Cancellative orders

David Easdown and Victoria Gould

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School of Mathematics and Statistics
University of Sydney
Sydney NSW 2006
Australia

Department of Mathematics
University of York
Heslington, York YO1 5DD
England

Abstract

A subsemigroup S of a semigroup Q is a *left order* in Q and Q is a *semigroup of left quotients* of S if every $q \in Q$ can be written as $q = a^*b$ for some $a, b \in S$, where a^* denotes the inverse of a in a subgroup of Q, and if, in addition, every square-cancellable element of S lies in a subgroup of Q. Perhaps surprisingly, a semigroup, even a commutative cancellative semigroup, can have non-isomorphic semigroups of left quotients. We show that if S is a cancellative left order in Q then Q is completely regular and the $\mathcal{D}$-classes of Q are left groups. The semigroup S is right reversible and its group of left quotients is the minimum semigroup of left quotients of S.

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1 Introduction

If S is a right reversible, cancellative semigroup, then as shown by Ore [CP], S has a group G of left quotients . That is, S is a subsemigroup of G and any element of G can be written as $a^{-1}b$ where a and b are elements of S. Inspired by methods of both classical ring and semigroup theory, Fountain and Petrich in 1982 [FP] extended the notion of *group* of left quotients of a semigroup S to that of *semigroup* of left quotients of S. Their approach, which we adopt here, is to consider inverses of elements in *any* subgroup of a semigroup and not just the group of units (which may not exist).

Let S be a subsemigroup of a semigroup Q. Then Q is a *semigroup of left quotients* of S if every $q \in Q$ can be written as $q = a^*b$ for some $a, b \in S$, where a^* denotes the inverse of a in a subgroup of Q and if, in addition, every element of S satisfying a weak cancellation condition known as square cancellability lies in a subgroup of Q. Clearly if S has a group of left quotients G then G is also a semigroup of left quotients of S. If Q is a semigroup of left quotients of S then, borrowing terminology from ring theory, we also say that S is a *left order* in Q. *Semigroups of right quotients* and *right orders* are defined dually. If S is both a left order and a right order in Q then S is an *order* in Q and Q is a *semigroup of quotients* of S.

Unlike the case for groups and rings, a semigroup can have non-isomorphic semigroups of left quotients [FP]. Indeed a commutative cancellative semigroup can be an order in a non-trivial semilattice of groups, as well as, of course, an order in a group [G]. The thrust of many earlier papers was to consider semigroups that are (left) orders in semigroups in a well known class, for example the class of completely 0-simple semigroups, and in particular to choose classes such that semigroups of (left) quotients taken from the class are unique up to isomorphism.

Here we take a different approach, fixing a property of the *orders* under consideration. Specifically, we consider semigroups that are cancellative left orders. It is easy to see that such a semigroup S must be right reversible, and thus have a group of left quotients. On the other hand we show that if S is a left order in Q then Q is completely regular (a union of groups).

Many left orders naturally satisfy an additional property called *straightness*. Straight left orders in completely regular semigroups have been characterised by Smith in [P]. Left orders in completely regular semigroups need not be straight, however *cancellative* left orders must be. This enables us to use the results of [P] to describe all semigroups of left quotients of a given cancellative left order.

After some preliminaries in Section 2, we study in Section 3 cancellative left orders. We show that if S is such an order in Q, then Q is completely regular. Further, the $\mathcal{D}$-classes of Q are *left groups*, that is, direct products of groups with left zero semigroups, and S is straight and *local* in Q. The group of left quotients of S is, in an established sense, the minimum semigroup of left quotients of S. If S is commutative then any semigroup of left quotients must be commutative [EG], so that if in addition S is cancellative, any semigroup of left quotients of S is a semilattice of commutative groups.

In the final section we specialise the result of [P] charactersising straight left orders in completely regular semigroups, to the case where the left order is cancellative. This enables us to present an example of a semigroup that is a left order in a free group on countably many generators and which is also a left order in a countable chain of left groups.

We would like to thank P.N. Anh for suggesting the topic of this paper.

2 Preliminaries

We assume the reader has a basic knowledge of algebraic semigroup theory, as may be found in the early chapters of [CP] or [H]. As far as possible we follow the notation and terminology of these references. However we deviate from the norm in denoting by a^* the inverse of an element a of a semigroup Q in a *subgroup containing* a. Thus if a^* exists, the $\mathcal{H}$-class H_a is a subgroup of Q and a^* is the inverse of a in H_a.

Recall that a semigroup S is *right reversible* if $Sa \cap Sb \neq \emptyset$ for all elements a, b of S. As shown in Theorem 1.24 of [CP], a semigroup S has a group G of left quotients if and only if it is right reversible and cancellative; transparently, all elements of S then lie in a subgroup of G. If a semigroup S has a *semigroup* of left quotients Q then only the square cancellable elements of S lie in subgroups of Q.

To define square cancellability, we make use of the generalisations $\mathcal{L}^*, \mathcal{R}^*$ and $\mathcal{H}^*$ of Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$. The relation $\mathcal{L}^*$ is defined on a semigroup S by the rule that $a\mathcal{L}^*b$ if and only if

$$ax = ay \text{ if and only if } bx = by$$

for all $x, y \in S^1$. Lemma 1.1 of [F] explains that $a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually and we put $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$. An element a of a semigroup S is *square cancellable* if $a\mathcal{H}^*a^2$. Being square cancellable is clearly a necessary condition for an element of S to lie in a subgroup of some oversemigroup. The definition of (left, right) semigroup of quotients insists that all square cancellable elements lie in subgroups of any semigroup of (left, right) quotients. If S is cancellative, then all elements of S are square cancellative and so if S is a left order in Q, all elements of S lie in subgroups of Q. P.N. Anh asked the natural question whether in this case Q must be completely regular, which we answer in the affirmitive in the next section.

In addition to Green's relations $\mathcal{L}$ and $\mathcal{R}$ it is useful for our purposes to consider the preorders $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ with which they are associated. If a and b are elements of a semigroup Q, then $a \leq_{\mathcal{L}} b$ if and only if $Q^1a \subseteq Q^1b$. Clearly $\leq_{\mathcal{L}}$ is a preorder right compatible with multiplication with associated equivalence relation $\mathcal{L}$. The preorder $\leq_{\mathcal{R}}$ is defined dually. Where S is a (left) order in Q, the relations $\mathcal{L}, \mathcal{R}, \leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ will always refer to relations *on* Q.

Observe that if a and b are elements of a semigroup Q and $a\mathcal{H}a^2$, then if $b \leq_{\mathcal{R}} a$ we have $aa^*b = b$ and so $b\mathcal{L}a^*b$. Now if S is a left order in Q then any $q \in Q$ can be written as $q = c^*d$ for some $c, d \in S$. Then $q = (c^*)^2cd = a^*b$ where $a = c^2, b = cd$ and $b \leq_{\mathcal{R}} a$ in Q. Thus $q\mathcal{L}b$ and S has non-empty intersection with every $\mathcal{L}$-class of Q. If S has the additional property that any $q \in Q$ can be written as $q = a^*b$ where $a, b \in S$ and $a\mathcal{R}b$ in Q, then we say that S is a *straight* left order in Q. Note that in this case $b\mathcal{R}a^*a\mathcal{R}a^*b = q$ so that $b\mathcal{H}q$ in Q and S has non-empty intersection with every $\mathcal{H}$-class of Q. Moreover S is *local* in Q [G2], that is, $S \cap H$ is a left order in H for each group $\mathcal{H}$-class H of Q. Many examples of straight left orders are known, indeed if S is a left order in Q and Q is a regular semigroup on which $\mathcal{H}$ is a congruence, then S is straight in Q [G2]. If S is a left order in a completely regular semigroup, then S need not be straight (see Example 4.2). However, we show in Theorem 3.3 that if S is a cancellative left order, then S must be straight.

3 Cancellative left orders

Our first aim is to show that if S is a cancellative left order in Q, then Q must be completely regular. As a preliminary we have

Lemma 3.1 *Let S be a cancellative left order in Q. Suppose that $a, b \in S$ and $b \leq_{\mathcal{R}} a$. Then $b \leq_{\mathcal{L}} a$.*

Proof Since S is cancellative, a^* exists in Q and $ba^* = c^*d$ for some $c, d \in S$. By a previous remark, we can assume that $d \leq_{\mathcal{R}} c$ in Q. Now $ba = ba^*a^2 = c^*da^2$ and then $cba = cc^*da^2 = da^2$; cancelling in S gives that $cb = da$. Further, b lies in a subgroup of Q so that $b\mathcal{H}b^2 \leq_{\mathcal{R}} ba\mathcal{R}ba^* \leq_{\mathcal{R}} b$. Thus $b\mathcal{R}ba^* = c^*d \leq_{\mathcal{R}} c$ and so $b = c^*cb = c^*da$ and $b \leq_{\mathcal{L}} a$ in Q.

As a consequence of this lemma, if S is a cancellative left order in Q, then $\mathcal{R} \cap (S \times S) = \mathcal{H} \cap (S \times S)$; we show later that S intersects every $\mathcal{H}$-class of Q, whence $\mathcal{R} = \mathcal{H}$ in Q. Note that, as in the lemma, if p and q are elements of a semigroup Q where $p\mathcal{H}p^2$ and $p \leq_{\mathcal{R}} q$, then $p\mathcal{R}pq$. We use this fact repeatedly.

Proposition 3.2 *Let S be a cancellative left order in Q. Then Q is completely regular.*

Proof Let $q \in Q$ and write $q = a^*b$ for some $a, b \in S$ with $b \leq_{\mathcal{R}} a$. Since $aa^*b = b$, $b^*aa^*bb^*a = b^*a$ and $a^*bb^*aa^*b = a^*b$ so that a^*bb^*a is idempotent, and $a^*b\mathcal{R}a^*bb^*a\mathcal{L}b^*a$. By the comment preceding the proposition we have $b\mathcal{R}ba$ in Q and so by Lemma 3.1, $b\mathcal{L}ba$. Hence

$$a^*b\mathcal{L}b\mathcal{L}ba\mathcal{L}b^*a\mathcal{L}a^*bb^*a.$$

This gives that $a^*b\mathcal{H}a^*bb^*a$ and so $q = a^*b$ lies in a subgroup of Q.

A completely regular semigroup has a decomposition as a semilattice of completely simple semigroups [H]. A completely simple semigroup is a *left group* if it is a single $\mathcal{L}$-class, that is, $\mathcal{R} = \mathcal{H}$. It is easy to see that a completely simple semigroup is a left group if and only if it is isomorphic to the direct product of a left zero semigroup with a group.

Theorem 3.3 *Let S be a cancellative left order in Q. Then*
(i) Q is a semilattice of left groups;
(ii) S is local and straight in Q.

Proof We know from Proposition 3.2 that Q is completely regular. Let $p, q \in Q$ be $\mathcal{D}$-related elements and write $p = a^*b, q = c^*d$ for some $a, b, c, d \in S$ with $b \leq_{\mathcal{R}} a$ and $d \leq_{\mathcal{R}} c$. Since $b\mathcal{L}p\mathcal{D}q\mathcal{L}d$ we have that b and d belong to the completely simple semigroup D_p. Hence $b\mathcal{R}bd\mathcal{L}d$ and Lemma 3.1 gives that $b\mathcal{L}bd\mathcal{L}d$ so that $p\mathcal{L}q$ and D_p is a left group.

To prove (ii) we first show that S has non-empty intersection with every $\mathcal{H}$-class of Q. Note that in Proposition 3.2 it is *not* necessarily the case that $a^*b\mathcal{H}b$, as illustrated by Example 4.6.

Let e be any idempotent of Q. Then $e = u^*v$ for some $u, v \in S$ with $v \leq_{\mathcal{R}} u$. From previous comments, $e\mathcal{L}v$ and $vu\mathcal{R}v$ in Q; from Lemma 3.1, $vu\mathcal{H}v$. Now $e = u^*v$ gives that $ue = v$ and so $v^2 = vue$. But $vu\mathcal{L}e$ so that $v^2 = vu$ and cancelling in S gives $v = u$. Thus $e = u^*u\mathcal{H}u$ and $S \cap H_e \neq \emptyset$.

Now let $q \in Q$ and choose $c \in S \cap H_q$. Then $cq = a^*b$ for some $a, b \in S$ with $b \leq_{\mathcal{R}} a$ so that also $q \leq_{\mathcal{R}} a$. Since Q is a semilattice of left groups, $\mathcal{H} = \mathcal{R}$ on Q and it follows that if $h, k \in Q$ and $h \leq_{\mathcal{R}} k$, then $hk\mathcal{H}h$ in Q. Here we have $q = c^*a^*b$ and $c^* \leq_{\mathcal{R}} a^*$ so that $c^*\mathcal{H}c^*a^*$ and $q = c^*a^*c^*cb$. Arguments that are now routine give that $c^*a^*c^*\mathcal{H}cac\mathcal{H}c$ and $c^*a^*c^* = (cac)^*$. Further, $b\mathcal{L}cq\mathcal{H}c$ so that $cb, cac \in H_q$ and $q = (cac)^*cb$. This shows that S is local in Q. Since Q is completely regular, certainly S is also straight in Q.

Note that in the above theorem, we cannot deduce (ii) immediately from (i), as $\mathcal{H}$ need not be a congruence on a semilattice of left groups.

Corollary 3.4 *Let S be a cancellative order in Q. Then Q is a semilattice of groups.*

We give examples in the next section to show that if S is a cancellative *left* order in Q, then the semilattice decomposition of Q can be non-trivial and we can have $\mathcal{D} \neq \mathcal{H}$ in Q.

Certainly the above corollary applies in the case where S is a commutative cancellative left order in Q, for by Corollary 3.2 of [EG], Q is also commutative. We remark that if S is a cancellative left order in Q then for S and Q to be commutative it is enough to assume that the $\mathcal{H}$-classes of Q are commutative groups. For in this case, if $u, v \in S$ and $u\mathcal{L}v$ in Q we have by Theorem 3.3 that $u\mathcal{H}uv$ and so by hypothesis, $u(uv) = (uv)u$. Cancelling in S gives $uv = vu$ and $u\mathcal{H}v$ in Q. Thus Q is a semilattice of commutative groups so that Q (and S) are commutative.

In the final result of this section we show that any cancellative left order S is right reversible so that by Ore's theorem, S has a group G of left quotients. In fact G is the *minimum* semigroup of quotients of S in the sense that if S is a left order in Q, then G is the image of Q under a homomorphism which restricts to the identity on S.

Proposition 3.5 *Let S be a cancellative left order. Then S is right reversible and the group G of left quotients of S is the minimum semigroup of left quotients of S.*

Proof Let S be a left order in Q and let $a, b \in S$. Then a^* exists in Q and $ba^* = c^*d$ for some $c, d \in S$ with $d \leq_{\mathcal{R}} c$ in Q. This equation yields $cba = da^2$ and we cancel in S to obtain $cb = da$. Thus S is right reversible and so by Ore's theorem, S has a group G of left quotients.

Suppose now that S is a left order in P. Trivially, if h, k are elements of S that are $\mathcal{R}$-related in P, then they are $\mathcal{R}$-related in G. Since S is straight in both P and G, an adaptation of Corollary 4.3 of [G4] (see [G5]) gives immediately that the mapping $a^*b \mapsto a^{-1}b$ where $a, b \in S$ and $b \leq_{\mathcal{R}} a$ in P is a homomorphism from P onto G which restricts to the identity on S. Thus G is the minimum semigroup of left quotients of S.

4 Semilattices of cancellative left orders

In the previous section we showed that if S is a cancellative left order in Q, then S is straight in Q and Q is completely regular, indeed, Q is a semilattice of left groups.

Straight left orders in completely regular semigroups are investigated in depth in [P]. If S is a straight left order in Q, where Q is a semilattice Y of completely simple semigroups $Q_\alpha, \alpha \in Y$, then it is easy to see that putting $S_\alpha = S \cap Q_\alpha, \alpha \in Y$, each S_α is a left order in Q_α and S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$.

On the other hand, given a semigroup S that is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$, where each S_α is a left order in a completely simple semigroup Q_α, under what conditions is S a straight left order in Q, where Q is a semilattice Y of semigroups $Q_\alpha, \alpha \in Y$? It is not difficult to see this is equivalent to being able to define an associative multiplication on $\bigcup\{Q_\alpha : \alpha \in Y\}$ which extends the multiplication on S and each Q_α. This question is answered in Theorem III 1.2 of [P], which we quote below. In fact we quote a slightly stronger result, the proof of which is implicit in [P]; it is also somewhat reworded. The statement involves the relation $\mathcal{R}'$ where $\mathcal{R}' = \bigcup\{\mathcal{R}'_\alpha : \alpha \in Y\}$ and for each $\alpha \in Y, \mathcal{R}'_\alpha = \mathcal{R}^{Q_\alpha} \cap (S_\alpha \times S_\alpha)$.

Theorem 4.1 *Let S be a semilattice Y of semigroups $S_\alpha, \alpha \in Y$, where for each $\alpha \in Y$, S_α is a left order in a completely simple semigroup Q_α. Put $Q = \bigcup\{Q_\alpha : \alpha \in Y\}$. Then Q is a semilattice of semigroups $Q_\alpha, \alpha \in Y$, if and only if conditions (I) to (VI) hold. In this case, S is a straight left order in Q.*
(I) For any $a \in S$, if $u, v \in S$ and $u\mathcal{R}'at$ for some $t \in S^1$, then $uv\mathcal{R}'as$ for some $s \in S^1$.
(II) The relation $\mathcal{R}'$ is contained in $\mathcal{R}^*$.
(III) If $a, b \in S_\alpha$, $aS_\alpha \cap bS_\alpha \neq \emptyset$ and $abx = aby$ for some $x, y \in S^1$, then $bx = by$.
(IV) The relation $\mathcal{R}'$ is a left congruence.
(V) If $a^2x, a^2y \in S$ and $a^2x\mathcal{R}'a^2y$, then $ax\mathcal{R}'ay$.
(VI) If $x \in S_\alpha, z \in S_\beta$ then there exist $u, v \in S_{\alpha\beta}$ with $u\mathcal{R}'v\mathcal{R}'xz$ and $uxz = vz^2$.

The example below shows that a left order in a completely regular semigroup Q need not be straight. This Q is also an example of a completely regular semigroup for which $\mathcal{R} = \mathcal{H}$ but $\mathcal{H}$ is not a congruence.

Example 4.2 .

Let V be the infinite cyclic group generated by an element a and let U be the subsemigroup of V consisting of the positive powers of a. Let G be a group and put $P = \mathbf{Z} \times G$, where $\mathbf{Z}$ is regarded as a left zero semigroup, so that P is a left group. Now let $Q = P \cup V$ and define a multiplication on Q extending that on P and V by

$$a^i(j, g) = (i + j, g),\ (j, g)a^i = (j, g)$$

for all $a^i \in V$ and $(j, g) \in P$. It is easy to see that under this multiplication Q is a completely regular semigroup. Indeed, Q is a chain of the group V and the left group P so that in particular, $\mathcal{H} = \mathcal{R}$ on Q. Note that $\mathcal{H}$ is not a congruence on Q since $a\mathcal{H}a^{-1}$ but

$$a(0, g) = (1, g) \not\mathcal{H} (-1, g) = a^{-1}(0, g).$$

Put $S = (\mathbf{N} \times G) \cup U$. Clearly S is a subsemigroup of Q and it is not difficult to show that S is a left order in Q. Since S does not intersect every $\mathcal{H}$-class of Q, S cannot be straight in Q.

We are interested here in applying Theorem 4.1 to *cancellative* left orders. From Theorem 3.3, cancellative left orders are always straight. Also from that theorem, if S is a cancellative left order in a completely simple semigroup Q, then Q is a left group.

Corollary 4.3 *Let S be a cancellative semigroup, and suppose that S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$, where for each $\alpha \in Y$, S_α is a left order in a left group Q_α. Put $Q = \bigcup\{Q_\alpha : \alpha \in Y\}$.*

Then Q is a semilattice of semigroups $Q_\alpha, \alpha \in Y$, if and only if conditions (I), (IV) and (V) hold. In this case, S is a straight left order in Q.

Proof Suppose that conditions (I), (IV) and (V) hold. Since $\mathcal{R}^*$ is universal on a cancellative semigroup, it is immediate that conditions (II) and (III) also hold. Let $x \in S_\alpha$ and $z \in S_\beta$. Then $z\mathcal{R}'z^3$ so that as $\mathcal{R}'$ is a left congruence, $xz\mathcal{R}'xz^3$. Thus in $Q_{\alpha\beta}$, $xz\mathcal{H}xz^3$ and as $S_{\alpha\beta}$ is a local left order in $Q_{\alpha\beta}$, there exist $u, w \in S_{\alpha\beta} \cap H_{xz}$ with $uxz = wxz^3$. Put $v = wxz$ so that $v \in S_{\alpha\beta} \cap H_{xz}, uxz = vz^2$ and $u\mathcal{R}'v\mathcal{R}'xz$. Thus condition (VI) holds. The corollary is now immediate from Theorem 4.1.

To apply Corollary 4.3 we need to know when a cancellative semigroup S is a left order in a left group. By Proposition 3.5, S must be right reversible.

Proposition 4.4 *Let S be a right reversible, cancellative semigroup. Then S is a left order in a left group $Q = L \times G$, where L is a left zero semigroup and G is a group, if and only if S is the disjoint union of right ideals $R_\ell, \ell \in L$. Moreover, if this holds then for any $a, b \in S$, $a\mathcal{R}b$ in Q if and only if $a, b \in R_\ell$ for some $\ell \in L$.*

Proof It is possible to deduce this result directly from the characterisations of left orders in completely simple semigroups given in [AFM] or [G3], however we give a short direct argument.

If S is a left order in $Q = L \times G$ then clearly S intersects every $\mathcal{H}$-class of Q, so that putting

$$R_\ell = S \cap \{(\ell, g) : g \in G\}, \ell \in L,$$

we have that S is a disjoint union of right ideals $R_\ell, \ell \in L$.

Conversely, suppose that S is a disjoint union of right ideals $R_\ell, \ell \in L$. Let G be the group of left quotients of S. It is easy to see that each R_ℓ is a left order in G. Put $Q = L \times G$ and define $\phi : S \to Q$ by $s\phi = (\ell, s)$, where $s \in R_\ell$. Then ϕ is an embedding of S into Q such that $S\phi$ is a left order in Q. Moreover, if $s, t \in S$, then $s\phi \,\mathcal{R}\, t\phi$ in Q if and only if $a, b \in R_\ell$ for some $\ell \in L$.

Example 4.5 .

Let G be the group semidirect product given by

$$G = \{(a, b) : a, b \in \mathbf{Q}, b \neq 0\}$$

with multiplication

$$(a, b)(c, d) = (a + bc, bd)$$

so that $(0, 1)$ is the identity of G and $(a, b)^{-1} = (-a/b, 1/b)$. Put

$$S = \{(a, b) : a, b \in \mathbf{Z}, b \neq 0\}.$$

Then S is a subsemigroup, indeed a left order in G. Let n be a positive integer. For any $(a, n), (c, d) \in S$,

$$(a, n)(c, d) = (a + nc, nd)$$

and $a + nc \equiv a$ modulo n. Thus $(0, n), (1, n), ..., (n - 1, n)$ generate disjoint right ideals. Putting

$$T = \bigcup\{(r, n)S : 0 \leq r \leq n - 1\}$$

we have that T satisfies the conditions of Proposition 4.3, where $L = \{0, 1, ..., n - 1\}$.

Fitting together Theorem 3.3, Corollary 4.3 and Proposition 4.4 we have the following picture for a cancellative semigroup S.

If S is a left order in Q, then Q is a semilattice Y of left groups $Q_\alpha, \alpha \in Y$. Putting $S_\alpha = S \cap Q_\alpha, \alpha \in Y$, we have that S_α is a left order in Q_α. Write $Q_\alpha = L_\alpha \times G_\alpha$ where L_α is a left zero semigroup and G_α is a group. For $\ell_\alpha \in L_\alpha$ put $R^\alpha_{\ell_\alpha} = S_\alpha \cap \{(\ell_\alpha, g_\alpha) : g_\alpha \in G_\alpha\}$. Then for each $\alpha \in Y$, S_α

is right reversible and is a disjoint union of right (S_α-)ideals $R^\alpha_{\ell_\alpha}, \ell_\alpha \in L_\alpha$. Defining the relation $\mathcal{R}'$ on S by the rule that for elements a, b of S we have $a\mathcal{R}'b$ if and only if $a, b \in R^\alpha_{\ell_\alpha}$ for some $\alpha \in Y$ and $\ell_\alpha \in L_\alpha$, we have that $\mathcal{R}' = \mathcal{R}^Q \cap (S \times S)$ and $\mathcal{R}'$ satisfies conditions (I), (IV) and (V).

Conversely, suppose that S is a semilattice Y of semigroups S_α where for each $\alpha \in Y$, S_α is right reversible and is a disjoint union of right (S_α-)ideals $R^\alpha_{\ell_\alpha}, \ell_\alpha \in L_\alpha$. Suppose also that the relation $\mathcal{R}'$ satisfies conditions (I), (IV) and (V), where $\mathcal{R}'$ is defined by the rule that $a\mathcal{R}'b$ if and only if $a, b \in R^\alpha_{\ell_\alpha}$ for some $\alpha \in Y$ and $\ell_\alpha \in L_\alpha$. Then each S_α is a left order in a left group $Q_\alpha, \alpha \in Y$, where for $a, b \in S_\alpha, a\mathcal{R}b$ in Q_α if and only if $a\mathcal{R}'b$. Further, $Q = \bigcup\{Q_\alpha : \alpha \in Y\}$ is a semilattice of semigroups $Q_\alpha, \alpha \in Y$ and S is a straight left order in Q.

In each case the relation $\mathcal{R}'$ is the restriction to S of the relation $\mathcal{R}$ in Q. Indeed a semigroup of left quotients Q of S is uniquely determined in the following sense by $\mathcal{R}'$. If Q_1 and Q_2 are semigroups of left quotients of S then by Theorem 3.3 S is straight in Q_1 and Q_2. Corollary 4.3 of [G4] yields that Q_1 is isomorphic to Q_2 under an isomorphism which restricts to the identity on S if and only if $\mathcal{R}^{Q_1} \cap (S \times S) = \mathcal{R}^{Q_2} \cap (S \times S)$.

We end this paper by presenting an example of a cancellative left order S that is also a left order in a chain of left groups.

Example 4.6 .

Let $F = \{a_i, a_i^{-1} : i \in \mathbf{N}\}^*$ be a free monoid and write $\overline{w}$ for the reduced form of any $w \in F$, so that $G = \{\overline{w} : w \in F\}$ is the free group on $\{a_i : i \in \mathbf{N}\}$ under the multiplication $\overline{w_1} \cdot \overline{w_2} = \overline{w_1 w_2}$ and $^- : F \to G$ is a homomorphism.

For each $i \in \mathbf{N}$ we define a homomorphism Δ_i from F to the integers under addition by

$$a_i\Delta_i = 1, a_i^{-1}\Delta_i = -1, a_j\Delta_i = a_j^{-1}\Delta_i = 0,\ i \neq j.$$

Call a word $w \in F$ *i-positive* if $w \neq 1$ and for all $u, v \in F$

$$u \neq 1,\ w = uv \text{ implies that } u\Delta_i > 0.$$

Put $T_0 = \{1\}$ and for $i \in \mathbf{N}$

$$T_i = \{a_1, ..., a_i, a_1^{-1}, ..., a_i^{-1}\}^*.$$

Now for $i \in \mathbf{N}$ let

$$U_i = \{t_{i-1}t_i : t_{i-1} \in T_{i-1}, t_i \in T_i, t_i \text{ } i\text{-positive}\}.$$

Claim 1 For each $i, j \in \mathbf{N}, U_iU_j \subseteq U_{max\{i,j\}}$ so that $U = \bigcup\{U_i : i \in \mathbf{N}\}$ is a subsemigroup of F.

Proof Let $u \in U_i, v \in U_j$ so that $u = s_{i-1}s_i, v = t_{j-1}t_j$ where $s_{i-1} \in T_{i-1}, s_i \in T_i, t_{j-1} \in T_{j-1}, t_j \in T_j, s_i$ is i-positive and t_j is j-positive.

If $i < j$ then $s_{i-1}s_it_{j-1} \in T_{j-1}$ so that $uv \in U_j$. Suppose now that $i \geq j$ so that $s_it_{j-1}t_j \in T_i$. Let $x \neq 1$ be a prefix of $s_it_{j-1}t_j$. If x is a prefix of s_i then certainly $x\Delta_i > 0$. If $x = s_iy$ where y is a prefix of T_{j-1}, then

$$x\Delta_i = s_i\Delta_i + y\Delta_i = s\Delta_i > 0.$$

If $x = s_it_{j-1}z$ where z is a prefix of t_j, then as above if $j < i$ we have $x\Delta_i = s_i\Delta_i > 0$; on the other hand if $j = i$ we have

$$x\Delta_i = s\Delta_i + t_{i-1}\Delta_i + z\Delta_i = s_i\Delta_i + z\Delta_i > 0.$$

Thus $s_it_{j-1}t_j$ is i-positive and $uv \in U_i$.

For $i \in \mathbf{N}$ let

$$S_i = \overline{U_i} = \{\overline{u_i} : u_i \in U_i\}$$

and put

$$S = \bigcup\{S_i : i \in \mathbf{N}\}.$$

By Claim 1, S is a subsemigroup of G. We shall show that S is the left order for which we are searching.

Claim 2 For all words $w \in F$, if w is i-positive, then $\overline{w}$ is i-positive.

Proof Suppose that $w = w_0a_j^{\epsilon}a_j^{-\epsilon}w_1$ is i-positive, where $\epsilon \in \{1, -1\}$. Note that $w_0w_1 \neq 1$. Let $v \neq 1$ be a prefix of w_0w_1. If v is a prefix of w_0 then clearly $v\Delta_i > 0$. Otherwise, $v = w_0z$ for some prefix z of w_1. Then

$$v\Delta_i = w_0\Delta_i + z\Delta_i = w_0\Delta_i + a_j^{\epsilon}\Delta_i + a_j^{-\epsilon}\Delta_i + z\Delta_i = (w_0a_j^{\epsilon}a_j^{-\epsilon}z)\Delta_i > 0.$$

Thus w_0w_1 is i-positive and the claim follows by induction.

For a word $w \in F$ we denote by $c(w)$ the *content* of w, that is, $c(w)$ is the set of *letters* occurring in w.

Claim 3 Each element of S has a unique representation as $\overline{s_{i-1}}\,\overline{s_i}$ (multiplication *in* F) where $s_{i-1} \in T_{i-1}, s_i \in T_i$ and s_i is i-positive.

Proof If $s \in S$ then by definition, $s = \overline{s_{i-1}s_i}$ for some $s_{i-1} \in T_{i-1}$ and i-positive $s_i \in T_i$. By Claim 2, $\overline{s_i}$ is i-positive, so that $\overline{s_i} = a_i x$ for some $x \in F$. But $a_i^{-1} \notin c(\overline{s_{i-1}})$, whence $s = \overline{s_{i-1}}\,\overline{s_i}$.

Suppose now that $\overline{s_{i-1}}\,\overline{s_i} = \overline{t_{j-1}}\,\overline{t_j}$ where $s_{i-1} \in T_{i-1}, s_i \in T_i$ is i-positive, $t_{j-1} \in T_{j-1}, t_j \in T_j$ is j-positive. As above, $\overline{s_i} = a_i x$ and $\overline{t_j} = a_j y$ for some $x, y \in F$. Thus $a_i \in c(\overline{t_{j-1}}\,\overline{t_j}) \subseteq \{a_1, ..., a_j, a_1^{-1}, ..., a_j^{-1}\}$ and so $i \leq j$. Dually, $j \leq i$ and so $i = j$. Now $\overline{s_{i-1}}a_i x = \overline{t_{i-1}}a_i y$ gives $\overline{s_{i-1}} = \overline{t_{i-1}}$ and $x = y$, since $a_i \notin c(\overline{s_{i-1}}) \cup c(\overline{t_{i-1}})$. The claim follows.

We say that an element s of S is in *normal form* if $s = \overline{s_{i-1}}\,\overline{s_i}$ where $s_{i-1} \in T_{i-1}, s_i \in T_i$ and s_i is i-positive.. Claim 3 says that each element of S has a unique normal form. Moreover it follows also that if $i \neq j$ then $S_i \cap S_j = \emptyset$. Together with Claim 1 this gives

Claim 4 The semigroup S is a chain $\mathbf{N}$ of semigroups $S_i, i \in \mathbf{N}$.

Consider now the product $s \cdot t$ of two elements $s = \overline{s_{i-1}}\,\overline{s_i}$ and $t = \overline{t_{j-1}}\,\overline{t_j}$ of S in normal form. We have that

$$s \cdot t = (\overline{s_{i-1}}\,\overline{s_i}) \cdot (\overline{t_{j-1}}\,\overline{t_j}) = \overline{s_{i-1}s_i} \cdot \overline{t_{j-1}t_j} = \overline{s_{i-1}s_it_{j-1}t_j}.$$

¿From the proof of Claim 1, if $i < j$ then

$$s \cdot t = \overline{s_{i-1}s_it_{j-1}}\,\overline{t_j}$$

is in normal form, whilst if $i \geq j$ then

$$s \cdot t = \overline{s_{i-1}}\,\overline{s_it_{j-1}t_j}$$

is in normal form.

Now fix $i \in \mathbf{N}$ and consider S_i. For $t_{i-1} \in T_{i-1}$ put

$$R^i_{\overline{t_{i-1}}} = \{\overline{t_{i-1}}\,\overline{t_i} : t_i \in T_i, t_i \text{ is } i\text{-positive}\}.$$

By the above, each $R^i_{\overline{t_{i-1}}}$ is a right ideal of S_i and by the uniqueness of normal forms,

$$S = \bigcup\{R^i_{\overline{t_{i-1}}} : \overline{t_{i-1}} \in \overline{T_{i-1}}\}$$

where the union is disjoint.

Claim 5 For $i \in \mathbf{N}$ the semigroup S_i is a left order in $\overline{T_i}$.
Proof Let $g \in \overline{T_i}$, so that $g = \overline{w}$ for some $w \in T_i$. Let $N = 1 + max\{|v\Delta_i| : v \text{ is a prefix of } w\}$. Then $a_i^N, a_i^N w$ are i-positive elements of T_i so that $a_i^N, a_i^N w \in U_i$ and

$$g = \overline{w} = \overline{a_i^{-N} a_i^N w} = \overline{a_i^{-N}} \cdot \overline{a_i^N w} = \overline{a_i^N}^{-1} \cdot \overline{a_i^N w}$$

as required.

The next claim is now immediate.

Claim 6 The semigroup S is a left order in G.

¿From Claim 5 and Proposition 4.4 it follows that for each $i \in \mathbf{N}$, S_i is a left order in a left group Q_i. Moreover, if $\overline{s_{i-1}}\,\overline{s_i}, \overline{t_{i-1}}\,\overline{t_i} \in S_i$ are in normal form, then $\overline{s_{i-1}}\,\overline{s_i} \mathcal{R} \overline{t_{i-1}}\,\overline{t_i}$ in Q_i if and only if $\overline{s_{i-1}} = \overline{t_{i-1}}$.

We wish to apply Corollary 4.3 to give that S is a left order in Q, where Q is a chain $\mathbf{N}$ of the semigroups $Q_i, i \in \mathbf{N}$. The relation $\mathcal{R}'$ in question is given by the rule that if $\overline{s_{i-1}}\,\overline{s_i}, \overline{t_{j-1}}\,\overline{t_j} \in S$ are in normal form, then they are $\mathcal{R}'$-related if and only if $i = j$ and $\overline{s_{i-1}} = \overline{t_{j-1}}$. It is straightforward to show that conditions (I), (IV) and (V) hold for $\mathcal{R}'$. Corollary 4.3 now gives our result.

Claim 7 The semigroup S is a left order in Q, where Q is a chain $\mathbf{N}$ of the left groups $Q_i, i \in \mathbf{N}$.

Note that the $\mathcal{R}$-classes of each Q_i are indexed by $\overline{T_{i-1}}$, so that Q_1 is a group, but for $i > 1, Q_i$ has infinitely many $\mathcal{R}$-classes.

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