A normal form for a class of monoids including the singular braid monoids.

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Abstract

The first part of this paper investigates a class of homogeneously presented monoids. Constructions which enable division and multiplication to be computed are described. The word problem and the division problem are solved, and a unique normal form is given for monoids in this class. The second part deals with the motivating examples of this class — the positive singular Artin monoids. Finally, the singular Artin monoids are introduced. The word and division problems are solved for all singular Artin monoids of finite type, including, for example, the singular braid monoids introduced by Baez and Birman.

1 Introduction and preliminaries

Baez [2] and Birman [3] introduced the singular braid monoids in 1992 and 1993. A solution to the word problem would seem to be a step towards determining the conjecture of Birman's that the singular braid monoid embeds in the group algebra of the braid group (still unproved, to this author's knowledge). Garside's solution [6] to the word problem in the braid groups indicates an approach to the problem in singular braid monoids. The work of Brieskorn and Saito [4] extending this to arbitrary Artin groups, with special results for those of finite type, suggests the definition of *singular Artin monoids.* Their techniques could be adapted to work in this new class of

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monoids. Some of the terminology (such as *a-chains*) is taken from that paper. It turns out that the singular Artin monoids and positive Artin monoids are leading examples of this class so far investigated, although various single relation monoids can be constructed with the desired property.

A second motivation for finding solutions to the word and related problems lies in singular knot and link theory. Armand-Ugon et al [1] showed that an arbitrary representative of a singular knot or link with double or triple points may be deformed to a closed singular braid. Bernd Gemein [7] proved an extended Markov's theorem for singular links: if β and γ are two singular braids whose closed braids are equivalent as singular links, then β and γ are related by a sequence of algebraic operations known as Markov's moves. If you can solve the word, division and conjugacy problem in the singular braid monoid, then you can determine if two singular braids are related by a Markov move. In this paper we solve the first two of those problems. A solution to the conjugacy problem is the subject of a paper in preparation by this author. Singular link invariants are of interest in various fields, such as conformal field theory, Vassiliev invariants, and loop representations of quantum gravity (see [1] for more details).

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{R}$ a finite subset of $\mathcal{A}^* \times \mathcal{A}^*$. Elements of $\mathcal{A}$ may be referred to as *letters*, and elements of $\mathcal{A}^*$ as *words*. Letterwise equality of words is denoted by $\equiv$. Let

$$\mathcal{R}^\Sigma := \{(U,V) \mid (U,V) \in \mathcal{R} \text{ or } (V,U) \in \mathcal{R}\}.$$

If $W \equiv X\rho_1 Y$ and $V \equiv X\rho_2 Y$ where $(\rho_1,\rho_2) \in \mathcal{R}^\Sigma$ then we say that V and W *differ by an elementary transformation*. If there is a sequence of words $W_0,\dots,W_k$ such that each pair W_i and W_{i+1} differ by an elementary transformation, then we say that W_0 and W_k are *equivalent*, and write $W_0 \sim W_k$. In this case, we also say W_0 and W_k differ by a transformation of length k. (The terminology *transformation* is according to [6] and [4]). The monoid presented by $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is the set of equivalence classes under $\sim$.

A monoid presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is said to have the *uniqueness property* if whenever there are letters a and b and words U, V, X, Y over A such that both $(aU, bV) \in \mathcal{R}^\Sigma$ and $(aX, bY) \in \mathcal{R}^\Sigma$, then $X \equiv U$ and $Y \equiv V$. It is said to have the *cancellative property* if for all words U, V over $\mathcal{A}^*$ with $(aU, bV) \in \mathcal{R}$, $a \not\equiv b$. It is said to have the *homogeneity property* if whenever (U,V) is in $\mathcal{R}$, U and V have the same length. It is said to have the *reduction property* if for all a, b in $\mathcal{A}$ and for all words X, Y in $\mathcal{A}^*$, $aX \sim bY$ implies there exist words U, V and W in $\mathcal{A}^*$ such that $X \sim UW$, $Y \sim VW$ and either $(aU, bV) \in \mathcal{R}^\Sigma$ or $aU \equiv bV$.

Lemma 1 *Suppose that $\langle \mathcal{A} \mid \mathcal{R} \rangle$ has the cancellative and reduction properties. Whenever there are words A, V and W such that $AV \sim AW$, then $V \sim W$.*

Proof. If $aV \sim aW$ for some letter a then, by the reduction property, there are words X_1, X_2 and Y such that $V \sim X_1Y$, $W \sim X_2Y$ and either $(aX_1, aX_2) \in \mathcal{R}^\Sigma$ or $aX_1 \equiv aX_2$. By the cancellative property, the first possibility fails, so $X_1 \equiv X_2$ and $V \sim X_1Y \sim W$. If $aUV \sim aUW$ for some letter a and word U, then by the previous observation, $UV \sim UW$, so by induction on the length of U, $V \sim W$. □

In Sections 2 and 3, we analyse the properties of a monoid presentation with the uniqueness, cancellative, homogeneity and reduction properties. The previous lemma shows that any such monoid is left cancellative. Because of the homogeneity requirement, all equivalent words have the same length. We define $\ell(V)$ to be the length of V, and so if $W \sim V$ then $\ell(W) = \ell(V)$.

Suppose V and W are words over $\mathcal{A}$. We say that V *divides* W if there exists a word U such that $W \sim VU$. We also say that W *is a multiple of* V. If W divides V and V divides W, then by homogeneity, $V \sim W$. Given a set of words V_i where i runs over some indexing set I, a *common multiple of* $\{V_i \mid i \in I\}$ is a word W such that every V_i divides W. A *least common multiple* is a common multiple which divides all other common multiples. If W and W' are both least common multiples of $\{V_i \mid i \in I\}$ then they divide one another and hence are equivalent. Thus, if a least common multiple exists, then it is unique up to equivalence.

2 Factorisation

Throughout this section we assume that $\langle \mathcal{A} \mid \mathcal{R} \rangle$ has the uniqueness, cancellative, homogeneity and reduction properties. Thus the monoid presented by $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is left cancellative, by Lemma 1.

Let C be a word and a, b be letters. We say C is a *simple a-chain* with *source a* and *target b* if there are (possibly empty) words U and V such that (aU, CbV) is in $\mathcal{R}^\Sigma$. Call C a *compound a-chain* if $C \equiv C_1 \cdots C_k$ for simple chains $C_1, \ldots, C_k$ such that the source of C_1 is a and the source of C_{i+1} is the target of C_i for $i = 1, \ldots, k-1$. In this case, the *source* and *target* of C are defined to be the source of C_1 and the target of C_k respectively.

Lemma 2 *Suppose C is an a-chain to b and W is a common multiple of a and C. Then W is a common multiple of a and Cb. In particular, a does not divide C.*

Proof. There is a word D such that $W \sim CD$. Write $C \equiv C_1 \cdots C_k$ where each C_i is simple, and suppose that d is the target of C_1, so $(aU, C_1dV) \in \mathcal{R}^\Sigma$ for some words U and V. By the cancellative property, C_1d does not begin with a. However a does divide CD. By the reduction property, there is a word Z such that $aUZ \sim CD \sim C_1dVZ$. If $k = 1$ then $b \equiv d$ and $Cb \equiv C_1d$ divides CD. Otherwise, after cancelling, dX, and hence d, divides $C_2 \cdots C_kD$, so by induction $C_2 \cdots C_kb$ divides $C_2 \cdots C_kD$, and whence Cb divides CD. This proves the first statement.

If a were to divide C, then C would be a common multiple of a and C, and so by the first statement, Cb would divide C. This contradicts homogeneity, and so a cannot divide C. $\square$

A simple a-chain C with target b is *imminent* if there is a word U such that (aU, Cb) is in $\mathcal{R}^\Sigma$. In this case define $C^+ :\equiv aU$.

Lemma 3 *Suppose that C is a simple a-chain. Then C is imminent with target b precisely when a divides Cb.*
If C is a compound a-chain such that a divides Cb then b is the target of C.

Proof. If C is a simple imminent a-chain to b, then $(aU, Cb) \in \mathcal{R}^\Sigma$ for some U. Thus a divides $aU \equiv C^+ \sim Cb$. On the other hand, if a divides Cb, then by the reduction and uniqueness properties, there are words U and V such that $(aU, CbV) \in \mathcal{R}^\Sigma$ and CbV divides Cb. By homogeneity, V must be empty, making C imminent with target b.

Now suppose $C \equiv C_1 \cdots C_{k-1}C_k$ where each C_i is a simple chain and such that a divides Cb. Suppose that d is the target of C_{k-1}, and hence the source of C_k. By Lemma 2, $C_1 \cdots C_{k-1}C_kb$ is a common multiple of a and $C_1 \cdots C_{k-1}d$, so by cancellativity, d divides the C_kb. By the previous part, the d-chain C_k has target b, and hence b is the target of C. $\square$

Letters a and b are said to be *incomparable* if for all words U and V, $(aU, bV) \notin \mathcal{R}^\Sigma$. If W is a word which begins with a then ${}^{a^{-1}}W$ denotes the word obtained by removing a from the beginning of W. For each letter a, we define an operator

$$K_a : \mathcal{A}^* \longrightarrow \mathcal{A}^* \dot\cup \infty$$

recursively. Let W be a word, and let

$$K_a(W) :\equiv \begin{cases} W & \text{if } W \text{ is empty, begins with } a \text{ or is a simple } a\text{-chain; or} \\ \infty & \text{if } a \text{ is incomparable with the first letter of } W. \end{cases}$$

Otherwise, write $W \equiv CD$ where C and D are non-empty words, and C is the largest prefix of W which is a simple a-chain, with target b, say. The rest of the definition of $K_a(W)$ is recursive on the lengths of W and D:

$$K_a(W) :\equiv \begin{cases} \infty & \text{if } K_b(D) \equiv \infty; \\ CK_b(D) & \text{if } K_b(D) \not\equiv \infty \text{ and} \\ & \qquad K_b(D) \text{ does not begin with } b; \\ C^{+\ b^{-1}} K_b(D) & \text{if } C \text{ is imminent and} \\ & \qquad K_b(D) \text{ begins with } b; \text{ or} \\ K_a(Cb^{\ b^{-1}} K_b(D)) & \text{otherwise.} \end{cases}$$

If $K_a(W) \not\equiv \infty$ we will say $K_a(W)$ *is defined.*

Lemma 4 *If $K_a(W)$ is defined then $K_a(W) \sim W$ and $K_a(W)$ either is empty, begins with a or is an a-chain. Furthermore, if a and W have a common multiple then $K_a(W)$ is defined. In particular, if a divides W then $K_a(W)$ begins with a.*

Proof. Suppose $K_a(W)$ is defined. If W is empty, begins with a or is a simple a-chain then $K_a(W) \equiv W$ and we are done.

Otherwise, write $W \equiv CD$ where C is the longest prefix of W which is a simple a-chain. Let b denote the target of C. Since $K_a(W) \not\equiv \infty$ then $K_b(D) \not\equiv \infty$, so by induction on the length of D, $K_b(D) \sim D$ and $K_b(D)$ is either empty, a b-chain or begins with b. If $K_b(D)$ is empty or a b-chain then $K_a(W) \equiv CK_b(D)$ which is an a-chain, and moreover $K_a(W) \sim CD \equiv W$.

Otherwise $K_b(D)$ begins with b. If C is imminent then $K_a(W) \equiv C^{+\ b^{-1}} K_b(D)$ which begins with a, and moreover,

$$K_a(W) \sim Cb^{\ b^{-1}} K_b(D) \equiv CK_b(D) \sim CD \equiv W.$$

Otherwise we have $K_b(D) \sim D$, $K_b(D)$ begins with b and C is not imminent; so

$$K_a(W) \equiv K_a(Cb^{\ b^{-1}} K_b(D)).$$

Fortunately Cb is a simple a-chain of length greater than the length of C so by another induction, $K_a(Cb\,{}^{b^{-1}}K_b(D))$ begins with a or is an a-chain, and

$$K_a(Cb\,{}^{b^{-1}}K_b(D)) \sim Cb\,{}^{b^{-1}}K_b(D) \equiv CK_b(D) \sim CD \equiv W.$$

This gives the first part of the lemma.

Now suppose that a and W have a common multiple. If W is empty, begins with a or is a simple a-chain then we are done.Otherwise, let U be a word such that a divides WU. By the reduction property the first letter of W is not incomparable with a. So W can be written $W \equiv CD$ where C is the longest prefix of W which is a simple a-chain. Let b be the target of C. Since a divides CDU then by Lemma 2, Cb divides CDU, so by cancellation, b divides DU. Then by induction, $K_b(D) \not\equiv \infty$. If $K_b(D)$ does not begin with b, or begins with b where C is imminent, then $K_a(W) \not\equiv \infty$, and we are done. Otherwise, $K_a(W) \equiv K_a(Cb\,{}^{b^{-1}}K_b(D))$ and again applying induction on the length of Cb (a simple a-chain of length greater than that of C), because a and $Cb\,{}^{b^{-1}}K_b(D) \sim W$ have a common multiple, $K_a(Cb\,{}^{b^{-1}}K_b(D)) \not\equiv \infty$. This gives the second part of the lemma.

Finally suppose that a divides W. Then W is a common multiple of a and W, so $K_a(W)$ is either an a-chain or begins with a. But Lemma 2 says an a-chain may not be divisible by a; thus $K_a(W)$ (which is equivalent to W and hence divisible by a) is not an a-chain and hence must begin with a. □

Let V and W be words. If W is empty then define $(V/W) \equiv V$. Otherwise write $W \equiv Ua$ for some word U and some letter a. We make the recursive definition:

$$(V/W) \equiv \begin{cases} \infty & \text{if } (V/U) \equiv \infty, \text{ or if} \\ & \qquad K_a(V/U) \text{ does not begin with } a; \text{ or} \\ {}^{a^{-1}}K_a(V/U) & \text{otherwise.} \end{cases}$$

The following result gives a solution to the division problem in $\langle \mathcal{A} \mid \mathcal{R} \rangle$.

Lemma 5 *The word W divides V precisely when $(V/W) \not\equiv \infty$, in which case $V \sim W(V/W)$.*

Proof. If W is empty then the result clearly holds. So we may write $W \equiv Ua$ for some word U and some letter a. Suppose W divides V, so there is a word X such that $WX \equiv UaX \sim V$. By induction $(V/U) \not\equiv \infty$ and $V \sim U(V/U)$.

By cancellation, $aX \sim (V/U)$, so a divides (V/U). By Lemma 4, $K_a(V/U)$ begins with a, so $(V/W) \not\equiv \infty$ and $(V/W) \sim X$.

On the other hand, suppose $(V/W) \not\equiv \infty$. Then $(V/U) \not\equiv \infty$, and in fact $K_a(V/U)$ has to begin with a. By induction $V \sim U(V/U)$, so

$$V \sim U(V/U) \sim UK_a(V/U) \sim Ua\ {}^{a^{-1}}K_a(V/U) \sim W(V/W)$$

by the definition of (V/W), and so W divides V. □

There is a solution to the word problem for any homogeneous presentation obtained by listing all words of given length and then searching to determine equivalent words. The machinery described above leads to a more sophisticated solution to the word problem.

Corollary 6 *Two words W and V are equivalent precisely when (V/W) is the empty word.*

3 Common multiples and a normal form

Let a be a letter and W a word. The a-series of W is a series $W_0^a, W_1^a, \ldots$ over $\mathcal{A}^* \dot\cup \infty$ defined as follows. Set $W_0^a :\equiv K_a(W)$, so by Lemma 4, either W_0^a is empty, an a-chain, begins with a or is not defined. Then for $i \geq 1$, define recursively

$$W_i^a :\equiv \begin{cases} a & \text{if } W_{i-1}^a \text{ is empty;} \\ W_{i-1}^a & \text{if } W_{i-1}^a \text{ begins with } a; \\ K_a(W_{i-1}^a b) & \text{if } W_{i-1}^a \text{ is an } a\text{-chain to } b\text{; or} \\ \infty & \text{otherwise.} \end{cases}$$

By Lemma 4, when it is defined W_i^a is either an a-chain or begins with a (or if $i = 0$, W_i^a may be empty). The a-series converges to a word W_k^a precisely when W_k^a begins with a. The following definition is intended to capture a notion of the limit of the a-series of W.

$$L(a, W) :\equiv \begin{cases} W_k^a & \text{if } W_k^a \equiv W_{k+1}^a\text{; or} \\ \infty & \text{otherwise.} \end{cases}$$

The following two examples are included to illustrate various ways in which $L(a, W) \equiv \infty$. The first example exhibits two a-series with members which are not defined, and in the second example the a-series is always defined but consists of words of strictly increasing length. The presentations are shown in Section 4 and [4] respectively to have the reduction property, and they clearly have the uniqueness, cancellative and homogeneity properties.

Example 7 We consider what is known as the positive singular braid monoid on three strings, which is denoted $\mathcal{S}^+_{\mathcal{A}_2}$ in Section 4. Let $\mathcal{A} = \{a, a', b, b'\}$ and $\mathcal{R} = \{(aba, bab), (a'ba, bab'), (b'ab, aba'), (aa', a'a), (bb', b'b)\}$. We first look at the a'-series of $V \equiv b'$. Since a' and b' are incomparable, $K_{a'}(b')$ is not defined and so $V_0^{a'} \equiv \infty$, so subsequently every member of the a'-series of V is not defined, giving $L(a', b') \equiv \infty$.

Now consider the a'-series of $W \equiv bb'$. Then $W_0^{a'} \equiv K_a(b') \equiv bb'$, $W_1^{a'} \equiv K_{a'}(bb'a) \equiv bK_a(b'a) \equiv bb'a$, but $W_2^{a'} \equiv K_{a'}(bb'ab) \equiv K_{a'}(baba') \equiv \infty$. Thus some members of the a' series of $W \equiv bb'$ are defined, but $L(a', bb') \equiv \infty$.

Example 8 Consider now the a-series of $W \equiv bc$ where $\mathcal{A} = \{a, b, c\}$ and $\mathcal{R} = \{(aba, bab), (bcb, cbc), (cac, aca)\}$. Observe that for any $k \geq 1$, $U_k \equiv (bacbac)^k$ is an a-chain with target a. The a-series starts with $W_0^a \equiv bc \equiv U_0bc$, and then for all $k \geq 0$,

$$\begin{array}{rclrclrcl} W^a_{6k} & \equiv & U_kbc, & W^a_{6k+2} & \equiv & U_kbaca, & W^a_{6k+4} & \equiv & U_kbacbab, \\ W^a_{6k+1} & \equiv & U_kbca, & W^a_{6k+3} & \equiv & U_kbacab, & W^a_{6k+5} & \equiv & U_kbacbabc, \end{array}$$

and so $W^a_{6k+6} \equiv U_kbacbacbc \equiv U_{k+1}bc$ and so on. In this case, the a-series is always defined but never converges to a word, and so $L(a, bc) \equiv \infty$.

The following result characterises the situation when $L(a, W) \not\equiv \infty$.

Lemma 9 *$L(a, W)$ is defined precisely when a and W have a common multiple, in which case $L(a, W)$ is a least common multiple of a and W and begins with a.*

Proof. If W is empty then $W_0^a \equiv W$ and $W_i^a \equiv a$ for all $i \geq 1$. Thus $L(a, W) \equiv a$, and so the result holds trivially. So we may suppose that W is nonempty.

Suppose that a and W have a common multiple M. By Lemma 4, $W_0^a \equiv K_a(W) \sim W$ is defined, and further, divides M. Since W is not empty, W_0^a either begins with a or is an a-chain, is a multiple of W and divides M. We will show that the same is true of all W_i^a, using induction on i. If W_i^a begins with a, then $W_j^a \equiv W_i^a$ for all $j \geq i$, and so we are done. Otherwise, W_i^a is an a-chain to b which is a multiple of W and divides M. So M is a common multiple of W_i^a and a, and thus by Lemma 2, M is a common multiple of $W_i^a b$ and a. Now by the second statement of Lemma 4, $W_{i+1}^a \equiv K_a(W_i^a b)$ is defined, and either begins with a or is an a-chain. Also M is a common multiple of $W_{i+1}^a \sim W_i^a b$ and a, and since W divides W_i^a, then W must also divide W_{i+1}^a. Thus we have shown that when a and W have a common multiple, every element of the a-series of W is defined, is a multiple of W, and divides M. Since elements of the a-series increase in length until an element begins with a, and since divisors of M cannot exceed M in length, eventually there is a first W_k^a which begins with a. Hence $L(a, W) \equiv W_k^a$. Furthermore, we have shown that $L(a, W)$ divides every common multiple M of a and W, making it a least common multiple.

On the other hand, suppose $L(a, W) \not\equiv \infty$. Then there is a number $k \geq 0$ such that the elements $W_0^a, \ldots, W_k^a$ of the a-series of W are defined, and W_k^a begins with a. If $k = 0$, then $L(a, W) \equiv W_0^a \sim W$. If $k > 0$ then by the definition of the a-series, there are letters $b_1, \ldots, b_k$ which are targets of the a-chains $W_0^a, \ldots, W_{k-1}^a$ respectively, and for each $i < k$, $W_i^a b_{i+1} \sim W_{i+1}$, so $L(a, W) \equiv W_k^a \sim W_0^a b_1 \cdots b_k \sim W b_1 \cdots b_k$. Hence $L(a, W)$ is a common multiple of a and W. $\square$

Let V and W be words. Define recursively:

$$L(V, W) :\equiv \begin{cases} W & \text{if } V \text{ is empty;} \\ aL(U, {}^{a^{-1}}L(a, W)) & \text{if } V \equiv aU,\ L(a, W) \not\equiv \infty \text{ and} \\ & \qquad\qquad L(U, {}^{a^{-1}}L(a, W)) \not\equiv \infty;\ \text{or} \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 10 *$L(V, W)$ is defined precisely when V and W have a common multiple, in which case $L(V, W)$ begins with V and is a least common multiple of V and W. Moreover, $L(V, W) \not\equiv \infty$ precisely when $L(W, V) \not\equiv \infty$, in which case $L(V, W) \sim L(W, V)$.*

Proof. If V is empty then the first statement holds trivially. Suppose then that $V \equiv aU$ for some letter a and word U.

Suppose that $L(V, W) \not\equiv \infty$. Then both $L(a, W)$ and $L(U, {}^{a^{-1}}L(a, W))$ are defined, so by Lemma 9 and induction, there are respectively least common multiples of a and W, and of U and ${}^{a^{-1}}L(a, W)$. Moreover, by induction, $L(U, {}^{a^{-1}}L(a, W))$ begins with U, so $L(V, W)$ begins with V. But

$aL(U,\ {}^{a^{-1}}L(a,W)) \sim a\ {}^{a^{-1}}L(a,W)Q \sim L(a,W)Q$ for some word Q. Thus W divides $L(V,W)$, making $L(V,W)$ a common multiple of V and W.

Suppose that $V \equiv aU$ and W have a common multiple $M \equiv aUN$, say. Then M is a common multiple of a and W, so by Lemma 9, $L(a,W)$ is defined, begins with a, and divides aUN. By cancellation, ${}^{a^{-1}}L(a,W)$ divides UN. Thus U and ${}^{a^{-1}}L(a,W)$ have a common multiple UN, and by induction, $L(U,\ {}^{a^{-1}}L(a,W))$ is defined and divides UN. Certainly then $L(V,W)$ is defined and $L(V,W) \equiv aL(U,\ {}^{a^{-1}}L(a,W))$ divides $aUN \equiv M$. By the first part this also proves $L(V,W)$ is a least common multiple of V and W.

Finally, by what we have just proved, $L(V,W) \not\equiv \infty$ if and only if V and W have a common multiple, which again is if and only if $L(W,V) \not\equiv \infty$. In this case, $L(V,W)$ and $L(W,V)$ divide one another, so by homogeneity are equivalent. □

Let $V_1, \ldots, V_m$ be words and let ε denote the empty word. Define recursively:

$$L(V_1,\ldots,V_m) :\equiv \begin{cases} \varepsilon & \text{if } m = 0, \\ V_1 & \text{if } m = 1, \\ \infty & \text{if } m \geq 2 \text{ and } L(V_2,\ldots,V_m) \equiv \infty, \\ L(V_1, L(V_2,\ldots,V_m)) & \text{if } m \geq 2 \text{ and } L(V_2,\ldots,V_m) \not\equiv \infty\ . \end{cases}$$

The following result follows by a simple induction on the previous lemma.

Lemma 11 *$L(V_1,\ldots,V_m)$ is defined precisely when $V_1,\ldots,V_m$ have a common multiple, in which case $L(V_1,\ldots,V_m)$ begins with V_1 and is a least common multiple of $V_1,\ldots,V_m$. Moreover, for any permutation σ of $\{1,\ldots,m\}$, $L(V_1,\ldots,V_m) \not\equiv \infty$ if and only if $L(V_{1\sigma},\ldots,V_{m\sigma}) \not\equiv \infty$, in which case $L(V_1,\ldots,V_m) \sim L(V_{1\sigma},\ldots,V_{m\sigma})$.* □

Corollary 12 *Let Ω be a finite set of words. Then Ω has a common multiple if and only if it has a least common multiple.* □

Since $\mathcal{A}$ is finite then an infinite subset of $\mathcal{A}^*$ must have elements of arbitrary length. By homogeneity, a common multiple must be at least as long as any of the factors. So an infinite set of words can have no common multiples. On the other hand, the empty word divides every other word, so an arbitrary set Ω of words has a common divisor. If D denotes the set of all common divisors of Ω, then D is finite by the preceding discussion. Since every element of Ω is a common multiple of D, then by Corollary 12, D has a least common multiple, which is a greatest common divisor of Ω. Thus greatest common divisors always exist.

Define a total order $<$ on $\mathcal{A}$. For any nonempty subset I of $\mathcal{A}$ let $\Delta_I \equiv L(a_1, \ldots, a_k)$ where $a_1 < \cdots < a_k$ and $I = \{a_1, \ldots, a_k\}$. Define $\Delta_\emptyset \equiv \varepsilon$. For a word W, let $\Lambda(W)$ be the set of letters which divide W. This is calculable, as $K_a(W)$ is calculable, and $\Lambda(W)$ is just the set of a in $\mathcal{A}$ such that $K_a(W)$ begins with a. Since W is a common multiple of $\Lambda(W)$, $\Delta_{\Lambda(W)}$ is defined and (as it as a least common multiple), divides W. Thus $(W/\Delta_{\Lambda(W)})$ is defined.

We are now in a position to define a normal form. Define an operator $N : \mathcal{A}^* \longrightarrow \mathcal{A}^*$ recursively by

$$N(W) :\equiv \begin{cases} W & \text{if } \ell(W) \leq 1 \text{ , and} \\ \Delta_{\Lambda(W)} N(W/\Delta_{\Lambda(W)}) & \text{otherwise.} \end{cases}$$

It is clear from this definition and Lemma 5 that $N(W) \sim W$. The following theorem shows that N produces a unique representative for all words equivalent to W.

Theorem 13 *For any two words V and W over $\mathcal{A}$, $V \sim W$ if and only if $N(V) \equiv N(W)$.*

Proof. By the preceding remark, if $N(V) \equiv N(W)$ then $V \sim N(V) \equiv N(W) \sim W$. Now suppose $V \sim W$. We first notice that if $\ell(V) \leq 1$ then the result follows. Otherwise we have

$$N(V) \equiv \Delta_{\Lambda(V)} N(V/\Delta_{\Lambda(V)}) \text{ and } N(W) \equiv \Delta_{\Lambda(W)} N(W/\Delta_{\Lambda(W)}).$$

Since $V \sim W$ then $\Lambda(V) = \Lambda(W)$, and so $\Delta_{\Lambda(V)} \equiv \Delta_{\Lambda(W)}$. Cancellation gives $(V/\Delta_{\Lambda(V)}) \sim (W/\Delta_{\Lambda(W)})$, so by induction $N(V/\Delta_{\Lambda(V)}) \equiv N(W/\Delta_{\Lambda(W)})$, and thus $N(V) \equiv N(W)$. □

4 Positive singular Artin monoids

Let Γ_M be a labelled graph with n vertices in one to one correspondence with a finite indexing set I, and with edge labels from the set $\{3, 4, 5, \ldots, \infty\}$. Let

m_{ij} denote the label of the edge between nodes i and j, or set $m_{ij} = 2$ if there is no such edge. Let $\langle xy \rangle^m$ denote the alternating product $xyx\cdots$ of length m. Let $S = \{\sigma_i \mid i \in I\}$ and $T = \{\tau_i \mid i \in I\}$. The *positive singular Artin monoid of type* M, denoted $\mathcal{S}_M^+$, is the monoid generated by $S \cup T$ subject to the relations $\mathcal{R}$ listed below:

$$\begin{aligned}
\langle \sigma_a \sigma_b \rangle^{m_{ab}} &= \langle \sigma_b \sigma_a \rangle^{m_{ab}} \text{ if } m_{ab} < \infty, \\
\tau_a \langle \sigma_b \sigma_a \rangle^{m_{ab}-1} &= \begin{cases} \langle \sigma_b \sigma_a \rangle^{m_{ab}-1} \tau_b \text{ if } m_{ab} < \infty \text{ and is odd, or} \\ \langle \sigma_b \sigma_a \rangle^{m_{ab}-1} \tau_a \text{ if } m_{ab} < \infty \text{ and is even,} \end{cases} \\
\tau_a \tau_b &= \tau_b \tau_a \text{ if } m_{ab} = 2, \text{ and} \\
\sigma_a \tau_a &= \tau_a \sigma_a \text{ for all } a \text{ in } I.
\end{aligned}$$

The relations of the first type involve only letters from S and are called the *braid relations*, and will be denoted $\mathcal{R}_1$. Relations of the last three types will be denoted $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$ respectively. It has been shown in [4] that the monoid generated by S and subject to the braid relations has the reduction property. In this section we show that positive singular Artin monoids have the reduction property. Inspection of the relations will assure the reader that the conditions for the uniqueness, cancellative and homogeneity properties are satisfied.

The monoid generated by S subject only to the relations $\mathcal{R}_1$ is called the *positive Artin monoid of type* M, and denoted $\mathcal{G}_M^+$. Define $\hat{}: S \cup T \to S$ to be the identity on S and to map τ_a to σ_a for all a. Observe that if $\rho_1 = \rho_2$ is any relation from $\mathcal{R}$ then either $\widehat{\rho_1} \equiv \widehat{\rho_2}$, or $\widehat{\rho_1} = \widehat{\rho_2}$ is a relation in $\mathcal{R}_1$. Thus $\hat{}$ defines a (surjective) homomorphism from $\mathcal{S}_M^+$ to $\mathcal{G}_M^+$. We will use the symbol $\overset{\mathcal{R}_1}{\sim}$ to indicate when two words over S are equivalent by the relations $\mathcal{R}_1$. The relation $\sim$ is with respect to the alphabet $S \cup T$ and the relations $\mathcal{R}$. Thus if $X \sim Y$, then $\widehat{X} \overset{\mathcal{R}_1}{\sim} \widehat{Y}$.

The *Coxeter group of type* M is the group generated by S subject to the relations $\mathcal{R}_1$ and the relations $\sigma_i^2 = 1$ for all $i \in I$. If the Coxeter group of type M is finite, then M is said to be a *finite type*. It is the case (see, for example, [8], Chapter 2) that a Coxeter group is finite precisely when its graph is a finite disjoint union of the graphs shown in Figure 1. (When $m_{ij} = 3$, the edge label is suppressed).

Another way to write the relations of types $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ is by using the variables x and y to represent whatever combination of σ or τ is desired, and then

$$x_a \langle \sigma_b \sigma_a \rangle^{m_{ab}-2} y_e = y_b \langle \sigma_a \sigma_b \rangle^{m_{ab}-2} x_d,$$

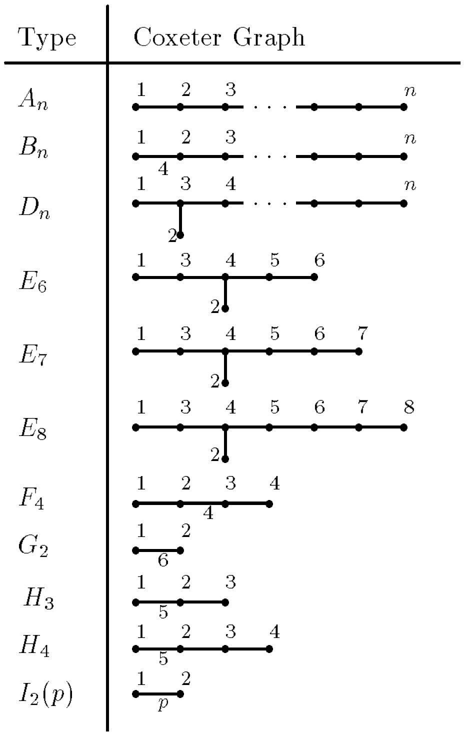

Figure 1: The irreducible Coxeter graphs of finite types.

where $e = a$ and $d = b$ if m_{ab} is odd, or $e = b$ and $d = a$ if m_{ab} is even, subject to the condition that if $m_{ab} > 2$ then at most one of x and y is τ.

For example if $m_{ab} = 3$ then the above becomes:

$$x_a \sigma_b y_a = y_b \sigma_a x_b,$$

which yields the relations:

$$\begin{aligned} \sigma_a \sigma_b \sigma_a &= \sigma_b \sigma_a \sigma_b, \\ \tau_a \sigma_b \sigma_a &= \sigma_b \sigma_a \tau_b \text{ and} \\ \sigma_a \sigma_b \tau_a &= \tau_b \sigma_a \sigma_b, \end{aligned}$$

using all possible substitutions of τ and σ for x and y, except $x = \tau = y$ which is forbidden because $m_{ab} = 3$. Under this notation the reduction property translates to the statement below.

Lemma 14 *Suppose that* $x_aX \sim y_bY$ *for some words* X *and* Y, *with* x_a *and* y_b *elements of* $S \cup T$. *If* $m_{ab} > 2$ *then at most one of* x *and* y *is* τ. *If* $x_a \equiv y_b$ *then* $X \sim Y$. *If* $x \neq y$ *but* $a = b$ *then there is a word* V *such that* $X \sim y_bV$ *and* $Y \sim x_aV$. *Otherwise there is a word* V *such that*

$$X \sim \langle \sigma_b \sigma_a \rangle^{m_{ab}-2} y_e V \text{ and } Y \sim \langle \sigma_a \sigma_b \rangle^{m_{ab}-2} x_d V,$$

where $e = a$ *and* $d = b$ *if* m_{ab} *is odd, or* $e = b$ *and* $d = a$ *if* m_{ab} *is even.*

Proof. The argument goes by induction on both the length of X and the length of the transformation from x_aX to y_bY. If X is empty then the result holds trivially. If $x_aX \equiv y_bY$ then there is nothing to prove. If x_aX differs from y_bY by an elementary transformation then the lemma holds.

Suppose then that the result is true for words of length less than $\ell(X)$, and for words of length equal to X which differ by a transformation of length at most t. The following result is a corollary of the reduction lemma restricted to words for which we assume the lemma holds:

Intermediate Lemma. *Suppose* $\ell(Z) \leq \ell(X)$. *Then*

(a) if $Z \sim MW \sim MU$ *then* $W \sim U$;

(b) if u *is a letter such that* $Z \sim uU \sim MW$ *and* $(uv, vu) \in \mathcal{R}^\Sigma$ *or* $u \equiv v$ *for all letters* v *appearing in the word* M, *then either* u *appears in* M *(and so divides it), or there is a word* V *such that* $U \sim MV$ *and* $W \sim uV$; *and*

(c) if $Z \sim \tau_i W \sim \tau_j V$ *where* $i \neq j$ *then* $m_{ij} = 2$.

Now suppose that x_aX differs from y_bY by a transformation of length $t+1$. Thus there is a letter z_c and a word Z such that x_aX differs from z_cZ by an elementary transformation and z_cZ and y_bY differ by a tranformation of length t. Thus the reduction property holds (by assumption) for $x_aX \sim z_cZ$ and $z_cZ \sim y_bY$.

There are six main cases. The reduction property will be frequently applied without comment whenever permitted by induction.

<u>**Case I.**</u> Suppose x_a, y_b and z_b are not distinct. If $x_a \equiv z_c$ or $y_b \equiv z_c$ then application of the reduction property gives $X \sim Z$ or $Y \sim Z$ respectively. By transitivity of the relation $\sim$, the result holds.

Now suppose $x_a \equiv y_b$, but that these are both distinct from z_c. So $x_aX \sim z_cZ$ by an elementary transformation, and $z_cZ \sim x_aY$ by a transformation of length t. Applying the reduction property to $x_aX \sim z_cZ$, there are words W_1, W_2 and W such that

$$(x_aW_1, z_cW_2) \in \mathcal{R}^\Sigma, \; X \sim W_1W \text{ and } Z \sim W_2W.$$

By the uniqueness property, the pairs (x_aW_1, z_cW_2) and (z_cW_2, x_aW_1) are the only ones in $\mathcal{R}^\Sigma$ beginning with the letters x_a and z_c. So applying the reduction property to $z_cZ \sim x_aY (\equiv y_bY)$, there is a word W' with $Z \sim W_2W'$ and $Y \sim W_1W'$. So $Z \sim W_2W \sim W_2W'$. By the cancellation part of the Intermediate Lemma, $W \sim W'$. Thus $X \sim W_1W \sim W_1W' \sim Y$, as required.

Case II. Here we consider when x_a, z_c and y_b are distinct, and (x_az_c, z_cx_a), $(y_bz_c, z_cy_b) \in \mathcal{R}^\Sigma$. Applying the reduction property, there are words W and W' such that

$$X \sim z_cW,\ Z \sim x_aW \sim y_bW' \text{ and } Y \sim z_cW'.$$

Now we can use induction on the length of Z. So there are words W_1, W_2 and U such that

$$(x_aW_1, y_bW_2) \in \mathcal{R}^\Sigma,\ W \sim W_1U \text{ and } Y \sim W_2U.$$

Inspection of the relations $\mathcal{R}$ indicates that all letters in W_1 and W_2 must have subscript a or b. Thus z_c commutes with W_1 and W_2, and so $X \sim z_cW \sim z_cW_1U \sim W_1z_cU$ and $Y \sim z_cW' \sim z_cW_2U \sim W_2z_cU$. Setting $V \equiv z_cU$, we have shown that there are words W_1, W_2 and V such that $X \sim W_1V$, $Y \sim W_2V$ and $(x_aW_1, y_bW_2) \in \mathcal{R}^\Sigma$, as required.

Case III. In this case we consider when x_a, y_b and z_c are distinct, but a, b and c are not distinct. There are three main subcases.

(1) The first case considered is when $a = b$. For x_a and y_b to be distinct, then one of x and y is σ and the other τ. If $m_{ac} = 2$, then (x_az_c, z_cx_a) and (y_az_c, z_cy_a) are in $\mathcal{R}^\Sigma$, which was already covered in Case II.

So suppose $m_{ac} > 2$. We know one of x and y is τ, so by application of the reduction property z must be σ. Moreover, there are words W and U such that

$$\begin{aligned} &X \sim \langle\sigma_c\sigma_a\rangle^{m_{ac}-1}W,\ Z \sim \langle\sigma_a\sigma_c\rangle^{m_{ac}-2}x_dW, \quad \text{and} \\ &Z \sim \langle\sigma_a\sigma_c\rangle^{m_{ac}-2}y_dU,\ Y \sim \langle\sigma_c\sigma_a\rangle^{m_{ac}-1}U, \end{aligned}$$

where $d = a$ if m_{ac} is even, and c otherwise. Now by induction on the length of Z, we can cancel $\langle\sigma_a\sigma_c\rangle^{m_{ac}-2}$, giving $x_dW \sim y_dU$. Because U is shorter than X, there is a word V such that $W \sim y_dV$ and $U \sim x_dV$. Then

$$\begin{aligned} X &\sim \langle\sigma_c\sigma_a\rangle^{m_{ac}-1}W \sim \langle\sigma_c\sigma_a\rangle^{m_{ac}-1}y_dV \sim y_a\langle\sigma_c\sigma_a\rangle^{m_{ac}-1}V, \quad \text{and} \\ Y &\sim \langle\sigma_c\sigma_a\rangle^{m_{ac}-1}U \sim \langle\sigma_c\sigma_a\rangle^{m_{ac}-1}x_dV \sim x_a\langle\sigma_c\sigma_a\rangle^{m_{ac}-1}V. \end{aligned}$$

(2) The second case is when $a = c$. So at most one of x and z is τ. If $m_{ab} = 2$ then $(x_a z_a, z_a x_a)$ and $(y_b z_a, z_a y_b)$ are in $\mathcal{R}^\Sigma$, which was already covered in Case II.

So suppose that $m_{ab} > 2$. By the reduction property for $z_a Z \sim y_b Y$, at most one of z and y is τ, and moreover there is a word U such that

$$Z \sim \langle \sigma_b \sigma_a \rangle^{m_{ab}-2} y_d U, \quad \text{and} \quad Y \sim \langle \sigma_a \sigma_b \rangle^{m_{ab}-2} z_e U,$$

where $d \equiv a$ if m_{ab} is odd, and b otherwise, and $e \equiv b$ if m_{ab} is odd, and a otherwise. Further, there is a word W such that $X \sim z_a W$ and $Z \sim x_a W$. Thus $x_a W \sim \langle \sigma_b \sigma_a \rangle^{m_{ab}-2} y_d U$, so by induction there is a word V_1 such that

$$W \sim \langle \sigma_a \sigma_b \rangle^{m_{ab}-1} V_1 \text{ and } \langle \sigma_a \sigma_b \rangle^{m_{ab}-3} y_d U \sim \langle \sigma_a \sigma_b \rangle^{m_{ab}-2} x_e V_1,$$

and cancelling from the left, we get $y_d U \sim \sigma_d x_e V_1$.

First, suppose that y is τ. So z must be σ, and to preserve distinctness, x must be τ also. So $y_d U \sim \sigma_d x_e V_1$ implies there is a word V_2 such that $U \sim \sigma_d V_2$ and $x_e V_1 \sim y_d V_2$. But since $m_{ab} > 2$ and $\{x_e, y_d\} = \{\tau_a, \tau_b\}$, there are no relations $(x_e W_1, y_d W_2)$ in $\mathcal{R}^\Sigma$, contradicting the reduction property applied to $x_e V_1 \sim y_d V_2$. So y cannot be τ.

Hence y must be σ. Cancelling gives $U \sim x_e V_1$. So

$$\begin{aligned} X &\sim z_a \langle \sigma_b \sigma_a \rangle^{m_{ab}-1} V_1 \sim \langle \sigma_b \sigma_a \rangle^{m_{ab}-1} z_e V_1 \sim \langle \sigma_b \sigma_a \rangle^{m_{ab}-2} y_d z_e V_1, \quad \text{and} \\ Y &\sim \langle \sigma_a \sigma_b \rangle^{m_{ab}-2} z_e U \sim \langle \sigma_a \sigma_b \rangle^{m_{ab}-2} z_e x_e V_1 \sim \langle \sigma_a \sigma_b \rangle^{m_{ab}-2} x_e z_e V_1, \end{aligned}$$

and we are done.

(3) The final subcase of Case III is when $b = c$. This can be done by the same argument as part (2), swapping a with b and x with y.

Completion of these three cases leaves us only with the cases when a, b and c are all distinct. By the comment at **2.2** of [4], if the subgraph $\Gamma_M\{a,b,c\}$ of Γ_M consisting of the vertices a, b and c and any edges joining them does not define a finite Coxeter group, then there are no words X', Y' and Z' such that $\sigma_a X' \overset{\mathcal{R}_1}{\sim} \sigma_c Z' \overset{\mathcal{R}_1}{\sim} \sigma_b Y'$. Since our assumption is that $x_a X \sim z_c Z \sim y_b Y$, then by application of the homomorphism $\widehat{\ }$, $\sigma_a \widehat{X} \overset{\mathcal{R}_1}{\sim} \sigma_c \widehat{Z} \overset{\mathcal{R}_1}{\sim} \sigma_b \widehat{Y}$. Thus $\Gamma_M\{a,b,c\}$ is a graph with 3 vertices which must define a finite Coxeter group. By inspecting Figure 1, we need only consider when $\{m_{ab}, m_{ac}, m_{bc}\}$ is either $\{2,2,p\}$ where $p = 2,3,4$ or 5, (some of which have already been considered in Case II), or $\{2,3,q\}$ where $q = 3,4$ or 5.

Case IV. Here we consider when $m_{ac} = 2$. The case when $m_{bc} = 2$ as well was covered in Case II, so we need only consider when $m_{bc} > 2$. Since there are no pairs of the form $(\tau_b W_1, \tau_c W_2)$ for $m_{bc} > 2$ in $\mathcal{R}^\Sigma$, at most one of y and z is τ. We consider the cases $m_{ab} = 2, 3, 4$ and 5 as separate subcases.

(1) Suppose $m_{ab} = 2$. There are words W and U such that $X \sim z_c W$ and $Z \sim x_a W$; and $Z \sim \langle \sigma_b \sigma_c \rangle^{m_{bc}-2} y_d U$ and $Y \sim \langle \sigma_c \sigma_b \rangle^{m_{bc}-2} z_e U$, where $d = c$ if m_{bc} is odd, and b otherwise, and $e = b$ if m_{bc} is odd, and c otherwise. Applying the Intermediate Lemma part (b) to the equation $Z \sim x_a W \sim \langle \sigma_b \sigma_c \rangle^{m_{bc}-2} y_d U$, there is a word V such that $W \sim \langle \sigma_b \sigma_c \rangle^{m_{bc}-2} y_d V$ and $U \sim x_a V$. Then since $m_{ab} = m_{ac} = 2$, and at most one of y and z is τ,

$$\begin{aligned} &X \sim z_c \langle \sigma_b \sigma_c \rangle^{m_{bc}-2} y_d V \sim y_b \langle \sigma_c \sigma_b \rangle^{m_{bc}-2} z_e V, \quad \text{and} \\ &Y \sim \langle \sigma_c \sigma_b \rangle^{m_{bc}-2} z_e x_a V \sim x_a \langle \sigma_c \sigma_b \rangle^{m_{bc}-2} z_e V. \end{aligned}$$

(2) In this case we consider when $m_{ab} = 3$. The situation that $m_{bc} = 2$ was completed in Case II. We consider the cases when $m_{bc} = 3, 4$ and 5 as separate subcases.

(a) Suppose $m_{bc} = 3$. Then at most one of z and y is τ. There are words W and U such that

$$X \sim z_c W \ \text{ and } \ Z \sim x_a W, \ \text{ and } Z \sim \sigma_b y_c U \ \text{ and } \ Y \sim \sigma_c z_b U.$$

Since $Z \sim x_a W \sim \sigma_b y_c U$ then there is a word V_1 such that $W \sim \sigma_b \sigma_a V_1$ and $y_c U \sim \sigma_a x_b V_1$. There is a word V_2 such that $U \sim \sigma_a V_2$ and $x_b V_1 \sim y_c V_2$. By application of the reduction property to this last equation, firstly note that at most one of x and y can be τ (since $m_{bc} > 2$), and secondly, there is a word V_3 such that $V_1 \sim \sigma_c y_b V_3$ and $V_2 \sim \sigma_b x_c V_3$. Reconstructing X and Y we get $X \sim z_c W \sim z_c \sigma_b \sigma_a \sigma_c y_b V_3$ and $Y \sim \sigma_c z_b U \sim \sigma_c z_b \sigma_a \sigma_b x_c V_3$. Straightforward application of the relations gives $X \sim \sigma_b y_a [\sigma_c \sigma_b z_a V_3]$ and $Y \sim \sigma_a x_b [\sigma_c \sigma_b z_a V_3]$, and we are done.

Cases (b) where $m_{bc} = 4$ and (c) where $m_{bc} = 5$ are similar but more complicated. The induction process is invoked seven times in case (b) and fourteen times in case (c). As in the above process, every step is forced.

(3) In this case we consider when $m_{ab} = 4$. This forces $m_{bc} = 3$. So at most one of y and z is τ, and moreover there are words W and U such that

$$X \sim z_c W \ \text{ and } \ Z \sim x_a W \sim \sigma_b y_c U \ \text{ and } \ Y \sim \sigma_c z_b U.$$

From successive applications of the reduction property, there are words V_1, V_2, V_3 and V_4 such that

$$\begin{gathered} W \sim \sigma_b\sigma_a\sigma_b V_1 \text{ and } y_c U \sim \sigma_a\sigma_b x_a V_1, \\ U \sim \sigma_a V_2 \text{ and } \sigma_b x_a V_1 \sim y_c V_2, \\ x_a V_1 \sim \sigma_c y_b V_3 \text{ and } V_2 \sim \sigma_b\sigma_c V_3, \\ V_1 \sim \sigma_c V_4 \text{ and } y_b V_3 \sim x_a V_4. \end{gathered}$$

Since $m_{ab} = 4$, at most one of x and y is τ, and there is a word V_5 such that

$$V_3 \sim \sigma_a\sigma_b x_a V_5 \text{ and } V_4 \sim \sigma_b\sigma_a y_b V_5.$$

Reconstructing X and Y we get

$$X \sim z_c\sigma_b\sigma_a\sigma_b\sigma_c\sigma_b\sigma_a y_b V_5 \text{ and } Y \sim \sigma_c z_b\sigma_a\sigma_b\sigma_c\sigma_a\sigma_b x_a V_5.$$

Application of the relations gives

$$X \sim \sigma_b\sigma_a y_b[\sigma_c\sigma_b\sigma_a\sigma_b z_c V_5] \text{ and } Y \sim \sigma_a\sigma_b x_a[\sigma_c\sigma_b\sigma_a\sigma_b z_c V_5],$$

and we are done.

(4) This last subcase of Case IV considers when $m_{ab} = 5$. We only need consider $m_{bc} = 3$, since $m_{bc} = 2$ was covered in Case II. This subcase is very similar to the previous subcase (3), except that induction must be invoked eleven times (where above it was invoked six times).

Case V. In this case we consider when $m_{bc} = 2$. This case proceeds identically to Case IV except for swapping a for b, and x for y.

Case VI. The final case considers when both m_{ac} and m_{bc} are larger than 2, which forces $m_{ab} = 2$. At most one of x and z is τ, and at most one of y and z is τ. The three subcases when m_{ac} is 3, 4 and 5 are considered separately.

(1) We consider first when $m_{ac} = 3$. Within this subcase, the situations when $m_{bc} = 3, 4$ or 5 are considered separately.

(a) Suppose $m_{bc} = 3$. Then there are words W and U such that

$$X \sim \sigma_c z_a W \text{ and } Z \sim \sigma_a x_c W \sim \sigma_b y_c U \text{ and } Y \sim \sigma_c z_b U.$$

By successive applications of the reduction property, there are words V_1, V_2, and V_3 such that

$$\begin{gathered} x_c W \sim \sigma_b V_1 \text{ and } y_c U \sim \sigma_a V_1, \\ W \sim \sigma_b\sigma_c V_2 \text{ and } V_1 \sim \sigma_c x_b V_2, \\ U \sim \sigma_a\sigma_c V_3 \text{ and } V_1 \sim \sigma_c y_a V_3. \end{gathered}$$

Thus $V_1 \sim \sigma_c x_b V_2 \sim \sigma_c y_a V_3$. Cancelling gives $x_b V_2 \sim y_a V_3$. Hence there is a word V_4 such that

$$V_2 \sim y_a V_4 \text{ and } V_3 \sim x_b V_4.$$

Now we can reconstruct X and Y to get $X \sim \sigma_c z_a \sigma_b \sigma_c y_a V_4$ and $Y \sim \sigma_c z_b \sigma_a \sigma_c x_b V_4$. Applying the relations (noting that at most one of y and z is τ, and at most one of x and z is τ), we get $X \sim y_b[\sigma_c \sigma_b \sigma_a z_c V_4]$ and $Y \sim x_a[\sigma_c \sigma_b \sigma_a z_c V_4]$, and we are done.

Cases (b) and (c) where m_{bc} is 4 and 5 respectively are very similar, but the induction step must be invoked in these cases nine and eighteen times respectively.

(2) This subcase deals with $m_{ac} = 4$. Thus $m_{bc} = 3$. The argument here proceeds as for Case VI (1) (b), except for swapping a for b and x for y throughout.

(3) This subcase deals with $m_{ac} = 5$. Again, $m_{bc} = 3$. This can be argued as for Case VI (1) (c), except for swapping a for b and x for y throughout. □

Thus positive singular Artin monoids have the uniqueness, cancellation, homogeneity and reduction properties. The results of Sections 1 and 2 apply, so we can solve the word and divisibion problems in positive singular Artin monoids, and in the notation of Section 2, the operator defined recursively by $N(W) \equiv \Delta_{\Lambda(W)} N(W/\Delta_{\Lambda(W)})$ yields a unique normal form for words over $S \cup T$.

To complete this section we include a few extensions of Sections 1 and 2 which are specific to positive singular Artin monoids which will be necessary in the following.

Lemma 15 *Let C be a word over S. If C is a σ_a-chain to σ_b such that $C\sigma_b \sim \sigma_a C$, then $C\tau_b \sim \tau_a C$.*

Proof. The proof is by induction on the length of C. If C is empty then $a = b$ and the result holds trivially. Otherwise, write $C \equiv \sigma_c V$ for some word V over S. So $\sigma_c V \sigma_b \sim \sigma_a \sigma_c V$. The reduction property implies there is a word W such that $\sigma_c V \sim \langle \sigma_c \sigma_a \rangle^{m_{ac}-1} W$. Since C is a σ_a-chain, it is not divisible by σ_a. Hence W cannot be divisible by σ_d, the target of $\langle \sigma_c \sigma_a \rangle^{m_{ac}-1}$. After substitution, $C\sigma_b \sim \sigma_a C$ yields

$$\langle \sigma_c \sigma_a \rangle^{m_{ac}-1} W \sigma_b \sim \sigma_a \langle \sigma_c \sigma_a \rangle^{m_{ac}-1} W \sim \langle \sigma_c \sigma_a \rangle^{m_{ac}} W,$$

so cancelling gives $\sigma_d W \sim W\sigma_b$. Since σ_d and W have a common multiple $W\sigma_b$ then by Lemma 4, $K_{\sigma_d}(W) \not\equiv \infty$, and since σ_d does not divide W, then $K_{\sigma_d}(W)$ must be a σ_d-chain. By the second part of Lemma 3, the target of $K_{\sigma_d}(W)$ is σ_b. Thus W is equivalent to a σ_d-chain $K_{\sigma_d}(W)$ to σ_b, and by the induction hypothesis, $\tau_d W \sim W\tau_b$. Hence

$$\tau_a C \sim \tau_a \langle \sigma_c \sigma_a \rangle^{m_{ac}-1} W \sim \langle \sigma_c \sigma_a \rangle^{m_{ac}-1} \tau_d W \sim \langle \sigma_c \sigma_a \rangle^{m_{ac}-1} W \tau_b \sim C\tau_b.$$

□

The final results in this section are about the positive Artin monoid, which is the monoid generated by S subject only to the braid relations (those denoted $\mathcal{R}_1$). Suppose that W and V are words over S such that $W \sim V$. Since the only relations involving only letters from S are the braid relations, it must be that $W \overset{\mathcal{R}_1}{\sim} V$. This shows that $\mathcal{G}_M^+$, the positive Artin monoid of type M, embeds in $\mathcal{S}_M^+$. This enables us to use many of the results obtained in [4] about words over S. For example, it is shown there that when M is of finite type, $L(V_1, \ldots, V_m) \not\equiv \infty$ for arbitrary words $V_1, \ldots, V_m$ over S. In particular, $\Delta_I \not\equiv \infty$ for all $I \subseteq S$. On the other hand, if M is not of finite type, then S has no common multiple (a corollary of [4] 2.2) and so $\Delta_S \equiv \infty$.

For the rest of this section we suppose that M is of finite type. The *fundamental element* Δ of type M is defined to be Δ_S. Recall that Δ_S is the word $L(\sigma_{a_1}, \ldots, \sigma_{a_n})$ where $\sigma_{a_1} < \cdots < \sigma_{a_n}$ for a fixed total order $<$ on S and where $S = \{\sigma_{a_1}, \ldots, \sigma_{a_n}\}$. We list below some properties of the fundamental element observed in [4]. The operator `rev` takes a word and writes it in reverse order.

Remark 16 *The following list of properties of the fundamental element is a distillation of results of [4], Sections 5 and 8.*
1. *Δ is a word over S,*
2. *Δ is both a left and right least common multiple of S,*
3. $\mathtt{rev}\Delta \sim \Delta$,
4. *There is a uniquely determined involutionary automorphism φ of S^* (the free monoid on S), with the properties that*
(a) *φ permutes S, and*
(b) *for all words W over S, $W\Delta \overset{\mathcal{R}_1}{\sim} \Delta\varphi(W)$.*
5. *The centre of the positive Artin monoid is infinite cyclic for all finite types, and the word Δ represents the generator of the centre of the positive*

Artin monoid for all finite types except types A_n for $n \geq 2$, D_{2k+1}, E_6 and $I_2(2q+1)$, in which cases Δ^2 represents the generator of the centre.

We extend φ to an automorphism $\tilde{\varphi}$ of $(S \cup T)^*$ by defining

$$\begin{array}{rcll} \tilde{\varphi}(\sigma_i) & \equiv & \varphi(\sigma_i), & \text{and} \\ \tilde{\varphi}(\tau_i) & \equiv & \tau_j & \text{where } \varphi(\sigma_i) = \sigma_j. \end{array}$$

It is straightforward to check that whenever (ρ_1, ρ_2) is in $\mathcal{R}$, then $\tilde{\varphi}(\rho_1) \sim \tilde{\varphi}(\rho_2)$. Thus if $W \sim V$ in $\mathcal{S}_M^+$, then $\tilde{\varphi}(W) \sim \tilde{\varphi}(V)$. Furthermore, $\tilde{\varphi}$ permutes T as well as S.

Lemma 17 *For each $a \in I$, both $\sigma_a\Delta \sim \Delta\tilde{\varphi}(\sigma_a)$ and $\tau_a\Delta \sim \Delta\tilde{\varphi}(\tau_a)$. So for all words W over $S \cup T$, $W\Delta \sim \Delta\tilde{\varphi}(W)$. Moreover, Δ represents the generator of the centre of the positive Artin monoid for all finite types except types A_n for $n \geq 2$, D_{2k+1}, E_6 and $I_2(2q+1)$, in which cases Δ^2 represents the generator of the centre.*

Proof. Consider any i. By Property 4 above, $\sigma_i\Delta \sim \Delta\varphi(\sigma_i)$, so $\sigma_i\Delta \sim \Delta\tilde{\varphi}(\sigma_i)$. Suppose $\sigma_k \sim \tilde{\varphi}(\sigma_i)$. Thus $\sigma_i\Delta \sim \Delta\sigma_k$. Let p be the largest integer such that $\Delta \sim \sigma_i^p W$ for some word W. Then $\sigma_i^{p+1}W \sim \sigma_i^p W\sigma_k$, so by cancelling, $\sigma_i W \sim W\sigma_k$. Since W and σ_i have a common multiple then $C \equiv K_{\sigma_i}(W)$ is defined, and since σ_i does not divide W (by maximality of p), C is a σ_i-chain. By the second statement of Lemma 3, C has target σ_k. By Lemma 15, $\tau_i C \sim C\tau_k$. Thus

$$\tau_i\Delta \sim \tau_i\sigma_i^p C \sim \sigma_i^p\tau_i C \sim \sigma_i^p C\tau_k \sim \Delta\tau_k.$$

Thus $\tau_i\Delta \sim \Delta\tilde{\varphi}(\tau_i)$. The last statement is immediate from the others. $\square$

5 Singular Artin monoids

Given a labelled graph Γ_M as described in the previous section, with vertices in correspondence with a finite indexing set I, let $S^{-1} = \{\sigma_i^{-1} \mid i \in I\}$, the set of formal inverses of S. Then the *singular Artin monoid of type M*, denoted $\mathcal{S}_M$, is the monoid generated by $S \cup S^{-1} \cup T$ and subject to the relations $\mathcal{R}$ and the free group relations:

$$\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1 \qquad \text{for each } i \text{ in } I.$$

If W and V are words over $S \cup S^{-1} \cup T$ which are equivalent under the relations $\mathcal{R}$ and the free group relations then write $W \approx V$. When $X \equiv \sigma_{a_1} \cdots \sigma_{a_k}$ is a word over S, we will denote by X^{-1} the word $\sigma_{a_k}^{-1} \cdots \sigma_{a_1}^{-1}$. Then $XX^{-1} \approx X^{-1}X \approx 1$.

The special case when $I = \{1, 2, \ldots, n\}$, $m_{ij} = 3$ when $|i - j| = 1$ and $m_{ij} = 2$ when $|i - j| \geq 1$, is the singular Artin monoid of type A_n, and was introduced in [2] and [3] as the *singular braid monoid on $n + 1$ strings*. In this section we solve the word and division problems in the singular Artin monoids of finite type, which includes the singular braid monoid. In these types, the fundamental element exists and plays a significant role. For the rest of this section we suppose that M is of finite type.

Let ζ be a word representing the generator of the centre of the positive singular Artin monoid — either Δ or Δ^2 by Lemma 17. Now $\zeta x \sim x\zeta$ for all x in $S \cup T$. Thus

$$\sigma_a^{-1}\zeta \approx \sigma_a^{-1}\zeta\sigma_a\sigma_a^{-1} \approx \sigma_a^{-1}\sigma_a\zeta\sigma_a^{-1} \approx \zeta\sigma_a^{-1}.$$

Thus ζ represents a central element in $\mathcal{S}_M$. Clearly, by similar reasoning, ζ^{-1} represents a central element in $\mathcal{S}_M$. Since ζ is divisible by all elements of S, $\zeta_a \equiv {}^{\sigma_a^{-1}}K_{\sigma_a}(\zeta)$ is defined. Centrality yields $\sigma_a\zeta_a\sigma_a \sim \zeta\sigma_a \sim \sigma_a\zeta \sim \sigma_a^2\zeta_a$, and cancelling σ_a then gives $\zeta_a\sigma_a \sim \zeta \sim \sigma_a\zeta_a$.

Let W be a word over $S \cup S^{-1} \cup T$. There are words U_i over $S \cup T$ such that $W \equiv U_0\sigma_{a_1}^{-1}U_1\sigma_{a_2}^{-1}U_2 \cdots U_{k-1}\sigma_{a_k}^{-1}U_k$. Then define

$$\theta_1(W) \equiv U_0\zeta_{a_1}U_1\zeta_{a_2}U_2 \cdots U_{k-1}\zeta_{a_k}U_k, \text{ and } \theta_2(W) = k.$$

So θ_2 counts the number of occurrences of letters from S^{-1} in W, and θ_1 replaces each with a corresponding ζ_a. Thus $\theta_1(W)$ is always a word over $S \cup T$, and θ_1 acts as the identity on words over $S \cup T$. Moreover, for any words X and Y, $\theta_1(XY) \equiv \theta_1(X)\theta_1(Y)$, and

$$\begin{aligned}
\zeta^{-\theta_2(W)}\theta_1(W) &\equiv \zeta^{-\theta_2(W)}U_0\zeta_{a_1}U_1\zeta_{a_2}U_2 \cdots U_{k-1}\zeta_{a_k}U_k, \\
&\approx U_0\zeta^{-1}\zeta_{a_1}U_1\zeta^{-1}\zeta_{a_2}U_2 \cdots U_{k-1}\zeta^{-1}\zeta_{a_k}U_k, \quad \text{by centrality,} \\
&\approx U_0\sigma_{a_1}^{-1}U_1\sigma_{a_2}^{-1}U_2 \cdots U_{k-1}\sigma_{a_k}^{-1}U_k, \\
&\equiv W.
\end{aligned}$$

Recall that two words V and W are said to *differ by an elementary transformation* if there are words X and Y and a relation $(\rho_1, \rho_2) \in \mathcal{R}^{\Sigma}$ such that $V \equiv X\rho_1Y$ and $W \equiv X\rho_2Y$. We say that a word V *is obtained from W by* *a trivial insertion* if there are words X and Y and a letter $x \in S \cup S^{-1}$, such that $W \equiv XY$ and $V \equiv Xxx^{-1}Y$. In this case we also say *W is obtained from V by a trivial deletion.*

Lemma 18 *Suppose that W and V are words over $S \cup S^{-1} \cup T$. If W and V differ by an elementary transformation then $\theta_1(W) \sim \theta_1(V)$. If V is obtained from W by a trivial insertion then $\zeta\theta_1(W) \sim \theta_1(V)$.*

Proof. If $W \equiv X\rho_1 Y$ and $V \equiv X\rho_2 Y$ where $(\rho_1, \rho_2) \in \mathcal{R}^\Sigma$, then $\theta_1(W) \equiv \theta_1(X)\rho_1\theta_1(Y) \sim \theta_1(X)\rho_2\theta_1(Y) \equiv \theta_1(V)$.

If $W \equiv XY$, $V \equiv X\sigma_a\sigma_a^{-1}Y$ and $U \equiv X\sigma_a^{-1}\sigma_a Y$, for some $a \in I$ and some words X and Y, then $\theta_1(V) \equiv \theta_1(X)\sigma_a\zeta_a\theta_1(Y) \sim \theta_1(X)\zeta\theta_1(Y) \sim \zeta\theta_1(X)\theta_1(Y) \equiv \zeta\theta_1(W)$, and similarly $\theta_1(U) \sim \zeta\theta_1(W)$. □

The proof of the following theorem is adapted from the proof in [5] that the positive braid monoid embeds in the braid group.

Theorem 19 *Let W and W' be words over $S \cup T$. Then $W \approx W'$ if and only if $W \sim W'$.*

Proof. It is clear that if $W \sim W'$ then $W \approx W'$. So suppose W and W' are words over $S \cup T$ such that $W \approx W'$. Then there is a sequence $W_1, \dots, W_k$ of words over $S \cup S^{-1} \cup T$ such that $W \equiv W_1 \approx W_2 \approx \cdots \approx W_k \equiv W'$ and W_{i+1} is obtained from W_i by an elementary transformation or by insertion or deletion of $\sigma_a\sigma_a^{-1}$ or $\sigma_a^{-1}\sigma_a$ for some a in I.

Let $p = \max\{\theta_2(W_i) \mid i = 1, \dots, k\}$. For each $i = 1, \dots, k$ define

$$V_i \equiv \zeta^{p-\theta_2(W_i)}\theta_1(W_i).$$

Each V_i is a word over $S \cup T$, and $V_i \approx \zeta^p\zeta^{-\theta_2(W_i)}\theta_1(W_i) \approx \zeta^p W_i$. If W_{i+1} is obtained from W_i by an elementary transformation, then by the previous result, $V_i \sim V_{i+1}$. If W_{i+1} is obtained from W_i by a trivial insertion, then $\theta_2(W_{i+1}) = \theta_2(W_i) + 1 \leq p$, so

$$V_i \equiv \zeta^{p-\theta_2(W_i)}\theta_1(W_i) \equiv \zeta^{p-\theta_2(W_{i+1})}\zeta\theta_1(W_i) \sim \zeta^{p-\theta_2(W_{i+1})}\theta_1(W_{i+1}) \equiv V_{i+1}$$

by an application of Lemma 18. Swapping $i+1$ and i in this argument yields the case when W_{i+1} is obtained from W_i by a trivial deletion. In all cases, $V_i \sim V_{i+1}$.

Thus we have $V_1 \sim V_k$. But W and W' have no letters from S^{-1}, so $V_1 \equiv \zeta^p W_1 \equiv \zeta^p W$ and $V_k \equiv \zeta^p W_k \equiv \zeta^p W'$. Hence $\zeta^p W \sim \zeta^p W'$, and cancelling ζ^p from the left yields $W \sim W'$. □

Corollary 20 *The positive singular Artin monoid of any finite type embeds in its corresponding singular Artin monoid.* $\square$

We can now construct a unique normal form for elements of the singular Artin monoid. Let $q(W)$ be the largest integer such that $(\theta_1(W)/\Delta^{q(W)})$ is defined. Since $0 \leq q(W) \leq \ell(\theta_1(W))$ this is calculable. Let $p(W)$ be the integer such that $\Delta^{p(W)} \approx \zeta^{-\theta_2(W)}\Delta^{q(W)}$. (This number depends on whether ζ is Δ or Δ^2, which in turn depends on the type). Thus $W \approx \Delta^{p(W)}(\theta_1(W)/\Delta^{q(W)})$.

Lemma 21 *If $W \approx V$ then $p(W) = p(V)$. Moreover, $p(W) \geq 0$ precisely when $W \approx W'$ for some word W' over $S \cup T$.*

Proof. Suppose $W \approx V$ for arbitrary words W and V, and that $k = \theta_2(W) - \theta_2(V) \geq 0$. Then

$$\theta_1(W) \approx \zeta^{\theta_2(W)}W \approx \zeta^{\theta_2(W)}V \equiv \zeta^k\zeta^{\theta_2(V)}V \approx \zeta^k\theta_1(V).$$

Thus by Theorem 19, $\theta_1(W) \sim \zeta^k\theta_1(V)$. Hence $\Delta^{q(W)} \equiv \zeta^k\Delta^{q(V)}$, so

$$\Delta^{p(W)} \approx \zeta^{-\theta_2(W)}\Delta^{q(W)} \approx \zeta^{-\theta_2(W)}\zeta^k\Delta^{q(V)} \approx \zeta^{-\theta_2(V)}\Delta^{q(V)} \approx \Delta^{p(V)}.$$

This implies that $\Delta^{p(W)} \sim \Delta^{p(V)}$, and hence that $p(W) = p(V)$.

If W' is a word over $S \cup T$ then $\theta_2(W') = 0$ so $p(W')$ is non negative, so by the first part, $W \approx W'$ implies $p(W)$ is non negative. Finally, if $p(W) \geq 0$, then $W \approx \Delta^{p(W)}(\theta_1(W)/\Delta^{q(W)})$, which is a word over $S \cup T$. $\square$

It was observed above that $W \approx \Delta^{p(W)}(\theta_1(W)/\Delta^{q(W)})$ for any word W over $S \cup S^{-1} \cup T$. Recall that the operator N defined in Section 3 acts on words over $S \cup T$, so since $(\theta_1(W)/\Delta^{q(W)})$ is a word over $S \cup T$, we can use the above formulation to extend N in order to act on arbitrary words W over $S \cup S^{-1} \cup T$. Define

$$\bar{N}(W) \equiv \begin{cases} N(\Delta^{p(W)}(\theta_1(W)/\Delta^{q(W)})) & \text{if } p(W) \geq 0; \text{ or} \\ \Delta^{p(W)}N(\theta_1(W)/\Delta^{q(W)}) & \text{otherwise.} \end{cases}$$

Observe that whenever $W \approx W'$ for some word W' over $S \cup T$, this definition yields the normal form for W' as defined in Section 3.

Theorem 22 *Let V and W be words over $S \cup S^{-1} \cup T$. Then $V \approx W$ precisely when $\bar{N}(V) \equiv \bar{N}(W)$.*

Proof. By the definition of $\bar{N}$, and the fact that for words W' over $S \cup T$, $N(W') \sim W'$, we have that

$$\bar{N}(W) \approx \zeta^{-\theta_2(W)}\Delta^{q(W)}(\theta_1(W)/\Delta^{q(W)}) \approx \zeta^{-\theta_2(W)}\theta_1(W) \approx W.$$

So if $\bar{N}(V) \equiv \bar{N}(W)$ then $V \approx W$. On the other hand, suppose $V \approx W$, and so $p(V) = p(W) = p$ by Lemma 21. If $p \geq 0$ then $\bar{N}(V) \equiv N(W') \equiv \bar{N}(W)$ for some word W' over $S \cup T$. Otherwise, $\bar{N}(V) \approx V \approx W \approx \bar{N}(W)$, so $\Delta^p N(\theta_1(V)/\Delta^{q(V)}) \approx \Delta^p N(\theta_1(W)/\Delta^{q(W)})$. But $(\theta_1(V)/\Delta^{q(V)})$ and $(\theta_1(W)/\Delta^{q(W)})$ are words over $S \cup T$, so by Theorem 19, $N(\theta_1(V)/\Delta^{q(V)}) \sim N(\theta_1(W)/\Delta^{q(W)})$, then by Theorem 13, $N(\theta_1(V)/\Delta^{q(V)}) \equiv N(\theta_1(W)/\Delta^{q(W)})$. Thus $\bar{N}(V) \equiv \Delta^p N(\theta_1(V)/\Delta^{q(V)}) \equiv \Delta^p N(\theta_1(W)/\Delta^{q(W)}) \equiv \bar{N}(W)$. □

Due to the addition of the free group relations, $\mathcal{S}_M$ does not have a homogeneous presentation. Thus the naive solution to the word problem via enumeration and searching described at the end of Section 2 will not work here, and we believe that the results in this paper give the only known solution to the word problem for singular Artin monoids, including the singular braid monoids. The following result solves the division problem in $\mathcal{S}_M$.

Theorem 23 *Let V and W be words over $S \cup S^{-1} \cup T$. Then V divides W in the singular Artin monoid precisely when $M \equiv L(\theta_1(W), \theta_1(V)) \not\equiv \infty$ and $(M/\theta_1(W))$ is a word over S.*

Proof. Suppose there is a word U such that $W \approx VU$. Then $\zeta^{-\theta_2(W)}\theta_1(W) \approx \zeta^{-\theta_2(V)}\theta_1(V)\zeta^{-\theta_2(U)}\theta_1(U)$ and so by centrality and Theorem 19,

$$\theta_1(W)\zeta^{\theta_2(V)+\theta_2(U)} \sim \theta_1(V)\theta_1(U)\zeta^{\theta_2(W)}.$$

Hence $\theta_1(W)$ and $\theta_1(V)$ have a common multiple, and so by Lemma 10, $M \equiv L(\theta_1(W), \theta_1(V)) \not\equiv \infty$, and M is a least common multiple, so it must divide $\theta_1(W)\zeta^{\theta_2(V)+\theta_2(U)}$. Thus $(M/\theta_1(W))$ must divide $\zeta^{\theta_2(V)+\theta_2(U)}$, and consequently is a word over S.

On the other hand, suppose $M \equiv L(\theta_1(W), \theta_1(V)) \not\equiv \infty$ and $X \equiv (M/\theta_1(W))$ is a word over S. Then

$$\begin{aligned} W &\approx \zeta^{-\theta_2(W)}\theta_1(W) \approx \zeta^{-\theta_2(W)}\theta_1(W)XX^{-1} \approx \zeta^{-\theta_2(W)}MX^{-1} \\ &\approx \zeta^{-\theta_2(W)}\theta_1(V)YX^{-1} \text{ because } M \sim \theta_1(V)Y \text{ for some } Y \text{ over } S \cup T, \\ &\approx \zeta^{-\theta_2(V)}\theta_1(V)YX^{-1}\zeta^{\theta_2(V)-\theta_2(W)} \approx VU \end{aligned}$$

where $U \equiv YX^{-1}\zeta^{\theta_2(V)-\theta_2(W)}$. □

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