TWO RESULTS CONCERNING DISTANCE-REGULAR DIRECTED GRAPHS

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Abstract. The study of distance-regular directed graphs can be reduced to that of short distance-regular directed graphs. We consider the eigenspaces of the intersection matrix of a short distance-regular directed graph and show that nearly all the eigenvalues are nonreal. Next we show that a nontrivial short distance-regular directed graph is primitive.

1. Introduction

Distance-regular and distance-transitive directed graphs are the directed version of distance-regular and distance-transitive (undirected) graphs. However, in the directed case, there are very few examples. Distance-transitive directed graphs were introduced by Lam [7] and have also been considered by others, for example in [1], [2], [3], [4], [8], [9], [10] and [11].

A *directed graph* or *digraph* Γ is a pair $\Gamma = (V, E)$ consisting of a finite set V of *vertices* and a set E of *edges*. The elements of E are ordered pairs of distinct elements of V. A (directed) *path* of *length* h from x to y is a sequence of vertices $x = x_0, x_1, \ldots, x_h = y$, such that $h > 0$ and $(x_i, x_{i+1}) \in E$ for $i = 0, 1 \ldots, h-1$. If $x = y$ then the path is a *circuit*. A digraph Γ is *strongly connected* if, for any $x, y \in V$, there is a path from x to y. The length of a shortest path from a vertex x to a vertex y is the *distance* from x to y, and is denoted $d(x, y)$. The *diameter* d of a strongly connected digraph is the maximum value taken by this distance function over all $x, y \in V$. The *girth* g is the minimum length of a circuit. Clearly $g \leq d+1$.

For any vertex x we define the ith *directed shell* $\Gamma_i(x)$ to be

$$\Gamma_i(x) = \{w \in V : d(x, w) = i\}.$$

A digraph Γ is *distance-regular* if it is strongly connected and for all vertices x and y, and for $0 \leq i, j \leq d$, $|\Gamma_i(x) \cap \Gamma_j(y)|$ depends only on i, j and $d(x, y)$.

An *automorphism* of a digraph is a permutation ρ of the vertices which preserves the edges, that is $(x, y) \in E$ if and only if $(x^\rho, y^\rho) \in E$. A strongly connected digraph is *distance-transitive* if, for any vertices x, y, z, w with $d(x,y) = d(z,w)$, there is an automorphism of the digraph such that $x^\rho = z$ and $y^\rho = w$. Clearly a distance-transitive digraph is distance-regular.

A directed circuit with n vertices is an example of a distance-transitive digraph with girth $g = d+1 = n$. For each prime power $q \equiv 3 \pmod 4$, the Paley tournament with q vertices is the digraph whose vertices are the elements of the finite field with q elements. There is an edge from x to y if and only if $y-x$ is a nonzero square in the field. A Paley tournament is a distance-transitive digraph with girth $g = d+1 = 3$.

Let Γ be a distance-regular digraph. A distance-regular digraph of girth 2 is essentially the same as the underlying undirected graph, so from now on we always assume Γ has girth at least 3. We define an involutory permutation on the set $\{0, 1, \ldots, g\}$, by setting $0^* = 0$, $g^* = g$ and $i^* = g - i$ for $0 < i < g$. Damerell [3] showed that for any vertices x, y of Γ, the distance $d(y,x) = d(x,y)^*$ and hence that the girth and diameter differ by at most 1. Damerell reduced the classification of distance-regular directed graphs to those which are *short*, that is those which satisfy $g = d + 1$. A distance-regular digraph in which $g \neq d + 1$ is called *long*.

Leonard and Nomura [8] showed that a short distance-regular directed graph which is not simply a directed circuit has girth at most 8, and, furthermore, that there always exist edges within the first directed shell of any vertex. There are examples of short distance-regular directed graphs of girth 3 or 4, see for example [7], [4], or [9]. Bannai, Cameron and Kahn [1] showed that a short distance-transitive directed graph which has an odd girth is a Paley tournament.

In §2 we recall definitions and results concerning the adjacency and intersection algebras of a short distance-regular digraph and prove the first result of this paper. This result was known () in the case where the digraph has a distance transitive group of automorphisms.

Theorem A. *Let C be the intersection matrix of a short distance-regular directed graph with girth g and valency k. Then C has g distinct eigenvalues, and*

i) if g is odd, then C has exactly one real eigenvalue k, and

ii) if g is even, then C has exactly two real eigenvalues, one of which is k, and the other of which is a negative real number.

In §3 we recall the definition of primitivity for a directed graph. Damerell showed that a long distance-regular digraph is imprimitive. It follows immediately that its automorphism group acts imprimitively on the vertices. For short distance-regular digraphs we prove the following theorem:

Theorem B. *Let Γ be a short distance-regular directed graph with valency k and n vertices.*

i) If $k = 1$ then Γ is a directed circuit with n vertices and is primitive if and only if n is prime.

ii) If $k > 1$ then Γ is primitive.

Corollary. *Let Γ be a short distance-regular directed graph with valency $k > 1$. Then any distance-transitive group of automorphisms acts primitively on the vertices.*

2. Adjacency and intersection matrices.

In this section we will always assume that Γ is a short distance-regular digraph with n vertices, diameter d, girth $g = d+1$, and labelled vertex set $V = \{x_1, x_2, \ldots, x_n\}$. For $i, j, k = 0, 1, \ldots d$, the *intersection numbers* $p_{i,j,k}$ are

$$p_{i,j,k} = |\Gamma_i(x) \cap \Gamma_{j^*}(y)|, \text{ for } x, y \text{ any vertices of } \Gamma \text{ with } d(x,y) = k.$$

For any matrix M of complex numbers, M^{T} denotes the transposed matrix, $\overline{M}$ denotes the complex conjugate.

For any digraph with vertex set $\{y_1, y_2, \ldots, y_m\}$ the *adjacency matrix* A is the matrix of 0s and 1s whose (r,s)-th entry is $(A)_{r,s}$, where

$$(A)_{r,s} = 1 \text{ if and only if } (y_r, y_s) \text{ is an edge.}$$

For $i = 0, 1, \ldots, d$, the *distance matrix* A_i of Γ is the matrix of 0s and 1s with

$$(A_i)_{r,s} = 1 \text{ if and only if } d(x_r, x_s) = i.$$

Thus A_0 is the identity matrix. The matrix $A = A_1$ is the *adjacency matrix* of the digraph. It is clear that in general $A_{i^*} = A_i^{\mathrm{T}}$, and that A_i is a symmetric matrix if and only if $i = 0$ or $g/2$.

By counting paths it is easy to see that, for $0 \leq i, h \leq d$, the distance matrices satisfy $A_i A_h = \sum_{j=0}^{d} p_{i,h,j} A_j$. Hence the linear span of $A_1, A_2, \ldots, A_d$, is closed under multiplication and is thus an algebra. This is called the *adjacency algebra* of the digraph. It is well known (see [7]) that the adjacency algebra is commutative with dimension $g = d+1$. Each of the sets $\{A_0, A_1, \ldots, A_d\}$ and $\{I, A^1, A^2, \ldots, A^d\}$ forms a basis, where the A_i are the distance matrices and the A^i are the powers of A.

Set $V_0 = 1$. For each $i = 0, 1, \ldots, d-1$, the product $A_i A = \sum_{j=0}^{i+1} p_{i,1,j} A_j$, and $p_{i,1,i+1} > 0$. For $i = 1, 2, \ldots d$, this equation recursively defines real polynomials $V_i(x)$ of degree i such that $V_i(A) = A_i$.

The algebra $\mathcal{A}$ acts on itself by multiplication on the right. For $h = 0, 1, \ldots, d$, the h-th *intersection matrix* C_h is defined to be the $(d+1) \times (d+1)$ matrix whose (i,j)-th entry is

$$(C_h)_{i,j} = p_{i,h,j} = p_{h,i,j}.$$

The matrix C_0 is the identity matrix. The matrix $C = C_1$ is called the *intersection matrix*. Right multiplication by A_h, regarded as a linear mapping of the adjacency algebra to itself with respect to the basis $\{A_0, A_1, \ldots, A_d\}$, can be faithfully represented by the transposed h-th intersection matrix, C_h^{T}.

The two matrices A and C have the same minimum polynomial and so have the same eigenvalues.

The matrix A is a real nonsymmetric matrix, which commutes with its transpose. Therefore A is a normal matrix and hence is diagonalizable and there is a basis of $\mathbb{C}^n$ which consists of eigenvectors of A. Since the minimum polynomial of A has degree $d+1$, A has $d+1$ distinct eigenvalues, which we denote by $\lambda_0 = 1, \lambda_1, \ldots, \lambda_d$.

We call an eigenvector of a matrix *standard* if its first coordinate is 1.

Lemma 2.1. *Let λ be an eigenvalue of C. Then*

i) we can construct a unique standard left eigenvector $\mathbf{w}(\lambda)$, and a unique standard right eigenvector $\mathbf{v}(\lambda)$ corresponding to λ,

ii) for $i = 0, 1, \dots, d$, the eigenvectors $\mathbf{w}(\lambda)$ and $\mathbf{v}(\lambda)$ are standard left and right eigenvectors of C_i with corresponding eigenvalue $V_i(\lambda)$,

iii) the eigenvalue λ is real if and only if $\mathbf{v}(\lambda)$ is a real vector if and only if $\mathbf{w}(\lambda)$ is a real vector, and

iv) $\mathbf{v}(\overline{\lambda}) = \overline{\mathbf{v}(\lambda)}$ and $\mathbf{w}(\overline{\lambda}) = \overline{\mathbf{w}(\lambda)}$.

Proof. Let λ be any eigenvalue of C. Then corresponding right and left eigenvectors in standard form can be constructed in the following manner:

If $\mathbf{v} = [v_0, v_1, \dots, v_d]^{\mathrm{T}}$ then the equation $C\mathbf{v} = \lambda\mathbf{v}$ becomes $\sum_{j=0}^{i+1} p_{i,1,j} v_j = \lambda v_i$. Setting $v_0 = 1$ we get the same system of equations as for the distance matrices. Therefore the entry $v_i = V_i(\lambda)$. The vector $\mathbf{v}(\lambda) = [1, \dots, V_i(\lambda), \dots, V_d(\lambda)]^T$ is the unique standard right eigenvector corresponding to λ.

Similarly a left eigenvector $\mathbf{w}(\lambda)$ corresponding to λ can be constructed. If $\mathbf{w} = [w_0, w_1, \dots, w_d]$ then the system $\mathbf{w}\lambda = \mathbf{w}C$ becomes $\sum_{i=0}^{d} w_i p_{1,i,j} = \lambda w_j$. This time each w_j is w_d times a polynomial in λ . Setting $w_d = \frac{\lambda}{k} \neq 0$, we get $\lambda w_0 = w_d p_{d,1,0} = w_d p_{1,d,0} = \frac{\lambda}{k} k = \lambda$. Therefore $w_0 = 1$, and $\mathbf{w}(\lambda) = [1, \dots, \frac{\lambda}{k}]$, the unique standard left eigenvector corresponding to λ.

For each i, $C_i = V_i(C)$, and so an eigenvector of C corresponding to eigenvalue λ, is an eigenvector of C_i with corresponding eigenvalue $V_i(\lambda)$.

Finally, since C is a real matrix, $\mathbf{v}(\overline{\lambda}) = \overline{\mathbf{v}(\lambda)}$ and $\mathbf{w}(\overline{\lambda}) = \overline{\mathbf{w}(\lambda)}$. Furthermore the eigenvalues corresponding to real eigenvectors must be real. Conversely, if λ is a real eigenvalue then for each i, the ith entry of $\mathbf{v}(\lambda)$ is $V_i(\lambda)$ which is real. Similarly the entries of $\mathbf{w}(\lambda)$ are the values of real polynomials evaluated at λ, and hence are real. Therefore the eigenvalue λ is real if and only if $\mathbf{v}(\lambda)$ is a real vector if and only if $\mathbf{w}(\lambda)$ is a real vector. □

The next lemma is a standard result.

Lemma 2.2. *If $\lambda_i \neq \lambda_j$ then $\mathbf{w}(\lambda_i)\mathbf{v}(\lambda_j) = 0$.* □

With respect to the eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_d$ of C, we define the *eigenmatrix* Λ of the digraph Γ to be $(d+1) \times (d+1)$ matrix whose jth column is the standard eigenvector $\mathbf{v}(\lambda_j)$. We denote the (i, j)th entry of Λ by λ_{ij}. By the previous construction, $\lambda_{ij} = V_i(\lambda_j)$, and the ith row of Λ consists of the eigenvalues of C_i. Row 0 of Λ consists of all 1s, and column 0 is $\mathbf{v}(k) = [k_0, k_1, \dots, k_d]^{\mathrm{T}}$. We prove first that Λ has at most two real columns.

Denote by $\mathbf{K}$ the diagonal matrix, $\mathbf{K} = \operatorname{diag}(k_0, k_1, \dots, k_d)$. For each i, we have the relationship between the distance matrices $A_{i^*} = A_i^{\mathrm{T}}$. The next lemma links C_{i^*}, C_i^{T} and $\mathbf{K}$.

Lemma 2.3. *For $i = 0, 1, \dots, d$, $C_{i^*}^{\mathrm{T}} = \mathbf{K}^{-1} C_i \mathbf{K}$.*

Proof. Let $h, j \in \{0, 1, \dots, d,\}$ and x be a vertex of Γ. We count in two ways the elements of the set $\mathbf{P} = \{(u, v) : u \in \Gamma_i(x),\ v \in \Gamma_j(x) \text{ and } d(u, v) = h\}$. For each

$u \in \Gamma_i(x)$ there are $p_{j,h^*,i} = p_{h^*,j,i}$ corresponding vertices v, and for each $v \in \Gamma_j(x)$ there are $p_{i,h,j} = p_{h,i,j}$ corresponding vertices u.

Therefore $|\mathbf{P}| = k_i p_{h^*,j,i} = k_j p_{h,i,j}$, and thus we have

$$(C_{h^*}^{\mathrm{T}})_{i,j} = p_{h^*,j,i} = k_j p_{h,i,j} k_i^{-1} = k_j (C_h)_{i,j} k_i^{-1}.$$

Hence $C_{i^*}^{\mathrm{T}} = \mathbf{K}^{-1} C_i \mathbf{K}$. $\square$

For each $i = 0, 1, \ldots, d$, the matrices C_i and C_{i^*} have the same set of eigenvalues and the same set of standard eigenvectors. The eigenvalues of C are distinct, and we define σ to be the permutation of $\{0, 1, \ldots, d\}$ such that $\lambda_{dj} = \lambda_{1j^\sigma}$ $(= \lambda_{j^\sigma})$. This gives rise to a permutation of the eigenvalues of C which is in fact complex conjugation.

Lemma 2.4. *For $j = 0, 1, \ldots, d$,*
(i) for $i = 0, 1, \ldots, d$, $\lambda_{ij^\sigma} = \overline{\lambda}_{ij} = \lambda_{i^*j}$, and
(ii) the standard right eigenvector $\mathbf{v}(\overline{\lambda_j}) = \mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}}$.

Proof. For $j = 0, 1, \ldots, d$, then we have

$$\begin{aligned} C\mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}} &= (\mathbf{K}C_d^{\mathrm{T}})\mathbf{w}(\lambda_j)^{\mathrm{T}} \\ &= \mathbf{K}(\mathbf{w}(\lambda_j)C_d)^{\mathrm{T}} \\ &= \mathbf{K}(\lambda_{dj}\mathbf{w}(\lambda_j))^{\mathrm{T}} \\ &= \lambda_{1j^\sigma}(\mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}}). \end{aligned}$$

Therefore $\mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}} = \mathbf{v}(\lambda_{j^\sigma})$, because the first entry in $\mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}}$ is 1.

Since $\mathbf{K}$ is a diagonal matrix with positive diagonal entries,

$$\mathbf{w}(\overline{\lambda}_j)\mathbf{v}(\lambda_{j^\sigma}) = \overline{\mathbf{w}(\lambda_j)}\mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}} > 0.$$

Hence $\overline{\lambda}_j = \lambda_{j^\sigma}$, and thus $\mathbf{v}(\overline{\lambda_j}) = \mathbf{v}(\lambda_{j^\sigma}) = \mathbf{K}\mathbf{w}(\lambda_j)^{\mathrm{T}}$.

Furthermore $\lambda_{ij^\sigma} = V_i(\lambda_{j^\sigma}) = V_i(\overline{\lambda}_j) = \overline{V_i(\lambda_j)} = \overline{\lambda}_{ij}$.

Finally, for $i, j \in \{0, 1, \ldots, d\}$, then

$$\begin{aligned} \lambda_{i^*j}\mathbf{w}(\lambda_j) =& \mathbf{w}(\lambda_j)C_{i^*} \\ =& \mathbf{v}(\overline{\lambda}_j)^{\mathrm{T}}\mathbf{K}^{-1}C_{i^*} \\ =& \mathbf{v}(\overline{\lambda}_j)^{\mathrm{T}}C_i^{\mathrm{T}}\mathbf{K}^{-1} \\ =& (C_i\mathbf{v}(\overline{\lambda}_j))^{\mathrm{T}}\mathbf{K}^{-1} \\ =& \lambda_{ij^\sigma}\mathbf{v}(\overline{\lambda}_j)^{\mathrm{T}}\mathbf{K}^{-1} \\ =& \overline{\lambda_{ij}}\mathbf{w}(\lambda_j) \end{aligned}$$

Therefore $\lambda_{i^*j} = \overline{\lambda}_{ij}$. $\square$

As a corollary to this lemma we have

Corollary 2.5. *The eigenmatrix Λ has at most two real rows.*

(i) If the girth g is odd, Λ has 1 real row, the 0th row.

(ii) If the girth g is even, Λ has 2 real rows, the 0th row and row $[g/2]$.

Proof. The eigenmatrix is nonsingular, since its columns are the $d+1$ standard eigenvectors corresponding to the $d+1$ distinct eigenvalues of C. Therefore the rows are certainly distinct. However, the previous proposition shows that row i^* and row i are conjugate. Therefore the ith row of Λ is real if and only if $i^* = i$. If g is odd, then $i^* = i$ if and only if $i = 0$. If g is even, then $i^* = i$ if and only if $i = 0$ or $[g/2]$. $\square$

We can now prove Theorem A.

Theorem A. *Let C be the intersection matrix of a short distance-regular directed graph with girth g and valency k. Then C has g distinct eigenvalues, and*

i) if g is odd, then C has exactly one real eigenvalue k, and

ii) if g is even, then C has exactly two real eigenvalues, one of which is k, and the other of which is a negative real number.

Proof. The eigenmatrix Λ is nonsingular. The permutation of the entries which takes $\lambda_{ij} \mapsto \lambda_{i^*j}$ is an involution, and comes from an involutory permutation of the rows of Λ. Let L be the permutation matrix such that $\Lambda \mapsto L\Lambda$ effects this permutation of the rows. Then $(L\Lambda)_{ij} = \lambda_{i^*j} = \overline{\lambda_{ij}}$.

The action on the columns of Λ, given by $\mathbf{v}(\lambda_j) \mapsto \mathbf{v}(\overline{\lambda_j})$ is also an involution. Let R be the permutation matrix such that $\Lambda \mapsto \Lambda R$ effects this permutation. Then the matrix entry $(\Lambda R)_{ij} = (\mathbf{v}(\overline{\lambda_j}))_i = \overline{\mathbf{v}(\lambda_j)_i} = \overline{\lambda_{ij}} = (L\Lambda)_{ij}$.

Therefore $L\Lambda = \Lambda R$, and so $L = \Lambda^{-1}R\Lambda$. The number of fixed points of a permutation is equal to the trace of the corresponding permutation matrix, and the traces of similar matrices are equal. Therefore L and R have the same number of fixed points in their actions.

By the previous corollary, there are at most two fixed rows, and so C has at most two real standard right eigenvectors and thus at most two real eigenvalues.

The eigenvalue $\lambda_0 = k$ is real and the nonreal eigenvalues occur in conjugate pairs. The number of eigenvalues is g. Therefore, if g is odd, then C has exactly one real eigenvalue. If g is even, then C has exactly two real eigenvalues, one of which is $\lambda_0 = k$, and the rest are pairs of complex conjugate eigenvalues. The product of the eigenvalues is the determinant of C which can be conveniently expanded down the first column. The determinant $\det(C) = -k\prod_{j=o}^{d-1} C_{j,j+1}$, which is $-k\times$ a product of positive integers, so the second real eigenvalue must be negative. $\square$

Note that the argument used in the proof, that L and R have the same number of fixed points in their actions, is a special case of a combinatorial lemma due to Brauer, to be found in Feit [5].

3. Primitivity

Throughout this section $\Gamma = (V, E)$ is a distance-regular digraph, not necessarily short. If the valency $k = 1$ then the digraph is called *trivial*, and in this case it is clear that it is simply a directed circuit.

Let $\mathbb{V} = \{I, E_1, \dots, E_d\}$ be the partition of V^2 defined by $(x, y) \in E_i$ if and only if $y \in \Gamma_i(x)$. For each i, we define Γ_i to be the digraph with vertex set V and edge set E_i, and so Γ_i is the directed graph with adjacency matrix A_i. If $i = g/2$ then Γ_i has girth 2. The digraphs Γ_i are not necessarily connected.

The digraph Γ is said to be *primitive* if the two trivial relations, I and V^2, are the only equivalence relations which are unions of members of $\mathbb{V}$, otherwise Γ is called *imprimitive.*

Lemma 3.1. *The digraph Γ is primitive if and only if each Γ_i is connected.*

Proof. Clear. $\square$

Lemma 3.2. *(Damerell* [3]*) A long distance-regular digraph is imprimitive.*

Proof. If Γ is a long distance-regular digraph, then $E = I \cup \Gamma_g$ is a nontrivial equivalence relation, so Γ is imprimitive. $\square$

Before we complete the proof of Theorem B, we need several results concerning edges and circuits within directed shells when Γ is nontrivial.

Lemma 3.3. *(Leonard and Nomura* [8]*). If Γ is a short and nontrivial then $p_{1,1,1} > 0$. That is, there are edges in the first directed shell.* $\square$

Corollary 3.4. *If Γ is a short and nontrivial, and x is any vertex, then there is a closed path entirely contained in $\Gamma_1(x)$.*

Proof. Let u_0 be any vertex in $\Gamma_1(x)$. Choose u_1 to be any out-neighbour of u_0 in $\Gamma_1(x)$. Choose u_2 to be any out-neighbour of u_1 in $\Gamma_1(x)$. Continue in this way constructing a path $u_0, u_1, \dots u_j$ in $\Gamma_1(x)$. Since $\Gamma_1(x)$ is finite, for some smallest m, $u_m = u_h$ for some $h < m$. Then $u_h \longrightarrow u_{h+1} \longrightarrow u_{h+2} \longrightarrow \cdots \longrightarrow u_m = u_h$ is a closed path with all u_j in $\Gamma_1(x)$. $\square$

As a further corollary we have that in the first directed shell of any vertex there are vertices y and z with $d(y, z) = j$ for any $j \le d$, that is the entries in column 1 of C are all nonzero.

Corollary 3.5. *If Γ is a short and nontrivial then $p_{1,i,1} > 0$ for $i = 0, 1, \dots, d$.* $\square$

Corollary 3.6. *If Γ is a short and nontrivial then $p_{d,1,i} > 0$ for $i = 0, 1, \dots, d$.* $\square$

Proof. Let x, y, z be vertices with $d(x, y) = 1$, $d(y, z) = i^*$ and $d(x, z) = 1$. These exist since $p_{1,i^*,1} > 0$. Then $d(z, x) = 1^* = d$, $d(x, y) = 1$ and $d(z, y) = (i^*)^* = i$. Therefore $p_{d,1,i} > 0$. $\square$

Lemma 3.7. *If Γ is a short nontrivial distance-regular digraph, and $0 < i \le \frac{g}{2}$, $x \in V$ and $y \in \Gamma_i(x)$, then there is a vertex $z \in \Gamma_i(x)$ such that $0 < d(y, z) < i$.*

Proof. Let $0 < i \le \frac{g}{2}$. We use the commutativity of the intersection matrices, and in particular $(C_1 C_d)_{1i} = (C_d C_1)_{1i}$.

$(C_1 C_d)_{1i} = \sum_{j=0}^{d} p_{1,1,j} p_{d,j,i} = p_{1,1,1} p_{d,1,i} + p_{1,1,2} p_{d,2,i}$.

$(C_d C_1)_{1i} = \sum_{j=0}^{d} p_{d,1,j} p_{1,j,i} = p_{d,1,i-1} p_{1,i-1,i} + p_{d,1,i} p_{1,i+1,i} \cdots + p_{d,1,d} p_{1,d,i}$.

Now $p_{1,1,1} = p_{d,1,d}$ and $p_{d,1,i} = p_{1,d,i}$. Therefore $p_{1,1,2}p_{d,2,i} \geq p_{d,1,i-1}p_{1,i-1,i} > 0$, and so $p_{d,2,i} > 0$. Therefore there are edges from $\Gamma_i(x)$ to $\Gamma_2(x)$.

Now suppose $y \in \Gamma_i(x)$. Since $p_{d,2,i} > 0$, there exists a w in $\Gamma_2(x) \cap \Gamma_1(y)$. Since $w \in \Gamma_2(x)$, there exists $z \in \Gamma_i(x)$, with $d(w,z) = i-2$.

Therefore $d(y,z) \leq d(y,w) + d(w,z) = i - 1 < i$. □

Setting $i = 2$ in this lemma we have that there are edges in the second directed shell.

Corollary 3.8. *If Γ is short and nontrivial then $p_{1,2,2} > 0$.* □

Corollary 3.9. *If Γ is short and nontrivial, and x is any vertex, then there is a closed path entirely contained in $\Gamma_2(x)$.* □

We can now prove Theorem B.

Theorem B. *Let Γ be a short distance-regular directed graph with valency k and n vertices.*

i) If $k = 1$ then Γ is a directed circuit with n vertices and is primitive if and only if n is prime.

ii) If $k > 1$ then Γ is primitive.

Proof.

i) If $k = 1$ then Γ is a directed circuit with n vertices. Clearly in this case Γ is primitive if and only if n is prime.

ii) Suppose $k > 1$ and that E is an equivalence relation on V which is a union of members of $\mathbb{V}$. Then for some ℓ, a divisor of g, $E = I \cup E_\ell \cup E_{2\ell} \cup \cdots \cup E_{g-\ell}$. Any two vertices y and z are in the same equivalence class if and only if ℓ divides $d(y,z)$.

Suppose $1 < \ell \leq \frac{g}{2}$, and $x \in V$. The equivalence class containing x also contains $\Gamma_\ell(x)$. By the previous lemma there are $y, z \in \Gamma_\ell(x)$ such that $0 < d(y,z) < \ell$. However, since y and z are in the same equivalence class ℓ divides $d(y,z)$. This is a contradiction, and so $\ell = 1$ and the equivalence relation is trivial.

Therefore any short nontrivial distance-regular digraph is primitive. □

A group G which acts transitively on a set acts *primitively* if the only partitions of the set which it preserves are the trivial ones, and so the corollary follows immediately.

Corollary. *Let Γ be a short distance-regular directed graph with valency $k > 1$. Then any distance-transitive group of automorphisms acts primitively on the vertices.*

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