groups. In our last section we turn our attention to restricted power semigroups of orders in semilattices of commutative torsion-free groups, and prove the results mentioned in the previous paragraph.

We comment that if S is an order in Q then S^0 (the semigroup S with a zero adjoined) is clearly an order in Q^0. Moreover it is easy to see that if T is an order in Q^0 then

$T = S^0$ where S is an order in Q. Including the empty set in $\mathcal{P}(S)$ would correspond to considering $\mathcal{P}(S)^0$. Thus excluding the empty set from $\mathcal{P}(S)$ does not affect our results in any essential way, it is merely convenient.

2 Preliminaries

We assume the reader has a basic knowledge of semigroup theory as in the early chapters of [1] or [4]. Any undefined notation or concepts may be found in these references. We deviate from standard notation in denoting by a^* the group inverse, where it exists, of an element a of a semigroup Q. That is, a^* exists if and only if a lies in a subgroup of Q, and the inverse of a in this subgroup is a^*. By a famous result of Green [4, Theorem II 2.5], a^* exists if and only if a is related to its square by the relation $\mathcal{H}$. Moreover, where a^* exists it is unique, being the inverse of a in the group $\mathcal{H}$-class H_a.

Let a be an element of a semigroup S. Then a is *square-cancellable* if for all $x, y \in S^1$

$$xa^2 = ya^2 \text{ implies } xa = ya$$

and

$$a^2x = a^2y \text{ implies } ax = ay.$$

It is clear that if a lies in a subgroup of an oversemigroup of S, then a is square-cancellable. We insist that if S is an order in Q, then all such elements lie in subgroups of Q. The set of square-cancellable elements of S is denoted by $\mathcal{S}(S)$.

Let Q be a semilattice Y of commutative groups $G_\alpha, \alpha \in Y$. We follow the usual practice of abbreviating $\{g\}H$ where $g \in Q$ and $H \in \mathcal{P}(Q)$ by gH. Further, we make the notational convention that e_α denotes the idempotent of $G_\alpha, \alpha \in Y$ and also e_i denotes the idempotent of $G_{\alpha_i}, \alpha_i \in Y, i \in \mathbf{N}$. For $X \in \mathcal{P}(Q)$ and $\alpha \in Y$ put

$$X_\alpha = X \cap G_\alpha$$

and

$$\text{supp } X = \{\alpha \in Y : X_\alpha \neq \emptyset\}, \ \max X = \ \max\{\text{supp } X\}.$$

Lemma 2.1 *For $H, K \in \mathcal{P}(Q)$, if $H \;\mathcal{H}\; K$ then supp H = supp K and $|H_\alpha| = |K_\alpha|$ for all $\alpha \in Y$.*

Proof Suppose that $H \ \mathcal{H} \ K$. Then $H = K$ (in which case the result is clearly true) or there exist $S, T \in \mathcal{P}(Q)$ with $H = SK$ and $K = TH$. Let $\alpha \in \max H$. Since $H_\alpha \subseteq SK$ there exists $\beta \in \max K$ with $\alpha \leq \beta$. But $K_\beta \subseteq TH$ so $\beta \leq \gamma$ for some $\gamma \in \max H$. Thus $\alpha = \beta = \gamma$ so that $\alpha \in \max K$. With the dual argument this gives that $\max H = \max K$.

Now let $\delta \in \text{supp } H$. Then $\delta \leq \alpha$ for some $\alpha \in \max K$. Since $K_\alpha \subseteq TH$ there exists $\epsilon \in \text{supp } T$ with $\alpha \leq \epsilon$. Let $t \in T_\epsilon$, then $tH_\delta \subseteq K_\delta$ so $\delta \in \text{supp } K$; hence $\text{supp } H = \text{supp } K$. Further, $|H_\delta| = |tH_\delta| \leq |K_\delta|$ and with the dual we obtain that $|H_\delta| = |K_\delta|$.

The following corollaries are now easy to see.

Corollary 2.2 *[5] Let G be a commutative group. Then for any $H, K \in \mathcal{P}(G)$, $H \ \mathcal{H} \ K$ if and only if $H = gK$ for some $g \in G$.*

Corollary 2.3 *[5] Let G be a commutative group. The following conditions are equivalent for $H \in \mathcal{P}(G)$:*
(i) $H \ \mathcal{H} \ H^2$;
(ii) $H^2 = gH$ for some $g \in G$;
(iii) $H^2 = hH$ for any $h \in H$;
(iv) $|H^2| = |H|$.

Corollary 2.4 *[5] Let G be a commutative group. Then $E \in \mathcal{P}(G)$ is idempotent if and only if E is a finite subgroup. Moreover if $H \in \mathcal{P}(G)$ then $H \ \mathcal{H} \ E$ if and only if H is a coset of E.*

We now return to the case where Q is a semilattice Y of commutative groups $G_\alpha, \alpha \in Y$. To adequately describe the idempotents of $\mathcal{P}(Q)$ and the elements of $\mathcal{P}(Q)$ lying in group $\mathcal{H}$-classes, we make henceforth the additional assumption that each G_α is torsion-free. Clearly then the idempotents of $\mathcal{P}(Q)$ are the finite subsemigroups of $E(Q)$. By Lemma 2.1, if $H \in \mathcal{P}(Q)$ and $H \ \mathcal{H} \ H^2$ then $|H_\alpha| = 1$ for all $\alpha \in \text{supp } H$. Also $\text{supp } H$ must be a subsemilattice of Y. We write $H = \{h_1, ..., h_n\}$ where $|H| = n$ and $h_i \in G_{\alpha_i}, 1 \leq i \leq n$. Then $H \ \mathcal{H} \ E$ where $E = \{e_1, ..., e_n\}$ and $e_i = e_i^2 \in G_{\alpha_i}, 1 \leq i \leq n$. Further, if $i, j \in \{1, ..., n\}$ and $\alpha_i \geq \alpha_j$ then from $EH = H$ we have $e_j h_i = h_j$. This leads us to the following definition.

An element H of $\mathcal{P}(Q)$ is *balanced* if $|H_\alpha| = 1$ for all $\alpha \in$ supp H and $\alpha, \beta \in$ supp H with $\alpha \geq \beta$ implies $e_\beta h_\alpha = h_\beta$ where $\{h_\alpha\} = H_\alpha$ and $\{h_\beta\} = H_\beta$.

Lemma 2.5 *An element H of $\mathcal{P}(Q)$ is in a subgroup of $\mathcal{P}(Q)$ if and only if supp H is a subsemilattice of Y and H is balanced. In this case, writing $H = \{h_1, ..., h_n\}$ where supp $H = \{\alpha_1, ..., \alpha_n\}$ and $H_{\alpha_i} = \{h_i\}, 1 \leq i \leq n$, then $H^* = \{h_1^*, ..., h_n^*\}$ and $HH^* = \{e_1, ..., e_n\}$.*

Proof We have seen that if $H \in \mathcal{P}(Q)$ and $H \; \mathcal{H} \; H^2$ then supp H is a subsemilattice of Y and H is balanced.

Conversely, suppose these conditions hold. Put $H = \{h_1, ..., h_n\}$ where supp $H = \{\alpha_1, ..., \alpha_n\}$ and $\{h_i\} = H_{\alpha_i}, 1 \leq i \leq n$. Certainly $E = \{e_1, ..., e_n\} \in E(\mathcal{P}(Q))$ and $H \subseteq EH$; consider $i, j \in \{1, ..., n\}$ and put $\alpha_k = \alpha_i\alpha_j$. Then $e_ih_j = e_ie_kh_j = e_kh_j = h_k$ as $\alpha_k \leq \alpha_j$ and H is balanced. So $EH \subseteq H$ and $EH = H$. Putting $K = \{h_1^*, ..., h_n^*\}$ we have K is balanced so that also $EK = K$.

Clearly $E \subseteq HK$. Let $i, j \in \{1, ..., n\}$ and again put $\alpha_k = \alpha_i\alpha_j$. Then $h_ih_j^* = h_ie_k(e_kh_j^*)$ and as Q is an inverse semigroup this gives $h_ih_j^* = h_ie_k(e_kh_j)^* = h_kh_k^* = e_k$ and it follows that $E = HK$. Thus H is in a group $\mathcal{H}$-class of $\mathcal{P}(Q)$ with $H^* = \{h_1^*, ..., h_n^*\}$ and $HH^* = \{e_1, ..., e_n\}$ as required.

3 Power semigroups of orders in commutative groups

In this section we show that if S is an order in a commutative group G then $\mathcal{P}(S)$ is an order in $\mathcal{P}(G)$. For such a G, $\mathcal{P}(G)$ has identity $\{e\}$, where $e^2 = e$, and group of units the singletons of G. If S is an order in G then as is well known the Common Denominator Theorem holds, that is, given any $g_1, ..., g_n \in G$ there exist $a, b_1, ..., b_n \in S$ with $g_i = a^{-1}b_i, 1 \leq i \leq n$. Thus we immediately have the following lemma.

Lemma 3.1 *Let S be an order in a commutative group G. Then $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(G)$.*

The more difficult task is to show that if S is an order in a commutative group G, then every square-cancellable element H of $\mathcal{P}(S)$ lies in a subgroup of $\mathcal{P}(G)$. By Corollary 2.3 this is equivalent to showing that $|H^2| = |H|$.

Theorem 3.2 *Let S be an order in a commutative group G. Then $\mathcal{P}(S)$ is an order in $\mathcal{P}(G)$.*

Proof From Lemma 3.1, $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(G)$. Clearly the singletons of S are square-cancellable and lie in a subgroup (the group of units) of $\mathcal{P}(G)$.

At this point it is useful to make the following notational convention. If $x \in X$ and $\bar{y} \in X^n$ where $X \in \mathcal{P}(S)$, then by saying that x is an X-*factor* of $\bar{y}$, we mean that there exist $x_1, ..., x_{n-1} \in X$ such that $xx_1...x_{n-1} = \bar{y}$.

Suppose now that $X = \{x, y\} \in \mathcal{S}(\mathcal{P}(S))$ where $x \neq y$. Every element of X^5 contains x^3 or y^3 as an X-factor. Indeed, $X^5 = \{x^3, y^3\}X^2$. Since X is square-cancellable, $X^4 = \{x^3, y^3\}X$. Looking at the element x^2y^2 of X^4 we have $x^2y^2 = x^3x, x^3y, y^3x$ or y^3y which each imply that $y^2 = x^2$. It follows that $X^3 = x^2X$ and so

$$|X| \leq |X^2| \leq |X^3| = |x^2X| = |X|$$

so that $|X| = |X^2|$ and by Corollary 2.3, X lies in a subgroup of $\mathcal{P}(G)$.

Now let $n \geq 3$ and suppose that $X = \{x_1, ..., x_n\} \in \mathcal{S}(\mathcal{P}(S))$ where $|X| = n$. Every element of X^{n^2} contains an X-factor x_i^n for some $i \in \{1, ..., n\}$. Hence

$$X^{n^2} = \{x_1^n, ..., x_n^n\}X^{n^2-n}$$

and using the fact that X is square-cancellable we obtain $X^{n+1} = \{x_1^n, ..., x_n^n\}X$.

Suppose that $X^{n+1} = \{x_1^n, ..., x_m^n\}X$ for some m with $1 < m \leq n$. We show that (with some re-labelling), $X^{n+1} = \{x_1^n, ..., x_{m-1}^n\}X$. Since it is certainly true that $X^{n+1} = \{x_1^n, ..., x_m^n\}X$ with $m = n$, we obtain in $n-1$ steps that $X^{n+1} = x_1^nX$. As above it follows that $|X| = |X^2|$ so that X lies in a subgroup of $\mathcal{P}(G)$.

Given $X^{n+1} = \{x_1^n, ..., x_m^n\}X$ with $1 < m \leq n$ we show that x_i is an X-factor of x_j^n for some $i, j \in \{1, ..., m\}$ with $i \neq j$. Consider first the element $x_1x_2...x_{m-1}x_m^2x_{m+1}..x_n$ of X^{n+1}. This element has the following possible forms:

$$x_1...x_{m-1}x_m^2x_{m+1}...x_n = \begin{cases} & x_i^{n+1} \quad \text{where} & 1 \leq i < m \\ \text{or} & x_i^nx_j & 1 \leq i < m, 1 \leq j \leq m, i \neq j \\ \text{or} & x_i^nx_m & 1 \leq i < m \\ \text{or} & x_i^nx_j & 1 \leq i < m, m < j \leq n \\ \text{or} & x_m^nx_i & 1 \leq i < m \\ \text{or} & x_m^{n+1} & \\ \text{or} & x_m^nx_j & m < j \leq n. \end{cases}$$

In the first four cases we have x_m is an X-factor of x_i^n for some $i \in \{1, ..., m-1\}$. In the last two, x_1 is an X-factor of x_m^n.

In the fifth case,

$$x_1 ... x_{m-1} x_m^2 x_{m+1} ... x_n = x_m^n x_i,$$

where $1 \leq i < m$, giving

$$x_1 ... x_{i-1} x_{i+1} ... x_{m-1} x_m^2 x_{m+1} ... x_n = x_m^n$$

so that, unless $m = 2$, we have x_j is an X-factor of x_m^n for some $j \in \{1, ..., m-1\}$. In the case where $m = 2$, $x_2^2 x_3 ... x_n = x_2^n$ so that $x_3 ... x_n = x_2^{n-2}$. At this point we look at the element $x_1^2 x_2 ... x_n$. Again using the hypothesis that $X^{n+1} = \{x_1^n, ..., x_m^n\}X$ we have that

$$x_1^2 x_2 ... x_n = \begin{cases} & x_1^{n+1} & & \\ \text{or} & x_1^n x_2 & & \\ \text{or} & x_1^n x_j & \text{where} & 2 < j \leq n \\ \text{or} & x_2^n x_1 & & \\ \text{or} & x_2^{n+1} & & \\ \text{or} & x_2^n x_j & & 2 < j \leq n. \end{cases}$$

Except for the second case, this yields that x_1 is an X-factor of x_2^n or x_2 is an X-factor of x_1^n. In the second case we have $x_3 ... x_n = x_1^{n-2}$. Since also we know that $x_3 ... x_n = x_2^{n-2}$ we have $x_1^{n-2} = x_2^{n-2}$ and so x_1 is certainly an X-factor of x_2^n.

We have now verified that there exist $i, j \in \{1, ..., m\}$ with $i \neq j$ such that x_i is an X-factor of x_j^n. For convenience we re-label $x_1, ..., x_m$ so that $i = 1$ and $j = m$, that is, x_1 is an X-factor of x_m^n. Since every product of $n+1$ elements of X contains an X-factor x_i^n for some $i \in \{1, ..., m\}$, by raising X to a high enough power t we have that every element of X^t contains an X-factor $x_i^{n^2}$ for some $i \in \{1, ..., m\}$. But if $i = m$ then as x_1 is an X-factor of x_m^n we have that x_1^n is an X-factor of $x_m^{n^2}$. Thus every element of X^t contains an X-factor x_i^n for some $i \in \{1, ..., m-1\}$. Then $X^t = \{x_1^n, ..., x_{m-1}^n\}X^{t-n}$ and as $X \in \mathcal{S}(\mathcal{P}(S))$, $X^{n+1} = \{x_1^n, ..., x_{m-1}^n\}X$. This completes the proof that X lies in a subgroup of $\mathcal{P}(G)$.

4 Power semigroups of orders in semilattices of torsion-free commutative groups

Let Q be a semilattice Y of torsion-free commutative groups $G_\alpha, \alpha \in Y$. We recall from Section 2 that $E \in \mathcal{P}(Q)$ is idempotent if and only if E is a finite subsemilattice of the idempotents of Q. Further, $H \in \mathcal{P}(Q)$ is in a subgroup of $\mathcal{P}(Q)$ if and only if supp H is a subsemilattice of Y and H is *balanced.*

Suppose now that S is an order in Q. As shown in Theorem 3.1 of [3], S is a semilattice Y of commutative cancellative semigroups $S_\alpha = S \cap G_\alpha$, and S_α is an order in $G_\alpha, \alpha \in Y$. It is not true that for any such S and Q, $\mathcal{P}(S)$ is an order in $\mathcal{P}(Q)$. In fact, as we show below, $\mathcal{P}(S)$ need not be a weak order in $\mathcal{P}(Q)$.

At this stage it is useful to make an elementary remark about weak orders, which we will use without further comment. Given a commutative semigroup Q we write $\leq_{\mathcal{H}}$ for the preorder associated with Green's relation $\mathcal{H}$. Now if S is a weak order in Q and $q \in Q$, then $q = a^*b$ for some $a, b \in S$. But then $q = (a^2)^*ab$ and $ab \leq_{\mathcal{H}} a^2$. Thus given $q \in Q$, we may write q as $q = c^*d$ for some $c, d \in S$ with $d \leq_{\mathcal{H}} c$ in Q. Further, if $h, k \in Q$ and k lies in a subgroup of Q then from $h \leq_{\mathcal{H}} k$ we deduce that $hk\mathcal{H}h$. So if $q = c^*d$ with $d \leq_{\mathcal{H}} c$ we have $q\mathcal{H}d$; it follows that S intersects every $\mathcal{H}$-class of Q.

Consider the three element semilattice $Q = \{\alpha, \beta, \gamma\}$ where $\alpha \leq \beta$ and $\alpha \leq \gamma$. Clearly Q is an order in Q; if $\mathcal{P}(Q)$ were a weak order in $\mathcal{P}(Q)$ then $\{\beta, \gamma\}$ could be written as $\{\beta, \gamma\} = U^*V$ for some $U, V \in \mathcal{P}(Q)$ with $V \leq_{\mathcal{H}} U$. As commented above, $V\mathcal{H}\{\beta, \gamma\}$ so that by Lemma 2.1, $V = \{\beta, \gamma\}$ and $\{\beta, \gamma\} = U\{\beta, \gamma\}$. But no such $U \in \mathcal{P}(Q)$ exists. Thus $\mathcal{P}(Q)$ is not a weak order in itself.

For the remainder of this section suppose that S is an order in Q, where Q is a semilattice Y of torsion-free commutative groups $G_\alpha, \alpha \in Y$. Our aim is to give necessary and sufficient conditions on S such that $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$. We then show that in this case, square-cancellable elements of $\mathcal{P}(S)$ lie in subgroups of $\mathcal{P}(Q)$ so that $\mathcal{P}(S)$ is an order in $\mathcal{P}(Q)$.

Proposition 4.1 *The semigroup $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$ if and only if (*) holds: (*) given any distinct $\alpha_1, ..., \alpha_n \in Y$ with $s_i \in S_{\alpha_i}, 1 \leq i \leq n$, there exists $\alpha \in Y$ with $\alpha \geq \alpha_i, 1 \leq i \leq n$ and elements $t_i \in S_{\alpha_i}, a \in S_\alpha$ such that $ae_{\alpha_i} = s_i t_i, 1 \leq i \leq n$.*

Proof Suppose that (*) holds and $X \in \mathcal{P}(Q)$; say $X = \{x_{ij} : 1 \leq i \leq n, 1 \leq j \leq m_i\}$ where $x_{i1}, ..., x_{im_i} \in G_{\alpha_i}, 1 \leq i \leq n$ and $\alpha_1, ..., \alpha_n$ are distinct elements of Y. Since S_{α_i} is an order in $G_{\alpha_i}, 1 \leq i \leq n$, there are elements $s_i, y_{i1}, ..., y_{im_i} \in S_{\alpha_i}$ with $x_{ij} = s_i^* y_{ij}, 1 \leq j \leq m_i$.

Let $\alpha \in Y$ and $a, t_1, ..., t_n \in S$ be chosen as in (*). For $i \in \{1, ..., n\}$ and $j \in \{1, ..., m_i\}$

we have

$$x_{ij} = s_i^* y_{ij} = s_i^* t_i^* t_i y_{ij} = (s_i t_i)^* t_i y_{ij} = (ae_{\alpha_i})^* t_i y_{ij} = a^* e_{\alpha_i} t_i y_{ij} = a^* t_i y_{ij}.$$

So

$$X = a^* \{t_i y_{ij} : 1 \leq i \leq n, 1 \leq j \leq m_i\},$$

giving that $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$.

Conversely, suppose that $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$. First we take mutually incomparable elements $\beta_1, ..., \beta_n$ in Y. We show that $\{\beta_1, ..., \beta_n\}$ has an upper bound in Y.

Pick $x_i \in S_{\beta_i}, 1 \leq i \leq n$. Then $\{x_1, ..., x_n\} = H^* K$ for some $H, K \in \mathcal{P}(S)$ with $K \leq_{\mathcal{H}} H$ in $\mathcal{P}(Q)$. Thus $H^* K \ \mathcal{H} \ K$ so that by Lemma 2.1, $K = \{y_1, ..., y_n\}$ where $y_i \in S_{\beta_i}, 1 \leq i \leq n$. Since $x_1 \in H^* K$ we have that $x_1 = pk$ for some $p \in H^*$ and $k \in K$; if $p \in G_\gamma$ then $\beta_1 \leq \gamma$. If $i \in \{2, ..., n\}$ then $py_i \in H^* K = \{x_1, ..., x_n\}$ and $py_i \in G_\delta$ where $\delta = \gamma \beta_i$. If $\beta_i \not\leq \gamma$ then $\delta < \beta_i$. But this is impossible since $\delta = \beta_j$ for some j and $\beta_1, ..., \beta_n$ are mutually incomparable. Thus $\beta_i \leq \gamma$ for all $i \in \{1, ..., n\}$.

Now suppose that $\alpha_1, ..., \alpha_n \in Y$ are distinct and $s_i \in S_{\alpha_i}, 1 \leq i \leq n$. By the above there exists $\gamma \in Y$ with $\gamma \geq \alpha_i, 1 \leq i \leq n$. Choose $x \in S_\gamma$, where if $\gamma = \alpha_i$ we take $x = s_i$. Then $\{x^*, s_1^*, ..., s_n^*\} = X^* Y$ for some $X, Y \in \mathcal{P}(S)$ with $Y \leq_{\mathcal{H}} X$ in $\mathcal{P}(Q)$, so that $Y \ \mathcal{H} \ \{x^*, s_1^*, ..., s_n^*\}$. From Lemma 2.1, $Y = \{y, t_1, ..., t_n\}$ where $y \in S_\gamma$ and $t_i \in S_{\alpha_i}, 1 \leq i \leq n$; if $\gamma = \alpha_i$ then also $y = t_i$.

Since $x^* \in X^* Y$ we have that $x^* = zw$ for some $z \in X^*$ and $w \in Y$. If $z \in G_\alpha$ then $\gamma \leq \alpha$ so that $\alpha_i \leq \alpha$ for all $i \in \{1, ..., n\}$. By Lemma 2.5, $z = a^*$ for some $a \in S_\alpha$. Now for $i \in \{1, ..., n\}$, $a^* t_i \in X^* Y = \{x^*, s_1^*, ..., s_n^*\}$ and $a^* t_i \in G_{\alpha_i}$ so that $a^* t_i = s_i^*$. The semigroup Q is inverse so that $a t_i^* = s_i$ and $a e_{\alpha_i} = a t_i^* t_i = s_i t_i, 1 \leq i \leq n$. Thus (*) holds.

We note that in the above proposition we showed that if $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$ then any $X \in \mathcal{P}(Q)$ can be written as $X = a^* Y$ where $a \in S_\alpha, Y \in \mathcal{P}(S)$ and $\beta \leq \alpha$ for all $\beta \in \operatorname{supp} Y$. Hence $X \ \mathcal{H} \ Y$ in $\mathcal{P}(Q)$. We now set out to prove that if $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$ then it is necessarily an order.

Lemma 4.2 *If $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$ and $X \in \mathcal{P}(S)$ is square-cancellable in $\mathcal{P}(S)$, then X is square-cancellable in $\mathcal{P}(Q)$.*

Proof Suppose that $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$ and $X \in \mathcal{S}(\mathcal{P}(S))$. Let $U, V \in \mathcal{P}(Q)$ with $UX^2 = VX^2$. We can express $U \cup V$ as $U \cup V = a^*W$ for some $a \in S_\alpha$ and $W \in \mathcal{P}(S)$ such that $\beta \leq \alpha$ for all $\beta \in$ supp W. Clearly there exist subsets W_1, W_2 of W such that $U = a^*W_1$ and $V = a^*W_2$. Now $a^*W_1X^2 = a^*W_2X^2$ so that $e_\alpha W_1X^2 = e_\alpha W_2X^2$; but e_α is an identity for all elements of W, so that $W_1X^2 = W_2X^2$. Since $X \in \mathcal{S}(\mathcal{P}(S))$ we have $W_1X = W_2X$ and so $a^*W_1X = a^*W_2X$ and $UX = VX$.

If $U \in \mathcal{P}(Q)$ and $UX^2 = X^2$, then in view of (*) $UX^2 = e_\alpha X^2$ where $\alpha \geq \beta$ for all $\beta \in$ supp X. Thus $UX = e_\alpha X = X$. Hence X is square-cancellable in $\mathcal{P}(Q)$.

For $X \in \mathcal{P}(Q)$ we put $\theta(X) = |\text{supp } X|$.

Lemma 4.3 *Suppose that $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$ and $X \in \mathcal{S}(\mathcal{P}(S))$. If $\alpha \in$ max X then $|X_\alpha| = 1$.*

Proof If $\theta(X) = 1$ then $X \subseteq S_\alpha$ for some $\alpha \in Y$ and certainly $X \in \mathcal{S}(\mathcal{P}(S_\alpha))$ so that by Theorem 3.2 X lies in a subgroup of $\mathcal{P}(G_\alpha)$. Since $\{e_\alpha\}$ is the only idempotent of $\mathcal{P}(G_\alpha)$ we have by Corollary 2.4 that $|X| = 1$.

Suppose now that $\theta(X) > 1$. Take $\alpha \in$ max X; we aim to show that $X_\alpha \in \mathcal{S}(\mathcal{P}(S_\alpha))$, so that as above $|X_\alpha| = 1$. Put $Y = X \setminus X_\alpha$ and note that if $y \in Y$ and $x \in X$ then $yx \notin X_\alpha$. Let $M =$ max YX_α. Suppose now that $U, V \in \mathcal{P}(S_\alpha)$ and $UX_\alpha^2 = VX_\alpha^2$. Put

$$\overline{U} = U \cup \bigcup\{e_\gamma U : \gamma \in M\} \cup \bigcup\{e_\gamma V : \gamma \in M\}$$

and

$$\overline{V} = V \cup \bigcup\{e_\gamma U : \gamma \in M\} \cup \bigcup\{e_\gamma V : \gamma \in M\}.$$

Then

$$\overline{U}X^2 = UX^2 \cup T = U(X_\alpha \cup Y)^2 \cup T = U(X_\alpha^2 \cup X_\alpha Y \cup Y^2) \cup T,$$

where

$$T = (\bigcup\{e_\gamma U : \gamma \in M\} \cup \bigcup\{e_\gamma V : \gamma \in M\})X^2.$$

If $\beta \in$ supp $UX_\alpha Y$ then $\beta \leq \gamma$ for some $\gamma \in M$ so that if $x \in (UX_\alpha Y)_\beta$ then $x = e_\gamma x \in T$. Similarly, $UY^2 \subseteq T$. Thus

$$\overline{U}X^2 = UX_\alpha^2 \cup T;$$

dually,

$$\overline{V}X^2 = VX_\alpha^2 \cup T$$

so that $\overline{U}X^2 = \overline{V}X^2$ and as $X \in \mathcal{S}(\mathcal{P}(S)), \overline{U}X = \overline{V}X$. It follows easily that $UX_\alpha = VX_\alpha$.

Finally, if $U \in \mathcal{P}(S_\alpha)$ and $UX_\alpha^2 = X_\alpha^2$, then taking $a \in S_\alpha$ we have $aUX_\alpha^2 = aX_\alpha^2$ and the above shows that $aUX_\alpha = aX_\alpha$; hence $UX_\alpha = X_\alpha$ and $X_\alpha \in \mathcal{S}(\mathcal{P}(S_\alpha)$.

We are now in a position to prove our final result.

Theorem 4.4 *The semigroup $\mathcal{P}(S)$ is an order in $\mathcal{P}(Q)$ if and only if S satisfies condition (*).*

Proof In view of Proposition 4.1 it only remains to show that if $\mathcal{P}(S)$ is a weak order in $\mathcal{P}(Q)$ then if $X \in \mathcal{S}(\mathcal{P}(S))$, X lies in a subgroup of $\mathcal{P}(Q)$. We use induction on $\theta(X)$ to show that X is balanced. If $\theta(X) = 1$ then $X \subseteq S_\alpha$ for some $\alpha \in Y$ and then from Theorem 3.2 X lies in a subgroup of $\mathcal{P}(Q)$ so that in particular, X is balanced by Lemma 2.5.

We make the inductive assumption that $\theta(X) > 1$ and if $Y \in \mathcal{S}(\mathcal{P}(S))$ with $\theta(Y) < \theta(X)$, then Y is balanced.

Let $\alpha \in \max X$; by Lemma 4.3, $|X_\alpha| = 1$. Suppose there exists $\beta \in \text{supp } X$ with $\beta < \alpha$. Write

$$X = \{s, t_1, ..., t_n, u_1, ..., u_m\}$$

where $s \in X_\alpha, t_1, ..., t_n$ are distinct elements of X_β and $u_1, ..., u_m$ are distinct elements of $X \setminus (X_\alpha \cup X_\beta)$. Pick any $a \in S_\beta$ and put $Z = aX$; since $\beta\alpha = \beta\beta$ we have that $\theta(Z) < \theta(X)$. It is then easy to see that $Z \in \mathcal{S}(\mathcal{P}(S))$ so that by the inductive assumption, Z is balanced. In particular, $|Z_\beta| = 1$. Thus $at_1 = ... = at_n$ and so $t_1 = ... = t_n$, that is, $|X_\beta| = 1$. Moreover $as = at_1$ so that $se_\beta = t_1$.

We can carry out this procedure for *any* $\alpha \in \max X$ and $\beta \in \text{supp } X$ with $\beta < \alpha$; it follows that $|X_\gamma| = 1$ for all $\gamma \in \text{supp } X$. Suppose now that $\beta, \gamma \in \text{supp } X$ and $\beta < \gamma$. Then $\beta < \gamma \leq \alpha$ for some $\alpha \in \max X$; from the above, if $x \in X_\beta, y \in X_\gamma$ and $z \in X_\alpha$, then $x = ze_\beta$ and $y = ze_\gamma$. Thus $ye_\beta = ze_\gamma e_\beta = ze_\beta = x$ and so X is balanced. By induction on $\theta(X)$, every element of $\mathcal{S}(\mathcal{P}(S))$ is balanced.

By Lemma 2.5, to show that $X \in \mathcal{S}(\mathcal{P}(S))$ lies in a subgroup of $\mathcal{P}(Q)$, it remains to show that supp X is a subsemilattice of Y. If $\theta(X) = n$ then supp X^n is a subsemilattice and also $X^n \in \mathcal{S}(\mathcal{P}(S))$. By the above, X^n is balanced so that by Lemma 2.5, X^n lies in a subgroup of $\mathcal{P}(Q)$. Further, if supp $X^n = \{\alpha_1, ..., \alpha_m\}$ then $EX^n = X^n$ where $E = \{e_1, ..., e_m\}$. We now call on Lemma 4.2 to give that $EX = X$. Let $\alpha, \beta \in$ supp X; then $\alpha = \alpha_i \in$ supp X^n and if $x \in X_\beta$ then $e_i x \in EX = X$ so that $\alpha\beta \in$ supp X. Thus supp X is a subsemilattice of Y and this completes the proof that X lies in a subgroup of $\mathcal{P}(Q)$. Hence $\mathcal{P}(S)$ is an order in $\mathcal{P}(Q)$.

Theorem 4.4 simplifies slightly in the case where Y is a chain.

Corollary 4.5 *If Y is a chain then $\mathcal{P}(S)$ is an order in $\mathcal{P}(Q)$ if and only if S satisfies condition (**):*
(**) for $\alpha, \beta \in Y$ with $\alpha > \beta$ and $x \in S_\alpha, y \in S_\beta$, there exists $p \in S_\alpha$ and $q \in S_\beta$ with $xpe_\beta = yq$.

Proof If $\mathcal{P}(S)$ is an order in $\mathcal{P}(Q)$ then by Theorem 4.4, (*) holds. Let $\alpha, \beta \in Y$ with $\alpha > \beta$ and $x \in S_\alpha, y \in S_\beta$. By (*) there exists $\gamma \in Y$ with $\gamma \geq \alpha > \beta$ and elements $p \in S_\alpha, q \in S_\beta$ and $a \in S_\gamma$ with $ae_\alpha = xp, ae_\beta = yq$. Then $xpe_\beta = ae_\alpha e_\beta = ae_\beta = yq$ and (**) holds.

Conversely, suppose that (**) holds. Let $\alpha_1, ..., \alpha_n$ be distinct elements of Y and $s_i \in S_{\alpha_i}, 1 \leq i \leq n$. Note that (*) holds trivially when $n = 1$. We assume inductively that $n > 1$ and (*) holds for all natural numbers less than n.

Without loss of generality we assume that $\alpha_1 > \alpha_2 > ... > \alpha_n$. By the inductive assumption there exists $\alpha' \in Y$ with $\alpha' \geq \alpha_2$ and elements $a' \in S_{\alpha'}, t'_i \in S_{\alpha_i}, 2 \leq i \leq n$ with $a'e_{\alpha_i} = s_i t'_i, 2 \leq i \leq n$. Let $b \in S_{\alpha_2}$; by (**) with $x = s_1, y = a'b$ there exist $p \in S_{\alpha_1}$ and $q \in S_{\alpha_2}$ with $s_1 p e_{\alpha_2} = a'bq$. Put $\alpha = \alpha_1, a = s_1 p, t_1 = p$ and $t_i = t'_i bq \in S_{\alpha_i}, 2 \leq i \leq n$. Then $ae_{\alpha_1} = s_1 p e_{\alpha_1} = s_1 p = s_1 t_1$ and for $i \in \{2, ..., n\}$ $ae_{\alpha_i} = s_1 p e_{\alpha_i} = s_1 p e_{\alpha_2} e_{\alpha_i} = a'bqe_{\alpha_i} = s_i t'_i bq = s_i t_i$. Thus (*) holds.

We end the paper by illustrating Theorem 4.4 and Corollary 4.5 with a number of examples.

Example 4.6 .

We consider several cases of orders in a two-element chain of torsion-free commutative groups.

Let G be the infinite cyclic group generated by an element z and let T be the infinite monogenic semigroup generated by z, so that T is an order in G.

Put $Q_1 = G^1$ and $S_1 = T^1$, so that S_1 is an order in Q_1. For any $z^n \in T$,

$$1\ 1\ z^0 \neq z\ z^n$$

so that (**) fails with $x = 1$ and $y = z$.

Now take $Q_2 = G^0$ and $S_2 = T^0$ so that again S_2 is an order in Q_2. Then for any $z^n \in T$,

$$z^n\ z\ 0 = 0\ 0$$

so that (**) holds.

With G and T as above let H be the infinite cyclic group generated by t and let U be the infinite monogenic semigroup generated by t, so that U is an order in H. Moreover if we let $\phi : G \to H$ be the trivial homomorphism then $S_3 = T \cup U$ is an order in $Q_3 = G \cup H$, where Q_3 is the two element chain of groups G and H with structure homomorphism ϕ. Now for any $r, s \geq 1$,

$$z\ z^r\ t^0 = t^0 \neq t\ t^s$$

so that (**) does not hold with $x = z$ and $y = t$.

On the other hand let $\psi : G \to H$ be the homomorphism given by $\psi(z) = t^n$ where $n \geq 1$. Put $S_4 = T \cup U$ and $Q_4 = G \cup H$ with structure homomorphism ψ. Again, S_4 is an order in Q_4. Let $r, s \geq 1$ and pick u with $n(r+u) > s$. Then

$$z^r\ z^u\ t^0 = z^{r+u}\ t^0 = \psi(z^{r+u})\ t^0 = (\psi(z))^{r+u}\ t^0 = t^{n(r+u)}\ t^0 = t^{n(r+u)} = t^s\ t^{n(r+u)-s}$$

which shows that (**) holds.

Moving now to the example of an order S in Q, where Q is an arbitrary semilattice Y of torsion-free commutative groups, we remark that by (*), if Y has a finite set of elements without an upper bound, then $\mathcal{P}(S)$ is not a weak order in $\mathcal{P}(Q)$. In particular, if Q is the three element semilattice with two incomparable elements then $\mathcal{P}(Q)$ is not a weak order in itself, as shown at the beginning of this section.

Example 4.7 .

Let Y be the four element semilattice $\{\alpha_1, ..., \alpha_4\}$ where $\alpha_1 \geq \alpha_2, \alpha_3 \geq \alpha_4$. For $i \in \{1, ..., 4\}$ let G_{α_i} be the infinite cyclic group generated by x_i and let T_{α_i} be the order in G_{α_i} consisting of the positive powers of x_i. Let Q be the semilattice Y of groups G_{α_i} with structure homomorphisms $\phi_{\alpha_i\alpha_j} : G_{\alpha_i} \to G_{\alpha_j}$ given by $\phi_{\alpha_i\alpha_j}(x_i) = x_j$. Putting S to be the subsemigroup of Q consisting of positive words we have that S is an order in Q.

Consider positive integers $a_1, ..., a_4$ and choose n with $n >$ max $\{a_1, ..., a_4\}$. Put $\alpha = \alpha_1, a = x_1^n$ and $t_i = x_i^{n-a_i}, 1 \leq i \leq 4$. Then

$$ae_i = x_1^n e_i = x_i^n = x_i^{a_i} x_i^{n-a_i} = x_i^{a_i} t_i,$$

thus showing that (*) holds.

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