LEFT CELLS IN THE WEYL GROUP OF TYPE E_7

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Abstract. By applying an algorithm, we get a representative set of the left cells and all the left cell graphs for the Weyl group W of type E_7. From this, we see that the generalized τ-invariants characterize all the non-exceptional left cells of W. We give a criterion to check the relation $\underset{L}{\sim}$ on the elements in the exceptional left cells of W.

§1. Introduction

Let $W=(W,S)$ be a Coxeter group with S its distinguished generator set. Cells of W (in the sense of D. Kazhdan and G. Lusztig [8]) have been studied extensively in the past two decades. It is of particular interest in the case where W is either a Weyl group or an affine Weyl group, where cells of W play an important role in the representation theory of the algebraic groups over the complex number field and over a p-adic number field. To understand well the relevant representation theory, it is desirable to have an explicit description of the cells of W and to know more about the properties of these cells. Let us restrict ourselves only to the case where W is an irreducible Weyl group. We know that all the cells of W have been described explicitly in the cases where W is of type not equal to E_7 and E_8. Cells in the Weyl groups of classical types were described by Kazhdan-Lusztig, Barbasch-Vogan and Garfinkle in terms of some combinatorial objects such as partitions, Young tableaux (see [1; 2; 5; 6; 7; 8]). The description of cells of

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the Weyl group W of type F_4, given by Takahashi, was based on the knowledge of all the Kazhdan-Lusztig polynomials of W; the latter were obtained by some computer programs (see [21]). The treatment for the Weyl group of type G_2 is easy. The one for the Weyl group of type E_6 by Tong relied on an algorithm (see Algorithm 3.6), by which computation of Kazhdan-Lusztig polynomials was almost avoided (see [22]). In the present paper, we shall describe all the left cells of $W = W(E_7)$, the Weyl group of type E_7. We achieve this by finding a representative set of left cells (an l.c.r. set for brevity), the latter occurs as a union of vertex sets of certain graphs (so called left cell graphs or essential graphs), by which we get an explicit presentation for the generalized τ-invariants of left cells of W. The generalized τ-invariants are very important invariants of left cells, which characterize left cells of W in many cases: types A_n ($n \geqslant 1$), B_m ($m \geqslant 2$), E_6, F_4, G_2, but not in the case of type D_k ($k \geqslant 4$). As a consequence of our results, we shall see that the generalized τ-invariants characterize most of the left cells of $W(E_7)$. We conjecture that such a phenomenon should also occur in the case of type E_8.

Our main tool in finding a representative set of left cells of $W(E_7)$ is Algorithm 3.6. This algorithm was first introduced in [16]. Then it was improved in [18] by refining the original process (**B**) into two processes (**B**) and (**C**) in the algorithm, and by setting the priority to (**A**), and then to (**B**), (**C**) in turn, according to the principle of easier-done-first. This enables us to avoid the complicated calculation of Kazhdan-Lusztig polynomials to a great extent. Algorithm 3.6 has been applied extensively to the cases of Weyl groups and affine Weyl groups (see [14; 16; 17; 18; 19; 22; 23]). In particular, in the case of a Weyl group, the number of left cells in a two-sided cell is known, the algorithm becomes much simplier, where we need only to perform Processes (**A**) and (**B**), but never (**C**). On the other hand, there are two important concepts: a left cell graph and an essential graph associated to an element of W in [18]. The left cell graphs exhibit the generalized τ-invariants for the left cells of W. The essential graphs can do more, whenever they exist, which provide an l.c.r. set of W in addition. Moreover, in the case when W has a simply-laced type (see 2.5), the left cells in the same left cell graph afford the equivalent cell representations of W (see [9]).

In order to make Algorithm 3.6 more efficient, we designed a series of computer programs, including the ones for the Processes (**A**) and (**B**) of the algorithm, and for the comparison of the generalized τ-invariants of elements of W. We are pleased to provide these programs upon request.

The contents of the paper are organized as follows. Section 2 consists of preliminaries; we collect some concepts and results on the cells of a Coxeter group, in particular of a Weyl group. We state the algorithm for finding an l.c.r. set in Section 3. Then in the subsequent sections we focus our attention on the Weyl group $W(E_7)$. Some known results on $W(E_7)$ are collected in Section 4. Section 5 is the main body of the paper. We obtain an l.c.r. set for each two-sided cell of $W(E_7)$. We also obtain the isomorphism classes of left cell graphs and some properties of the left cells of $W(E_7)$. Finally, we provide an effective method to check the relation $\underset{\mathrm{L}}{\sim}$ on a pair of elements of $W(E_7)$, in particular, in the case where the elements concerned are both in the exceptional left cells (see 5.11). We attach all the left cell graphs of $W(E_7)$ (up to isomorphism) in the Appendix.

§2. Preliminaries

2.1. Let $W = (W, S)$ be a Coxeter group with S its distinguished generator set. Let $\leqslant$ be the Bruhat order and $\ell(x)$ the length function on W. Let $\mathcal{A} = \mathbf{Z}[q]$ be the polynomial ring in an indeterminate q with integer coefficients. To each pair $x, y \in W$, we associate a polynomial $P_{x,y} \in \mathcal{A}$, called a Kazhdan-Lusztig polynomial (see [8]), which has the properties: $\deg P_{x,y} \leqslant (1/2)(\ell(y) - \ell(x) - 1)$ for $x < y$, $P_{x,x} = 1$ and $P_{x,y} = 0$ for $x \nleqslant y$. Two elements $x, y \in W$ are jointed, written x—y, if either $\deg P_{x,y}$ or $\deg P_{y,x}$ reaches its upper-bound $(1/2)(|\ell(y) - \ell(x)| - 1)$, where $|z|$ is the absolute value of z. Generally speaking, checking the relation x—y is not an easy task in practice since the computation of Kazhdan-Lusztig polynomials is very complicated. But we have the following simple and useful fact: if $x < y$ and $\ell(y) = \ell(x) + 1$, then x—y.

2.2. Let $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$ for any $w \in W$.

For $x, y \in W$, we write $x \underset{\mathrm{L}}{\leqslant} y$ (resp. $x \underset{\mathrm{R}}{\leqslant} y$) if there exists a sequence of elements $x = x_0, x_1, \cdots, x_n = y$ in W such that x_{i-1}—x_i and $\mathcal{L}(x_{i-1}) \not\subseteq \mathcal{L}(x_i)$ (resp. $\mathcal{R}(x_{i-1}) \not\subseteq \mathcal{R}(x_i)$) for each i, $1 \leqslant i \leqslant n$. We write $x \underset{\mathrm{LR}}{\leqslant} y$ if there exists a sequence of elements $x = x_0, x_1, \cdots, x_n = y$ in W such that either $x_{i-1} \underset{\mathrm{L}}{\leqslant} x_i$ or $x_{i-1} \underset{\mathrm{R}}{\leqslant} x_i$ holds for each i, $1 \leqslant i \leqslant n$. We write $x \underset{\mathrm{L}}{\sim} y$ (resp. $x \underset{\mathrm{R}}{\sim} y$, resp. $x \underset{\mathrm{LR}}{\sim} y$) if $x \underset{\mathrm{L}}{\leqslant} y \underset{\mathrm{L}}{\leqslant} x$ (resp. $x \underset{\mathrm{R}}{\leqslant} y \underset{\mathrm{R}}{\leqslant} x$, resp. $x \underset{\mathrm{LR}}{\leqslant} y \underset{\mathrm{LR}}{\leqslant} x$). Then $\underset{\mathrm{L}}{\sim}$ (resp. $\underset{\mathrm{R}}{\sim}$, resp. $\underset{\mathrm{LR}}{\sim}$) is an equivalence relation on W. The corresponding equivalence classes are called left (resp. right, resp. two-sided) cells of W.

The preorder relation $\underset{\mathrm{L}}{\leqslant}$ (resp. $\underset{\mathrm{R}}{\leqslant}$, resp. $\underset{\mathrm{LR}}{\leqslant}$) on the elements of W gives rise to a partial order relation on the left cells (resp. right cells, resp. two-sided cells) of W. By [8, Proposition 2.4], for $x, y \in W$, if $x \underset{\mathrm{L}}{\leqslant} y$ (resp. $x \underset{\mathrm{R}}{\leqslant} y$), then $\mathcal{R}(y) \subseteq \mathcal{R}(x)$ (resp. $\mathcal{L}(y) \subseteq \mathcal{L}(x)$). In particular, if $x \underset{\mathrm{L}}{\sim} y$ (resp. $x \underset{\mathrm{R}}{\sim} y$), then $\mathcal{R}(x) = \mathcal{R}(y)$ (resp. $\mathcal{L}(x) = \mathcal{L}(y)$). So for any left cell Γ of W, we may define $\mathcal{R}(\Gamma)$ to be $\mathcal{R}(x)$ for any $x \in \Gamma$.

2.3. From now on, we always assume that (W, S) is a Weyl group unless otherwise specified.

Lusztig defined in [10] a function $a : W \longrightarrow \mathbf{N}$, which satisfies the following properties.

(1) $a(z) \leqslant |\Phi^+|$ for any $z \in W$, where Φ^+ is a positive root system associated to W.

(2) $x \underset{\mathrm{LR}}{\leqslant} y$ implies $a(x) \geqslant a(y)$. In particular, $x \underset{\mathrm{LR}}{\sim} y$ implies $a(x) = a(y)$. So we can define the a-value $a(\Gamma)$ of a left (resp. right, resp. two-sided) cell Γ of W by $a(x)$ for any $x \in \Gamma$.

(3) For any $I \subseteq S$, let w_I be the longest element in the subgroup W_I of W generated by I. Then $a(w_I) = \ell(w_I)$.

The above properties of the function a were shown by Lusztig in his papers [10; 11].

Let $W_{(i)} = \{w \in W \mid a(w) = i\}$ for any nonnegative integer i. Then by (2), $W_{(i)}$ is a union of some two-sided cells of W.

(4) If $W_{(i)}$ contains an element of the form w_I for some $I \subseteq S$, then $\{w \in W_{(i)} \mid \mathcal{R}(w) = I\}$ forms a single left cell of W.

Let G be the connected reductive complex algebraic group with type dual to that of W. Then the following result is due to Barbasch, Vogan and Lusztig.

2.4. Theorem (cf. [1; 2; 3; 9]). *(1) There exists a bijection $\Omega \mapsto \mathbf{u}(\Omega)$ from the set of two-sided cells in W to the set of special unipotent conjugacy classes in G. This bijection satisfies the equation $a(\Omega) = \dim \mathfrak{B}_u$, where u is an element in $\mathbf{u}(\Omega)$ and $\dim \mathfrak{B}_u$ is the dimension of the variety of Borel subgroups of G containing u.*

(2) The number $n(\Omega)$ of left cells in a two-sided cell Ω of W is equal to the degree $d(\Omega)$ of the corresponding special representation of W.

We know all the special unipotent classes of G, all the numbers $\dim \mathfrak{B}_u$ and $d(\Omega)$. So the above theorem provides us with some useful information on the cells of Weyl groups.

2.5. Now assume that (W, S) is an irreducible Weyl group of simply-laced type, that is, for any $s \neq t$ in S, the order $o(st)$ of the product st is not greater than 3, or equivalently, W is of type A, D or E.

Given $s \neq t$ in S with $o(st) = 3$, a set of the form $\{ys, yst\}$ is called an $\{s,t\}$-string (or just called a string), if $\mathcal{R}(y) \cap \{s,t\} = \emptyset$.

Two elements $x, y \in W$ form a primitive pair, if there exist two sequences of elements $x_0 = x, x_1, \cdots, x_n$ and $y_0 = y, y_1, \cdots, y_n$ in W such that the following conditions are satisfied.

(a) For each i, $1 \leqslant i \leqslant n$, there exist some s_i, $t_i \in S$ with $o(s_i t_i) = 3$ such that both $\{x_{i-1}, x_i\}$ and $\{y_{i-1}, y_i\}$ are $\{s_i, t_i\}$-strings.

(b) x_i—y_i for some (and then for all, under the condition (a)) i, $0 \leqslant i \leqslant n$.

(c) Either $\mathcal{R}(x) \not\subseteq \mathcal{R}(y)$ and $\mathcal{R}(y_n) \not\subseteq \mathcal{R}(x_n)$, or $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $\mathcal{R}(x_n) \not\subseteq \mathcal{R}(y_n)$ hold.

We have $x \underset{\mathrm{R}}{\sim} y$ if $\{x, y\}$ is a primitive pair.

2.6. For each element $x \in W$, we denote by $M(x)$ the set of all elements y such that there is a sequence of elements $x = x_0$, x_1, $\cdots$, $x_n = y$ in W with some integer $n \geqslant 0$, where $\{x_{i-1}, x_i\}$ is a string for each i, $1 \leqslant i \leqslant n$.

In the description of left cells, we need some concepts of graphs introduced in [18; 19].

2.7. By a graph $\mathfrak{M}$, we mean that a set M of vertices together with a set of edges, where each edge is a two-element subset of M, and each vertex is labelled by some subset of S.

Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two graphs with the vertex sets M and M' respectively. They are isomorphic, written $\mathfrak{M} \cong \mathfrak{M}'$, if there exists a bijective map η from M to M' satisfying the following conditions.

(a) The labelling of x is the same as that of $\eta(x)$ for any $x \in M$.

(b) For $x, y \in M$, $\{x, y\}$ is an edge of $\mathfrak{M}$ if and only if $\{\eta(x), \eta(y)\}$ is an edge of $\mathfrak{M}'$.

This is an equivalence relation on graphs.

2.8. We define a graph $\mathfrak{M}(x)$ associated to an element $x \in W$ as follows. Its vertex set is $M(x)$ and its edge set consists of all the strings in $M(x)$. Each vertex $y \in M(x)$ is labelled by the set $\mathcal{R}(y)$. Clearly, for any $x \in W$, the graph $\mathfrak{M}(x)$ is always connected.

A left cell graph associated to an element $x \in W$, written $\mathfrak{M}_L(x)$, is by definition a graph, whose vertex set $M_L(x)$ consists of all left cells Γ of W with $\Gamma \cap M(x) \neq \emptyset$. Two vertices $\Gamma, \Gamma' \in M_L(x)$ are joined by an edge, if there are two elements $y \in M(x) \cap \Gamma$ and $y' \in M(x) \cap \Gamma'$ with $\{y, y'\}$ an edge of $\mathfrak{M}(x)$. Each vertex Γ of $\mathfrak{M}_L(x)$ is labelled by the common labelling of elements of $M(x) \cap \Gamma$. Clearly, the graph $\mathfrak{M}_L(x)$ is always connected.

A subgraph $\mathfrak{M}$ of $\mathfrak{M}(x)$ ($x \in W$) is said to be essential, if there exists an isomorphism η from $\mathfrak{M}$ to $\mathfrak{M}_L(x)$ such that each vertex y of $\mathfrak{M}$ is contained in the left cell $\eta(y)$. Note that an essential subgraph of $\mathfrak{M}(x)$ does not always exist.

2.9. Let $\mathfrak{N}$ and $\mathfrak{N}'$ be two graphs with N and N' the corresponding vertex sets. We say that $\mathfrak{N}'$ is opposed to $\mathfrak{N}$ (up to isomorphism), written $\mathfrak{N}' \cong \mathfrak{N}^{\mathrm{op}}$, if there exists a bijective map ϕ from N to N' satisfying that for any $x, y \in N$,

(a) $\mathcal{R}(\phi(x)) = S \setminus \mathcal{R}(x)$;

(b) $\{x, y\}$ is an edge of $\mathfrak{N}$ if and only if $\{\phi(x), \phi(y)\}$ is an edge of $\mathfrak{N}'$.

Clearly, the relation of two graphs being opposed is symmetric: $\mathfrak{N} \cong (\mathfrak{N}^{\mathrm{op}})^{\mathrm{op}}$.

2.10. By a path in the graph $\mathfrak{M}(x)$, we mean a sequence of vertices $z_0, z_1, \ldots, z_n$ in $M(x)$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathfrak{M}(x)$ for any i, $1 \leqslant i \leqslant n$. Two elements $x, x' \in W$ have the same generalized τ-invariants, if for any path $z_0 = x, z_1, \ldots, z_n$ in the graph $\mathfrak{M}(x)$, there is a path $z'_0 = x', z'_1, \ldots, z'_n$ in $\mathfrak{M}(x')$ with $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for every i, $0 \leqslant i \leqslant n$, and if the same condition holds when we interchange the roles of x with x'.

Then we have the following known result.

2.11. Proposition ([18, Proposition 3.10])**.** *(a) The elements in the same left cell of W have the same generalized τ-invariants.*

(b) If $x \underset{\mathrm{L}}{\sim} y$ in W, then the left cell graphs $\mathfrak{M}_L(x)$ and $\mathfrak{M}_L(y)$ are isomorphic.

§3. Algorithm

In this section, we assume that (W, S) is an irreducible Weyl group of simply-laced type (see 2.5).

3.1. A set Σ of left cells of W is represented by a set $K \subseteq W$ if Σ is the set of all left cells Γ of W with $\Gamma \cap K \neq \emptyset$. We call K a representative set for Σ, if K represents Σ

with $|K \cap \Gamma| = 1$ for any $\Gamma \in \Sigma$.

The algorithm is based on the the following result.

3.2. Theorem ([16, Theorem 3.1]). *Let Ω be a two-sided cell of W. Then a non-empty subset $K \subseteq \Omega$ is a representative set of left cells (an l.c.r. set for short) of W in Ω if K satisfies the following conditions.*

(1) $x \underset{L}{\not\sim} y$ for any $x \neq y$ in K;

(2) If an element $y \in W$ is such that there is some element $x \in K$ with $y — x$, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$, then $y \underset{L}{\sim} z$ for some $z \in K$.

3.3. A subset $K \subseteq W$ is distinguished if $K \neq \emptyset$ and $x \underset{L}{\not\sim} y$ for any $x \neq y$ in K.

3.4. By [8, 2.3f], we know that the relations $y — x$ and $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ hold if and only if one of the following cases occurs:

(a) $\{x, y\}$ is a string (see 2.5);

(b) $y = x \cdot s$ for some $s \in S$ with $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$, where by the notation $a = b \cdot c$ $(a, b, c \in W)$, we mean $a = bc$ and $\ell(a) = \ell(b) + \ell(c)$;

(c) $y < x$, $y — x$ and $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$.

According to the above facts, the following three processes were designed on a non-empty subset P in a two-sided cell of W (see [18]).

(**A**) Find a largest possible subset Q from the set $\underset{x \in P}{\cup} M(x)$ (see 2.6) with Q distinguished.

(**B**) For each $x \in P$, let

$$B_x = \{y \in W \mid y^{-1}x \in S, \mathcal{R}(y) \supsetneqq \mathcal{R}(x), a(y) = a(x)\}$$

and $B = P \cup (\cup_{x \in P} B_x)$. Take a largest possible subset Q from B with Q distinguished.

(**C**) For each $x \in P$, let

$$C_x = \{y \in W \mid y < x, y — x, \mathcal{R}(y) \supsetneqq \mathcal{R}(x), a(y) = a(x)\}$$

and define $C = P \cup (\cup_{x \in P} C_x)$. Take a largest possible subset Q from C with Q distinguished.

3.5. A subset P of W is **A**-saturated (resp. **B**-saturated, resp. **C**-saturated), if the Process (**A**) (resp. (**B**), resp. (**C**)) on P cannot produce any element z satisfying $z \underset{L}{\not\sim} x$ for all $x \in P$.

Clearly, a set of the form $\bigcup_{x\in K} M(x)$ for any $K \subseteq W$ is always **A**-saturated.

It follows from Theorem 3.2 that an l.c.r. set of W in a two-sided cell Ω is exactly a distinguished subset of Ω which is **A**, **B** and **C**-saturated. In order to get such a subset, we perform the following algorithm.

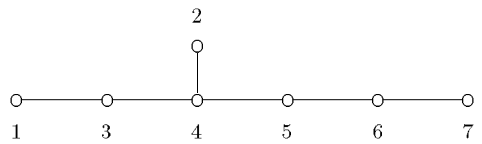
3.6. Algorithm ([18, 2.7]).

(1) Find a non-empty subset P of Ω (we take P to be distinguished to avoid unnecessary complication whenever possible);

(2) Perform Processes (**A**), (**B**) and (**C**) alternately on P until the resulting distinguished set cannot be further enlarged by these three processes.

§4. Some results on the Weyl group $W(E_7)$

4.1. In the sequel, we assume $W = W(E_7)$, the Weyl group of type E_7. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$, where the indices of simple reflections are compatible with the following Dynkin diagram



We shall denote s_i by $\mathbf{i}$ $(1 \leqslant i \leqslant 7)$ for brevity. For any subset $I = \{\mathbf{i_1}, \cdots, \mathbf{i_k}\}$ of S, the notation $w_{\mathbf{i_1}\cdots\mathbf{i_k}}$ stands for the longest element in the subgroup W_I of W generated by I.

4.2. By Theorem 2.4 and the results in [3; 9], we have $W_{(i)} \neq \emptyset$ if and only if $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 20, 21, 22, 25, 30, 36, 37, 46, 63\}$. There are in total 35 two-sided cells in W. In Table 4.2.1, we list for each two-sided cell Ω of W the value $a(\Omega)$ (see 2.3 (2)), the number $n(\Omega)$ of left cells contained in Ω, and the corresponding special unipotent class $\mathbf{u}(\Omega)$ of the algebraic group of type E_7, the notations of the latter are adopted from [3, Chapter 13].

4.3. Theorem (cf. [3; 8]). *(1) The map $\rho : x \mapsto w_0x$ induces an order-reversing involutive permutation (also denoted by ρ) on the set of left cells (resp. right cells, resp. two-sided cells) of W with respect to the partial order $\underset{\mathrm{L}}{\leqslant}$ (resp. $\underset{\mathrm{R}}{\leqslant}$, resp. $\underset{\mathrm{LR}}{\leqslant}$) (see 2.2), where w_0 is the longest element in W.*

(2) The ρ-orbits in the set of two-sided cells of W are as below.

$\{W_{(0)}, W_{(63)}\}$, $\{W_{(1)}, W_{(46)}\}$, $\{W_{(2)}, W_{(37)}\}$, $\{\Omega_{3,1}, W_{(30)}\}$, $\{\Omega_{3,2}, W_{(36)}\}$, $\{W_{(4)}, W_{(25)}\}$, $\{W_{(5)}, W_{(22)}\}$, $\{\Omega_{6,1}, \Omega_{21,3}\}$, $\{\Omega_{6,2}, \Omega_{21,2}\}$, $\{\Omega_{6,3}, \Omega_{21,1}\}$, $\{\Omega_{7,1}, W_{(16)}\}$, $\{\Omega_{7,2}, W_{(20)}\}$, $\{W_{(8)}, \Omega_{15,1}\}$, $\{W_{(9)}, W_{(14)}\}$, $\{\Omega_{10,1}, \Omega_{13,1}\}$, $\{\Omega_{10,2}, \Omega_{13,2}\}$, $\{W_{(12)}, \Omega_{15,2}\}$, $\{W_{(11)}\}$.

(3) ρ maps an l.c.r. set of a two-sided cell Ω to an l.c.r. set of $\rho(\Omega)$.

(4) ρ induces a bijective map from the vertex set M_L of a left cell graph $\mathfrak{M}_L$ in a two-sided cell Ω to the vertex set $\rho(M_L)$ of a left cell graph $\rho(\mathfrak{M}_L)$ in the two-sided cell $\rho(\Omega)$ such that $\{\Gamma, \Gamma'\}$ is an edge of $\mathfrak{M}_L$ if and only if $\{\rho(\Gamma), \rho(\Gamma')\}$ is an edge of $\rho(\mathfrak{M}_L)$. Moreover, we have $\mathcal{R}(w_0x) = S \setminus \mathcal{R}(x)$ for any $x \in W$, and so $\rho(\mathfrak{M}_L) \cong \mathfrak{M}_L^{\mathrm{op}}$.

Ω	$\mathbf{u}(\Omega)$	$a(\Omega)$	$n(\Omega)$	Ω	$\mathbf{u}(\Omega)$	$a(\Omega)$	$n(\Omega)$
$W_{(0)}$	E_7	0	1	$W_{(63)}$	1	63	1
$W_{(1)}$	$E_7(a_1)$	1	7	$W_{(46)}$	A_1	46	7
$W_{(2)}$	$E_7(a_2)$	2	27	$W_{(37)}$	$2A_1$	37	27
$\Omega_{3,1}$	$E_7(a_3)$	3	56	$W_{(30)}$	A_2	30	56
$\Omega_{3,2}$	E_6	3	21	$W_{(36)}$	$(3A_1)''$	36	21
$W_{(4)}$	$E_6(a_1)$	4	120	$W_{(25)}$	$A_2 + A_1$	25	120
$W_{(5)}$	$E_7(a_4)$	5	189	$W_{(22)}$	$A_2 + 2A_1$	22	189
$\Omega_{6,1}$	$D_6(a_1)$	6	210	$\Omega_{21,3}$	A_3	21	210
$\Omega_{6,2}$	$D_5 + A_1$	6	168	$\Omega_{21,2}$	$2A_2$	21	168
$\Omega_{6,3}$	A_6	6	105	$\Omega_{21,1}$	$A_2 + 3A_1$	21	105
$\Omega_{7,1}$	$E_7(a_5)$	7	315	$W_{(16)}$	$D_4(a_1)$	16	315
$\Omega_{7,2}$	D_5	7	189	$W_{(20)}$	$(A_3 + A_1)''$	20	189
$W_{(8)}$	$E_6(a_3)$	8	405	$\Omega_{15,1}$	$D_4(a_1) + A_1$	15	405
$W_{(9)}$	$D_5(a_1) + A_1$	9	378	$W_{(14)}$	$A_3 + A_2$	14	378
$\Omega_{10,1}$	$A_4 + A_2$	10	210	$\Omega_{13,1}$	$A_3 + A_2 + A_1$	13	210
$\Omega_{10,2}$	$D_5(a_1)$	10	420	$\Omega_{13,2}$	A_4	13	420
$W_{(12)}$	A_5''	12	105	$\Omega_{15,2}$	D_4	15	105
$W_{(11)}$	$A_4 + A_1$	11	512				

Table 4.2.1

By the above theorem, it suffices to consider one two-sided cell in each ρ-orbit, or equivalently, all the two-sided cells Ω of W with $a(\Omega) \leqslant 12$.

4.4. We know that in Algorithm 3.6, Process (**A**) (resp. Process (**C**)) is the simplest (resp. the most complicated) of the three processes. So in applying Algorithm 3.6, we prefer to perform Process (**A**) first, then to perform Process (**B**) when Process (**A**) becomes ineffective. Process (**C**) is performed only when both Processes (**A**) and (**B**) are ineffective. We want to find an l.c.r. set in a two-sided cell Ω of W by applying Algorithm 3.6. As the number $n(\Omega)$ of left cells in each two-sided cell Ω of W is known, we need only find a distinguished subset (see 3.4) in Ω with the cardinality $n(\Omega)$. This can be achieved by only applying Processes (**A**) and (**B**) in practice. We have designed a series of computer programs to perform Processes (**A**) and (**B**). These programs are so efficient that (for example) one need only take a couple of minutes to get the graph $\mathfrak{M}(x)$ from an element $x \in W$.

§5. Left cells of $W(E_7)$

In this section, we shall apply Algorithm 3.6 to find an l.c.r. set in each two-sided cell Ω of the Weyl group $W = W(E_7)$, which occurs as the vertex sets of certain graphs. From the latter we also get all the left cell graphs for each Ω. As mentioned in 4.4, we need only find a distinguished subset in each Ω with the cardinality $n(\Omega)$ (see 4.4). We always use the following criterion for a set E being distinguished: if the elements of E all have different generalized τ-invariants, then the set E is distinguished. By Theorem 4.3, we need only deal with the two-sided cells Ω with $a(\Omega) \leqslant 12$. For a given two-sided cell Ω of W, we always choose the elements of the form w_I, $I \subseteq S$, as the initial elements of Algorithm 3.6 whenever it is possible. We shall see that in our case, we need only perform Processes (**A**) and (**B**) but never (**C**). We usually get the set $M(x)$ (see 2.6) from $x \in \Omega$ by Process (**A**). Then Process (**B**) is often applied to get some element y from $x \in \Omega$ with $y \in \Omega \setminus M(x)$. We usually make $\{x, y\}$ to be a primitive pair (see 2.5).

5.1. $\mathbf{a}(\Omega) \leqslant 4$. Observe that $W_{(i)}$ is a single two-sided cell of W for $i = 0, 1, 2, 4$, and $W_{(3)}$ is a union of two two-sided cells $\Omega_{3,1}$, $\Omega_{3,2}$. Further $M(e)$ (e is the identity element of W), $M(\mathbf{1})$ and $M(\mathbf{12})$ are l.c.r. sets of $W_{(0)}$, $W_{(1)}$ and $W_{(2)}$ respectively by the fact that these sets are all distinguished and by comparing the cardinalities of these sets with the numbers of left cells in the corresponding two-sided cells of W (see Table 4.2.1). For the two-sided cell $W_{(4)}$, take $w_{\mathbf{1257}}, w_{\mathbf{123}}$ in $W_{(4)}$. Then $|M(w_{\mathbf{1257}})| = 15$ and $|M(w_{\mathbf{123}})| = 210$ (the notation $|X|$ stands for the cardinality of a set X). By 2.3 (4) and by closely observing the graph $\mathfrak{M}(w_{\mathbf{123}})$, we can pick out a maximal distinguished subset $M'(w_{\mathbf{123}})$ of $M(w_{\mathbf{123}})$ with cardinality 105. We see that $M(w_{\mathbf{1257}}) \cup M'(w_{\mathbf{123}})$ is distinguished with cardinality 120, equal to the number of left cells in $W_{(4)}$, which hence forms an l.c.r. set of $W_{(4)}$. We know that the numbers of left cells in the two-sided cells

$\Omega_{3,1}$, $\Omega_{3,2}$ are 56, 21, respectively. Take $w_{\mathbf{13}}$ in $W_{(3)}$. Then $|M(w_{\mathbf{13}})| = 42$. By 2.3 (4) and by closely observing the graph $\mathfrak{M}(w_{\mathbf{13}})$, we can pick out a maximal distinguished subset $M'(w_{\mathbf{13}})$ of $M(w_{\mathbf{13}})$ of cardinality 21. Take $x_0 = w_{\mathbf{13}} \cdot \mathbf{45}$ in $M(w_{\mathbf{13}})$. Let $y_0 = x_0 \cdot \mathbf{2}$. Then $\{x_0, y_0\}$ is a primitive pair. We have $y_0 \underset{\mathrm{L}}{\sim} w_{\mathbf{125}}$ by 2.3 (4) and by the fact $\mathcal{R}(y_0) = \{\mathbf{1}, \mathbf{2}, \mathbf{5}\}$. We see that the set $M'(w_{\mathbf{13}}) \cup M(w_{\mathbf{125}})$ is distinguished with cardinality 56. Hence it must be contained in the two-sided cell $\Omega_{3,1}$ as an l.c.r. set. On the other hand, we have $w_{\mathbf{257}} \in W_{(3)} \setminus \Omega_{3,1}$ since $\ell(w_{\mathbf{257}}) = 3$ and there is no element x in $M'(w_{\mathbf{13}}) \cup M(w_{\mathbf{125}})$ with $\mathcal{R}(x) = \{\mathbf{2}, \mathbf{5}, \mathbf{7}\}$. That is, $w_{\mathbf{257}} \in \Omega_{3,2}$. The set $M(w_{\mathbf{257}})$ is distinguished with cardinality 21. Hence it forms an l.c.r. set of $\Omega_{3,2}$.

5.2. $\Omega = \mathbf{W}_{(5)}$**.** Take $w_{\mathbf{1237}}$ in $W_{(5)}$. Then $|M(w_{\mathbf{1237}})| = 174$. Take $x_1 = w_{\mathbf{1237}} \cdot \mathbf{4265643452413}$ in $M(w_{\mathbf{1237}})$ (see 3.5 (b) for the notation of a dot) . Let $y_1 = x_1 \cdot \mathbf{7}$. Then $\{x_1, y_1\}$ is a primitive pair with $|M(y_1)| = 15$. The set $M(w_{\mathbf{1237}}) \cup M(y_1)$ is distinguished with cardinality 189, which forms an l.c.r. set of $W_{(5)}$ by Table 4.2.1.

5.3. $\Omega \subset \mathbf{W}_{(6)}$**.** The numbers of left cells in the two-sided cells $\Omega_{6,1}$, $\Omega_{6,2}$ and $\Omega_{6,3}$ are 210, 168 and 105, respectively. Take $w_{\mathbf{134}}$ in $W_{(6)}$. Then $M(w_{\mathbf{134}})$ is distinguished with $|M(w_{\mathbf{134}})| = 210$. So $M(w_{\mathbf{134}})$ must be contained in $\Omega_{6,1}$ as an l.c.r. set. We have $w_{\mathbf{1356}} \in W_{(6)} \setminus \Omega_{6,1}$ since $\ell(w_{\mathbf{1356}}) = 6$ and there is no element x in $M(w_{\mathbf{134}})$ with $\mathcal{R}(x) = \{\mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{6}\}$. The set $M(w_{\mathbf{1356}})$ is distinguished with $|M(w_{\mathbf{1356}})| = 168$. So we must have $M(w_{\mathbf{1356}})$ in $\Omega_{6,2}$ as an l.c.r. set. We have $w_{\mathbf{12357}} \in W_{(6)} \setminus (\Omega_{6,1} \cup \Omega_{6,2})$ since $\ell(w_{\mathbf{12357}}) = 6$ and there is no element x in $M(w_{\mathbf{134}}) \cup M(w_{\mathbf{1356}})$ with $\mathcal{R}(x) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}, \mathbf{7}\}$, The set $M(w_{\mathbf{12357}})$ is distinguished with cardinality 90. Take $x_2 = w_{\mathbf{12357}} \cdot \mathbf{4265643452413}$ in $M(w_{\mathbf{12357}})$. Let $y_2 = x_2 \cdot \mathbf{7}$. Then $\{x_2, y_2\}$ is a primitive pair with $|M(y_2)| = 15$. The set $M(w_{\mathbf{12357}}) \cup M(y_2)$ is distinguished with cardinality 105 and so it is an l.c.r. set of $\Omega_{6,3}$.

5.4. $\Omega \subset \mathbf{W}_{(7)}$**.** The numbers of left cells in the two-sided cells $\Omega_{7,1}$, $\Omega_{7,2}$ are 315, 189, respectively. We have $w_{\mathbf{1347}}, w_{\mathbf{12356}}, w_{\mathbf{2457}} \in W_{(7)}$ with $|M(w_{\mathbf{1347}})| = 210$, $|M(w_{\mathbf{12356}})| = 70$ and $|M(w_{\mathbf{2457}})| = 189$. The set $M(w_{\mathbf{1347}}) \cup M(w_{\mathbf{12356}}) \cup M(w_{\mathbf{2457}})$ is distinguished. This implies that $w_{\mathbf{1347}} \notin \Omega_{7,2}$ and hence $w_{\mathbf{2457}} \notin \Omega_{7,1}$. So $w_{\mathbf{1347}} \in \Omega_{7,1}$ and $w_{\mathbf{2457}} \in \Omega_{7,2}$. In particular, this implies that $M(w_{\mathbf{2457}})$ is an l.c.r. set of $\Omega_{7,2}$. Hence $M(w_{\mathbf{12356}}) \subset \Omega_{7,1}$. Take $x_3 = w_{\mathbf{1347}} \cdot \mathbf{56424534}$ in $M(w_{\mathbf{1347}})$. Let $y_3 = x_3 \cdot \mathbf{7}$. Then $\{x_3, y_3\}$ is a primitive pair with $|M(y_3)| = 105$. We see that the generalized τ-invariant of an element in $M(y_3)$ is different from that of an element in $M(w_{\mathbf{1347}}) \cup M(w_{\mathbf{12356}})$

and that there is a distinguished subset $M'(y_3)$ of $M(y_3)$ consisting of 35 elements. That is, $M(w_{\mathbf{1347}}) \cup M(w_{\mathbf{12356}}) \cup M'(y_3)$ is a distinguished subset of $\Omega_{7,1}$ with cardinality 315 and so it forms an l.c.r. set of $\Omega_{7,1}$.

5.5. $\Omega = \mathbf{W}_{(8)}$**.** Take $w_{\mathbf{12457}}$ in $W_{(8)}$. Then $M(w_{\mathbf{12457}})$ is distinguished with $|M(w_{\mathbf{12457}})| = 216$. Take $x_4 = w_{\mathbf{12457}} \cdot \mathbf{653143254}$ in $M(w_{\mathbf{12457}})$. Let $y_4 = x_4 \cdot \mathbf{2}$. Then $\{x_4, y_4\}$ is a primitive pair and $|M(y_4)| = 189$. We see that $M(w_{\mathbf{12457}}) \cup M(y_4)$ is distinguished with cardinality 405, exactly equal to the number of left cells in $W_{(8)}$. So it forms an l.c.r. set of $W_{(8)}$.

5.6. $\Omega = \mathbf{W}_{(9)}$**.** Take $w_{\mathbf{13567}}$ in $W_{(9)}$ and $x_5 = w_{\mathbf{13567}} \cdot \mathbf{42563142534}$ in $M(w_{\mathbf{13567}})$. Let $y_5 = x_5 \cdot \mathbf{1}$. Then $\{x_5, y_5\}$ is a primitive pair. We have $|M(w_{\mathbf{13567}})| = 84$, $|M(y_5)| = 294$ and that $M(w_{\mathbf{13567}}) \cup M(y_5)$ is distinguished. So $M(w_{\mathbf{13567}}) \cup M(y_5)$ forms an l.c.r. set of $W_{(9)}$ by Table 4.2.1.

5.7. $\Omega \subset \mathbf{W}_{(10)}$**.** There are exactly two two-sided cells $\Omega_{10,1}$, $\Omega_{10,2}$ in $W_{(10)}$, which contain 420, 210 left cells, respectively. Take $w_{\mathbf{1234}}$ in $W_{(10)}$. Then $|M(w_{\mathbf{1234}})| = 672$. There are two elements, say x, y, in $M(w_{\mathbf{1234}})$ satisfying $\mathcal{R}(x) = \mathcal{R}(y) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$, and there is a distinguished subset $M'(w_{\mathbf{1234}})$ in $M(w_{\mathbf{1234}})$ consisting of 336 elements. So by 2.3 (4) and by closely observing the graph $\mathfrak{M}(w_{\mathbf{1234}})$, we see that $M'(w_{\mathbf{1234}})$ is a maximal distinguished subset of $M(w_{\mathbf{1234}})$. Take $x_6 = w_{\mathbf{1234}} \cdot \mathbf{54316542476534}$ in $M(w_{\mathbf{1234}})$. Let $y_6 = x_6 \cdot \mathbf{1}$. Then $\{x_6, y_6\}$ is a primitive pair with $|M(y_6)| = 84$. The set $M(w_{\mathbf{1234}}) \cup M(y_6)$ is distinguished with cardinality 420, equal to the number of left cells in $\Omega_{10,1}$. This implies that $M(w_{\mathbf{1234}}) \cup M(y_6)$ is contained in $\Omega_{10,1}$ as an l.c.r. set. We have $w_{\mathbf{123567}} \in W_{(10)} \setminus \Omega_{10,1}$ since $\ell(w_{\mathbf{123567}}) = 10$ and there is no element x in $M(w_{\mathbf{1234}}) \cup M(y_6)$ satisfying $\mathcal{R}(x) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}, \mathbf{6}, \mathbf{7}\}$. Then $w_{\mathbf{123567}} \in \Omega_{10,2}$. The set $M(w_{\mathbf{123567}})$ is distinguished with cardinality 70. Take $x_7 = w_{\mathbf{123567}} \cdot \mathbf{42563142534}$ in $M(w_{\mathbf{123567}})$. Let $y_7 = x_7 \cdot \mathbf{1}$. Then $\{x_7, y_7\}$ is a primitive pair. The set $M(w_{\mathbf{123567}}) \cup M(y_7)$ is distinguished with cardinality 210, equal to the number of left cells of $\Omega_{10,2}$. So it forms an l.c.r. set of $\Omega_{10,2}$.

5.8. $\Omega = \mathbf{W}_{(11)}$**.** Take $w_{\mathbf{12346}}$ in $W_{(11)}$. Then $|M(w_{\mathbf{12346}})| = 884$. There are exactly two elements x, y in $M(w_{\mathbf{12346}})$ with $\mathcal{R}(x) = \mathcal{R}(y) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}\}$, and there is a distinguished subset $M'(w_{\mathbf{12346}})$ of $M(w_{\mathbf{12346}})$ consisting of 442 elements. By 2.3 (4) and by closely observing the graph $\mathfrak{M}(w_{\mathbf{12346}})$, we see that $M'(w_{\mathbf{12346}})$ is a maximal distinguished subset of $M(w_{\mathbf{12346}})$. Take $x_8 = w_{\mathbf{12346}} \cdot \mathbf{54231675643254}$ in

$M(w_{\mathbf{12346}})$. Let $y_8 = x_8 \cdot \mathbf{1}$. Then $\{x_8, y_8\}$ is a primitive pair with $|M(y_8)| = 70$. The set $M'(w_{\mathbf{12346}}) \cup M(y_8)$ is distinguished with cardinality 512, equal to the number of left cells in $W_{(11)}$, which thus forms an l.c.r. set of $W_{(11)}$.

5.9. $\Omega = \mathbf{W_{(12)}}$. Take $w_{\mathbf{2345}}$ in $W_{(12)}$. Then $M(w_{\mathbf{2345}})$ is distinguished with cardinality 105, equal to the number of left cells of $W_{(12)}$. So it is an l.c.r. set of $W_{(12)}$.

5.10. By the above discussion, we get Table 5.10.1 for the left cell graphs $\mathfrak{M}_L$ (up to isomorphism, see 2.8) in the two-sided cells Ω of W, where $\mathfrak{M}_1 = \mathfrak{M}_L(\mathbf{1})$, $\mathfrak{M}_2 = \mathfrak{M}_L(\mathbf{12})$, $\mathfrak{M}_3 = \mathfrak{M}_L(\mathbf{125})$, $\mathfrak{M}_4 = \mathfrak{M}_L(w_{\mathbf{13}})$, $\mathfrak{M}_5 = \mathfrak{M}_L(\mathbf{257})$, $\mathfrak{M}_6 = \mathfrak{M}_L(\mathbf{1257})$, $\mathfrak{M}_7 = \mathfrak{M}_L(w_{\mathbf{123}})$, $\mathfrak{M}_8 = \mathfrak{M}_L(w_{\mathbf{1237}})$, $\mathfrak{M}_9 = \mathfrak{M}_L(y_1)$ (see 5.2), $\mathfrak{M}_{10} = \mathfrak{M}_L(w_{\mathbf{134}})$, $\mathfrak{M}_{11} = \mathfrak{M}_L(w_{\mathbf{1356}})$, $\mathfrak{M}_{12} = \mathfrak{M}_L(w_{\mathbf{12357}})$, $\mathfrak{M}_{13} = \mathfrak{M}_L(y_2)$ (see 5.3), $\mathfrak{M}_{14} = \mathfrak{M}_L(w_{\mathbf{1347}})$, $\mathfrak{M}_{15} = \mathfrak{M}_L(w_{\mathbf{12356}})$, $\mathfrak{M}_{16} = \mathfrak{M}_L(y_3)$ (see 5.4), $\mathfrak{M}_{17} = \mathfrak{M}_L(w_{\mathbf{2457}})$, $\mathfrak{M}_{18} = \mathfrak{M}_L(w_{\mathbf{12457}})$, $\mathfrak{M}_{19} = \mathfrak{M}_L(y_4)$ (see 5.5), $\mathfrak{M}_{20} = \mathfrak{M}_L(y_5)$ (see 5.6), $\mathfrak{M}_{21} = \mathfrak{M}_L(w_{\mathbf{13567}})$, $\mathfrak{M}_{22} = \mathfrak{M}_L(w_{\mathbf{1234}})$, $\mathfrak{M}_{23} = \mathfrak{M}_L(y_6)$ (see 5.7), $\mathfrak{M}_{24} = \mathfrak{M}_L(w_{\mathbf{123567}})$, $\mathfrak{M}_{25} = \mathfrak{M}_L(y_7)$ (see 5.7), $\mathfrak{M}_{26} = \mathfrak{M}_L(w_{\mathbf{12346}})$, $\mathfrak{M}_{27} = \mathfrak{M}_L(y_8)$ (see 5.8) and $\mathfrak{M}_{28} = \mathfrak{M}_L(w_{\mathbf{2345}})$.

Ω	$\mathfrak{M}_L$	Ω	$\mathfrak{M}_L$
$W_{(0)}$	$\boxed{\emptyset}$	$W_{(63)}$	$\boxed{\mathbf{1234567}}$
$W_{(1)}$	$\mathfrak{M}_1$	$W_{(46)}$	$\mathfrak{M}_1^{\mathrm{op}}$
$W_{(2)}$	$\mathfrak{M}_2$	$W_{(37)}$	$\mathfrak{M}_2^{\mathrm{op}}$
$\Omega_{3,1}$	$\mathfrak{M}_3$, $\mathfrak{M}_4$	$W_{(30)}$	$\mathfrak{M}_3^{\mathrm{op}}$, $\mathfrak{M}_4^{\mathrm{op}}$
$\Omega_{3,2}$	$\mathfrak{M}_5$	$W_{(36)}$	$\mathfrak{M}_5^{\mathrm{op}}$
$W_{(4)}$	$\mathfrak{M}_6$, $\mathfrak{M}_7$	$W_{(25)}$	$\mathfrak{M}_6^{\mathrm{op}}$, $\mathfrak{M}_7^{\mathrm{op}}$
$W_{(5)}$	$\mathfrak{M}_8$, $\mathfrak{M}_9$	$W_{(22)}$	$\mathfrak{M}_8^{\mathrm{op}}$, $\mathfrak{M}_9^{\mathrm{op}}$
$\Omega_{6,1}$	$\mathfrak{M}_{10}$	$\Omega_{21,3}$	$\mathfrak{M}_{10}^{\mathrm{op}}$
$\Omega_{6,2}$	$\mathfrak{M}_{11}$	$\Omega_{21,2}$	$\mathfrak{M}_{11}^{\mathrm{op}}$
$\Omega_{6,3}$	$\mathfrak{M}_{12}$, $\mathfrak{M}_{13}$	$\Omega_{21,1}$	$\mathfrak{M}_{12}^{\mathrm{op}}$, $\mathfrak{M}_{13}^{\mathrm{op}}$
$\Omega_{7,1}$	$\mathfrak{M}_{14}$, $\mathfrak{M}_{15}$, $\mathfrak{M}_{16}$	$W_{(16)}$	$\mathfrak{M}_{14}^{\mathrm{op}}$, $\mathfrak{M}_{15}^{\mathrm{op}}$, $\mathfrak{M}_{16}^{\mathrm{op}}$
$\Omega_{7,2}$	$\mathfrak{M}_{17}$	$W_{(20)}$	$\mathfrak{M}_{17}^{\mathrm{op}}$
$W_{(8)}$	$\mathfrak{M}_{18}$, $\mathfrak{M}_{19}$	$\Omega_{15,1}$	$\mathfrak{M}_{18}^{\mathrm{op}}$, $\mathfrak{M}_{19}^{\mathrm{op}}$
$W_{(9)}$	$\mathfrak{M}_{20}$, $\mathfrak{M}_{21}$	$W_{(14)}$	$\mathfrak{M}_{20}^{\mathrm{op}}$, $\mathfrak{M}_{21}^{\mathrm{op}}$
$\Omega_{10,1}$	$\mathfrak{M}_{22}$, $\mathfrak{M}_{23}$	$\Omega_{13,1}$	$\mathfrak{M}_{22}^{\mathrm{op}}$, $\mathfrak{M}_{23}^{\mathrm{op}}$
$\Omega_{10,2}$	$\mathfrak{M}_{24}$, $\mathfrak{M}_{25}$	$\Omega_{13,2}$	$\mathfrak{M}_{24}^{\mathrm{op}}$, $\mathfrak{M}_{25}^{\mathrm{op}}$
$W_{(12)}$	$\mathfrak{M}_{28}$	$\Omega_{15,2}$	$\mathfrak{M}_{28}^{\mathrm{op}}$
$W_{(11)}$	$\mathfrak{M}_{26}$, $\mathfrak{M}_{27}$		

Table 5.10.1

All the left cell graphs of W (up to isomorphism) are displayed in Appendix.

There exist essential graphs (see 2.8) corresponding to all the above left cell graphs with the following exceptions: $\mathfrak{M}_4$, $\mathfrak{M}_7$, $\mathfrak{M}_{16}$, $\mathfrak{M}_{24}$, $\mathfrak{M}_{26}$ and their opposed counterparts.

5.11. In the remaining part of the paper, we shall discuss the way to check the relation $\underset{\mathrm{L}}{\sim}$ among the elements of W.

Given a left cell Γ of a Coxeter group W', we say that the generalized τ-invariants characterize Γ, or that the generalized τ-invariants are the complete invariants of Γ, if for any $x \in \Gamma$ and $y \in W'$, we have $x \underset{\mathrm{L}}{\sim} y$ if and only if x, y have the same generalized τ-invariants.

A left cell graph of W is exceptional, if it is one of the graphs $\mathfrak{M}_6$, $\mathfrak{M}_9$, $\mathfrak{M}_{13}$, $\mathfrak{M}_{21}$, $\mathfrak{M}_{23}$, $\mathfrak{M}_{24}$, $\mathfrak{M}_{27}$ and their opposed counterparts. A left cell of W is exceptional, if it is a vertex of some exceptional left cell graph.

From Table 5.10.1, we get the following

5.12. Theorem. *In the group W, the generalized τ-invariants characterize all the non-exceptional left cells.*

By this theorem, we see that the generalized τ-invariants are enough in checking the relation $\underset{\mathrm{L}}{\sim}$ on any pair $x, y \in W$ when at least one of x, y is in a non-exceptional left cell. Thus in the subsequent discussion, we shall always assume that both elements $x, y \in W$ are in the exceptional left cells unless otherwise specified.

We have the following graph isomorphisms: $\mathfrak{M}_6 \cong \mathfrak{M}_9 \cong \mathfrak{M}_{13}$, $\mathfrak{M}_{21} \cong \mathfrak{M}_{23}$, $\mathfrak{M}_{24} \cong \mathfrak{M}_{27} \cong \mathfrak{M}_{24}^{\mathrm{op}}$, $\mathfrak{M}_6^{\mathrm{op}} \cong \mathfrak{M}_9^{\mathrm{op}} \cong \mathfrak{M}_{13}^{\mathrm{op}}$ and $\mathfrak{M}_{21}^{\mathrm{op}} \cong \mathfrak{M}_{23}^{\mathrm{op}}$. No pair of these graphs belong to the same two-sided cell of W.

By observing the left cell graphs listed in Table 5.10.1, we have

5.13. Lemma. *Assume that x, y are two elements of W in the exceptional left cells, having the same generalized τ-invariant. Then $x \underset{\mathrm{L}}{\sim} y$ if and only if $a(x) = a(y)$.*

5.14. Hence to check the relation $\underset{\mathrm{L}}{\sim}$ on the elements x, y in Lemma 5.13, it suffices to compare their a-values. Instead of directly computing these a-values, we shall adopt the following method. We select some exceptional left cells Γ, one in each exceptional left cell graph, with the property that $\mathcal{R}(\Gamma)$ is unique among the labels of the vertices in the left cell graph containing Γ. We find the element sets of these left cells. We express these elements in the alcove forms (see [15] for the definition) and store them in a disk. The a-values of these elements are known by the previous results. For an arbitrary element

x in an exceptional left cell, we take an element z in $M(x)$ with $\mathcal{R}(z)$ being the specified one. Thus z is in one of the selected left cells. So we get the a-value of x from that of z.

5.15. Some exceptional left cells of W. For $x \in W$, we denote the left cell containing x by Γ_x. We obtain the element sets of some exceptional left cells of W in terms of **A**-saturated subsets. Here we should first introduce some elements in W which will be used frequently in the sequel: $c_1 = \mathbf{4235654}$; $c_2 = \mathbf{2354137}$; $c_3 = \mathbf{4235}$; $c_4 = \mathbf{46351}$ and $c_5 = \mathbf{1674235}$.

Let $z_0 = w_{\mathbf{1257}}$, $z_1 = \mathbf{21317} \cdot c_1 \cdot \mathbf{3} \cdot z_0$, $z_2 = \mathbf{5} \cdot z_1$. Then $a(z_0) = 4$, $a(z_1) = 5$ and $a(z_2) = 6$. Thus z_i $(i = 0, 1, 2)$ is the unique element x in $M(z_i)$ with $\mathcal{R}(x) = \{\mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{7}\}$ (see Graph $\mathfrak{M}_6$ in Appendix). Let $b_1 = z_0 \cdot \mathbf{431}$, $b_2 = z_0 \cdot \mathbf{3} \cdot c_3$, $b_3 = z_0 \cdot \mathbf{3} \cdot c_1 \cdot c_2$ and $b_4 = z_2^{-1} \cdot c_1 \cdot c_2$. Then $\Gamma_{z_0} = M(z_0)^{-1} \cup M(b_1)^{-1} \cup M(b_2)^{-1}$, $\Gamma_{z_1} = M(z_1^{-1})^{-1} \cup M(b_3)^{-1}$ and $\Gamma_{z_2} = M(z_2^{-1})^{-1} \cup M(b_4)^{-1}$.

Let $z_3 = w_{\mathbf{13567}}$ and $z_4 = \mathbf{4234} \cdot c_4^{-1} \cdot \mathbf{254} \cdot z_3$. Then $a(z_3) = 9$ and $a(z_4) = 10$. Thus z_j $(j = 3, 4)$ is the unique element x in $M(z_j)$ with $\mathcal{R}(x) = \{\mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{6}, \mathbf{7}\}$ (see Graph $\mathfrak{M}_{21}$ in Appendix). Let $b_5 = z_3 \cdot c_3 \cdot \mathbf{16} \cdot c_3 \cdot \mathbf{14}$, $b_6 = z_3 \cdot c_3 \cdot \mathbf{1} \cdot c_4 \cdot \mathbf{67} \cdot c_3 \cdot c_4 \cdot \mathbf{67}$ and $b_7 = z_3 \cdot c_3 \cdot \mathbf{1} \cdot c_3 \cdot c_5 \cdot \mathbf{6} \cdot c_3 \cdot \mathbf{4}$. Then $\Gamma_{z_3} = M(z_3)^{-1} \cup M(b_5)^{-1}$ and $\Gamma_{z_4} = M(z_4^{-1})^{-1} \cup M(b_6)^{-1} \cup M(b_7)^{-1}$.

Let $z_5 = w_{\mathbf{123567}}$, $z_6 = z_4 \cdot \mathbf{2}$ and $z_7 = c_3 \cdot (c_5)^3 \cdot \mathbf{16} \cdot c_3 \cdot \mathbf{4} \cdot z_5$. Then $a(z_5) = 10$, $a(z_6) = 11$ and $a(z_7) = 13$. Thus z_k $(k = 5, 6, 7)$ is the unique element x in $M(z_k)$ with $\mathcal{R}(x) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}, \mathbf{6}, \mathbf{7}\}$ (see Graph $\mathfrak{M}_{24}$ in Appendix). Let $b_8 = \mathbf{2} \cdot b_5$, $b_9 = b_8 \cdot \mathbf{767} \cdot c_3^{-1} \cdot \mathbf{31}$, $b_{10} = b_8 \cdot \mathbf{76} \cdot c_3^{-1}$ and $b_{11} = z_5 \cdot c_3 \cdot (c_5)^2 \cdot \mathbf{67}$. Then $\Gamma_{z_5} = M(z_5)^{-1} \cup M(b_8)^{-1}$, $\Gamma_{z_6} = M(z_6^{-1})^{-1} \cup M(b_9)^{-1} \cup M(b_{10})^{-1}$ and $\Gamma_{z_7} = M(z_7^{-1})^{-1} \cup M(b_{11})^{-1}$.

Based on the knowledge of the above exceptional left cells, we obtain the following criterion on the a-value of an element of W in an exceptional left cell.

5.16. Theorem. *Assume that $x \in W$ satisfies $\mathfrak{M}(x) \cong \mathfrak{M}_6$ (resp. $\mathfrak{M}_{21}$, $\mathfrak{M}_{24}$). Then there is a unique element y in $M(x)$ with $\mathcal{R}(y) = \{\mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{7}\}$ (resp. $\{\mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{6}, \mathbf{7}\}$, $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}, \mathbf{6}, \mathbf{7}\}$). We have*

$$a(x) = \begin{cases} 4, & \text{if } y \in \Gamma_{z_0}, \\ 5, & \text{if } y \in \Gamma_{z_1}, \\ 6, & \text{if } y \in \Gamma_{z_2}. \end{cases}$$

(*resp.*

$$a(x)=\begin{cases}9, & \text{if } y\in\Gamma_{z_3}\ ,\\ 10, & \text{if } y\in\Gamma_{z_4}\ .\end{cases}\qquad a(x)=\begin{cases}10, & \text{if } y\in\Gamma_{z_5}\ ,\\ 11, & \text{if } y\in\Gamma_{z_6}\ ,\\ 13, & \text{if } y\in\Gamma_{z_7}\ .\end{cases}\quad)$$

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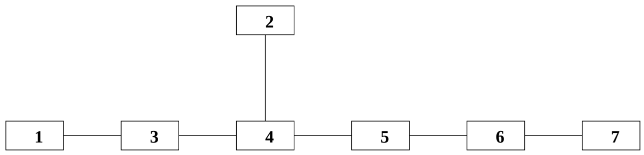
Appendix

Here we display all the left cell graphs (up to isomorphism) of the Weyl group of type E_7. Some of the graphs $\mathfrak{M}_x$ are divided into several separated parts only for the typewriting reason, which are denoted by $\mathfrak{M}_{x,A}$, $\mathfrak{M}_{x,B}$, etc. There are some edges joining pairs of vertices belonging to the different parts, which are recorded by some upper-left and upper-right labels of the involving vertices rather than by the concrete lines. For example, by the upper-right label $B2$ of the vertex ${}^{1}\boxed{\mathbf{16}}^{B2}$ in $\mathfrak{M}_{7,A}$, we mean that this vertex is joined by an edge to the vertex in $\mathfrak{M}_{7,B}$ with the upper-left label 2.



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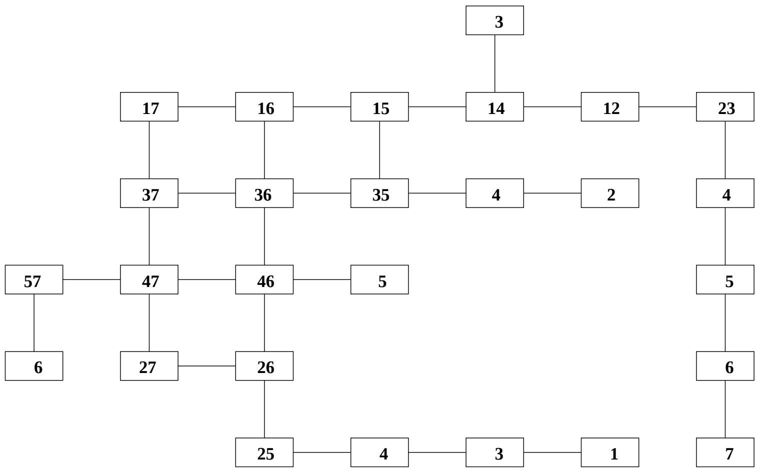

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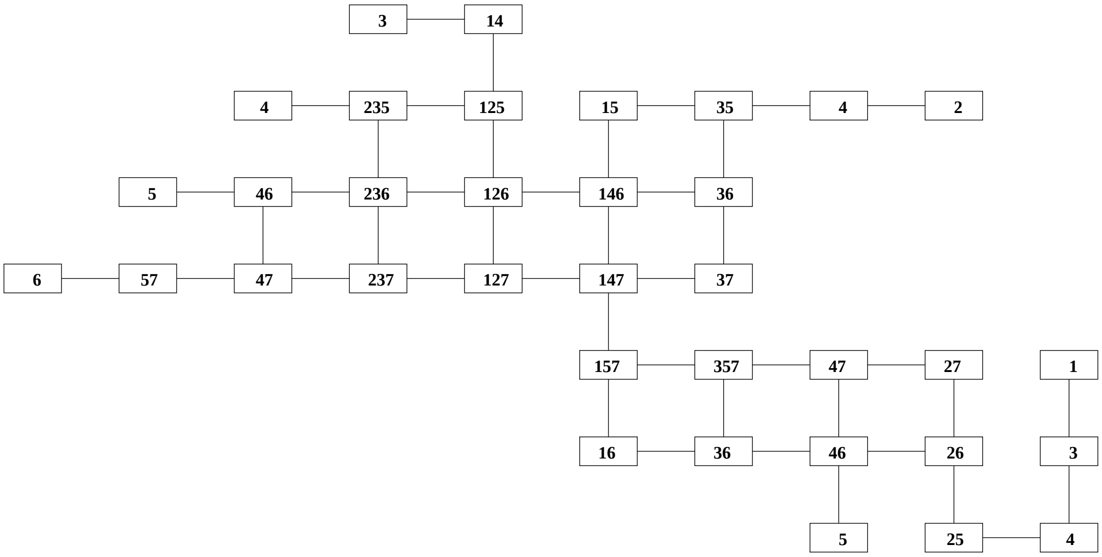

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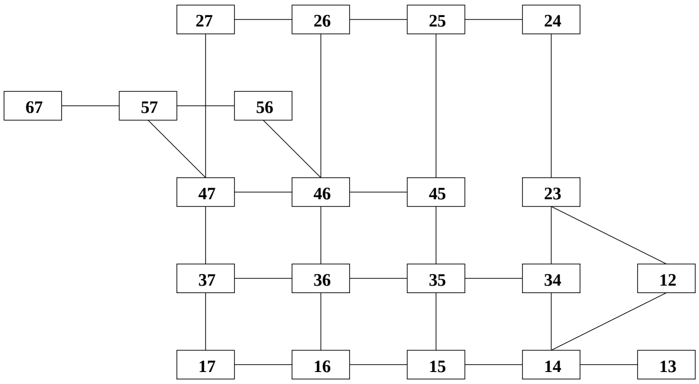
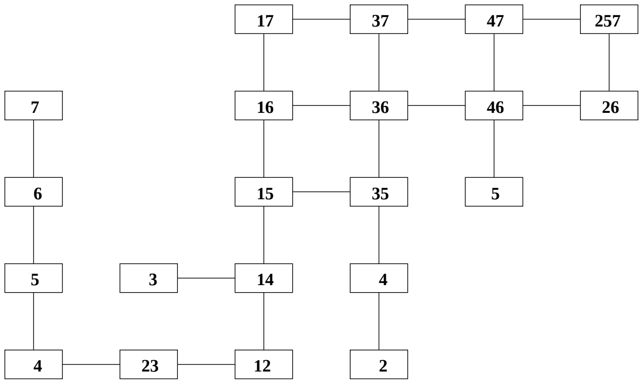
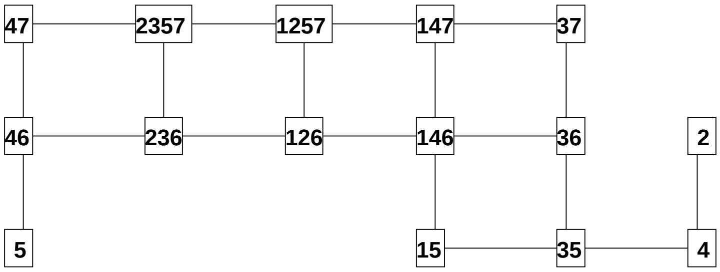


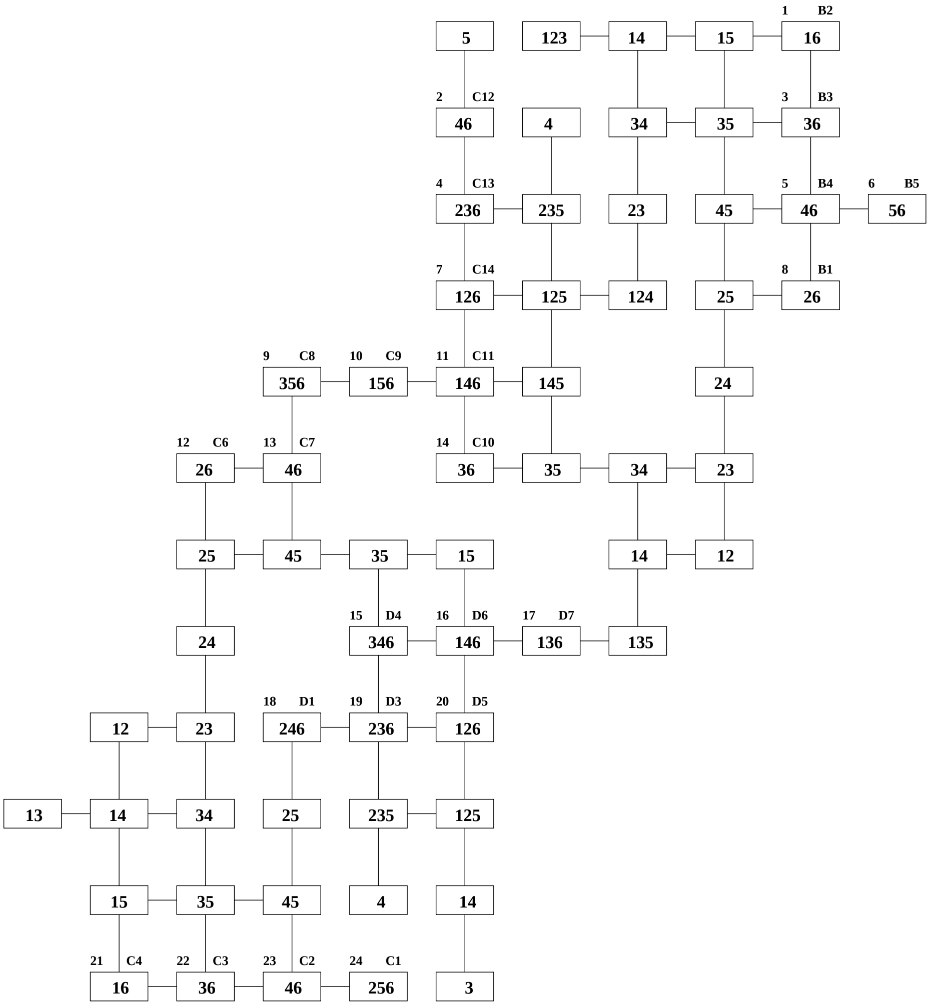
$\mathfrak{M}_4$



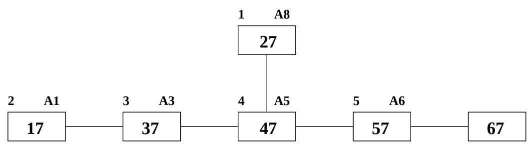
$\mathfrak{M}_5$



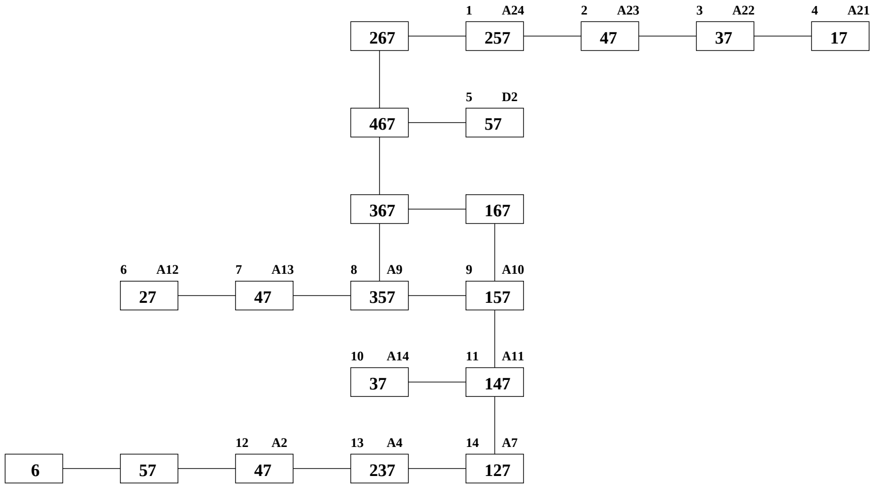
$\mathfrak{M}_6$



$\mathfrak{M}_{7,A}$



$\mathfrak{M}_{7,B}$



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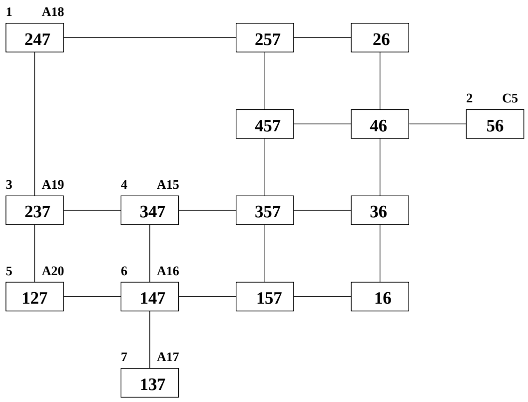
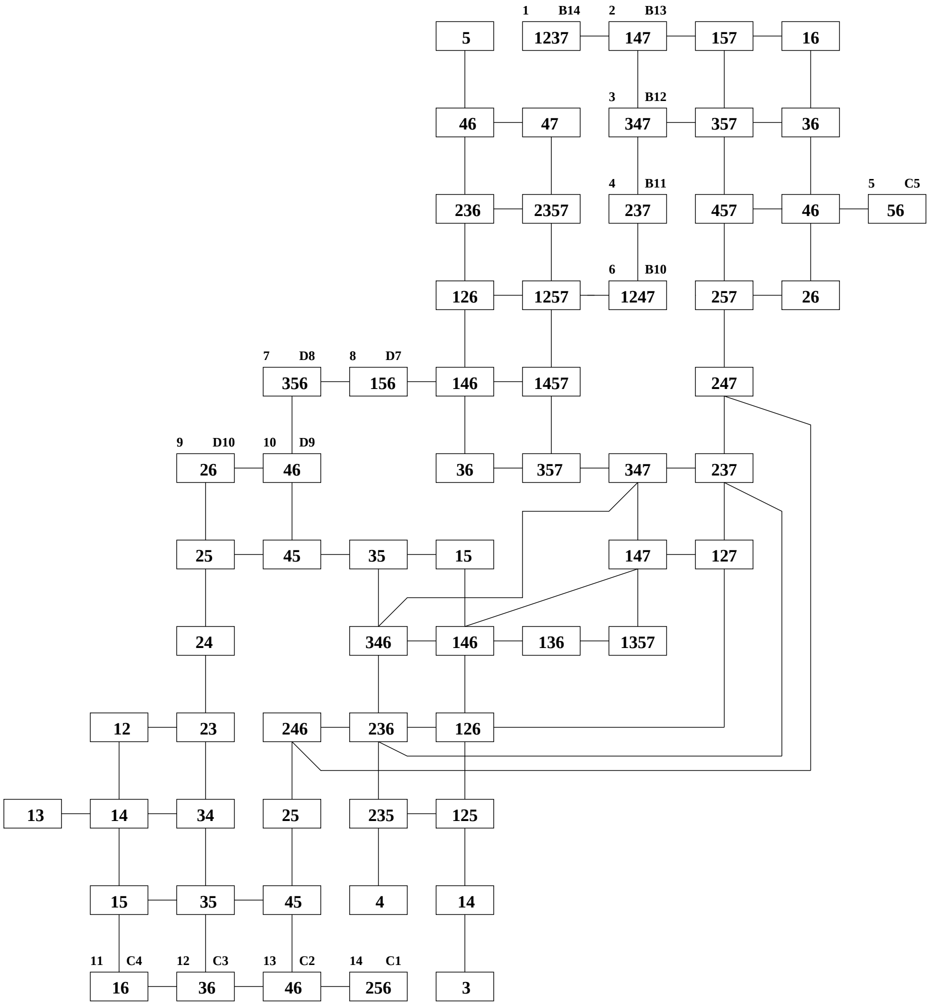
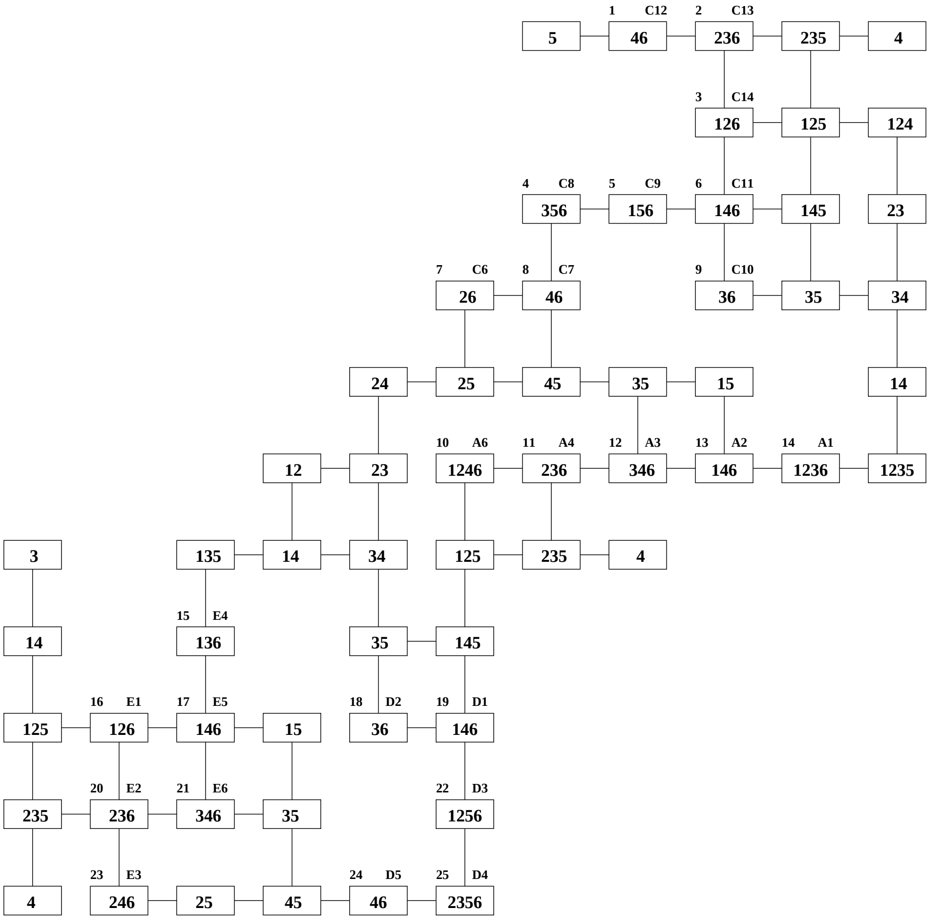


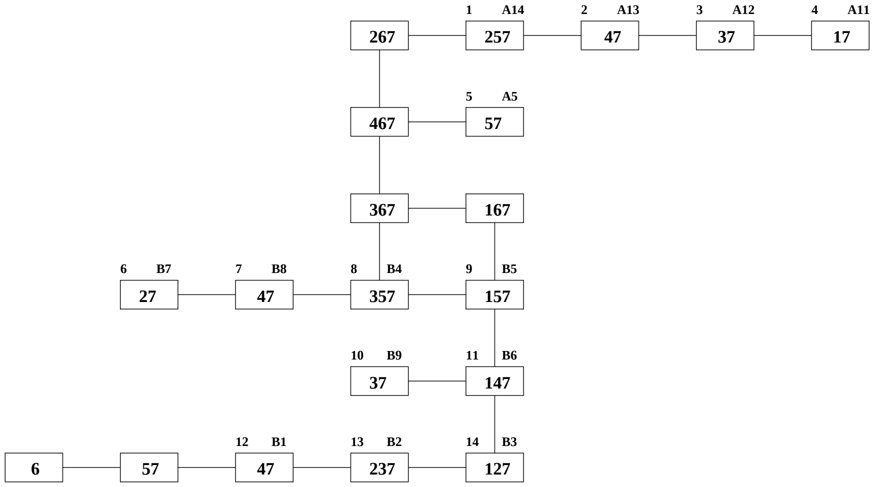
$\mathfrak{M}_{7,D}$



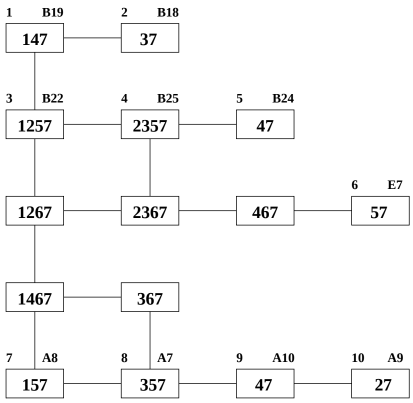
$\mathfrak{M}_{8,A}$



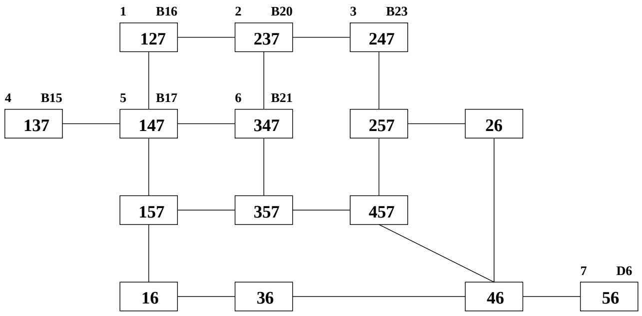
$\mathfrak{M}_{8,B}$



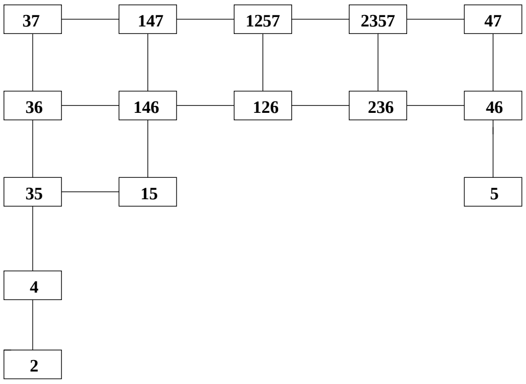
$\mathfrak{M}_{8,C}$



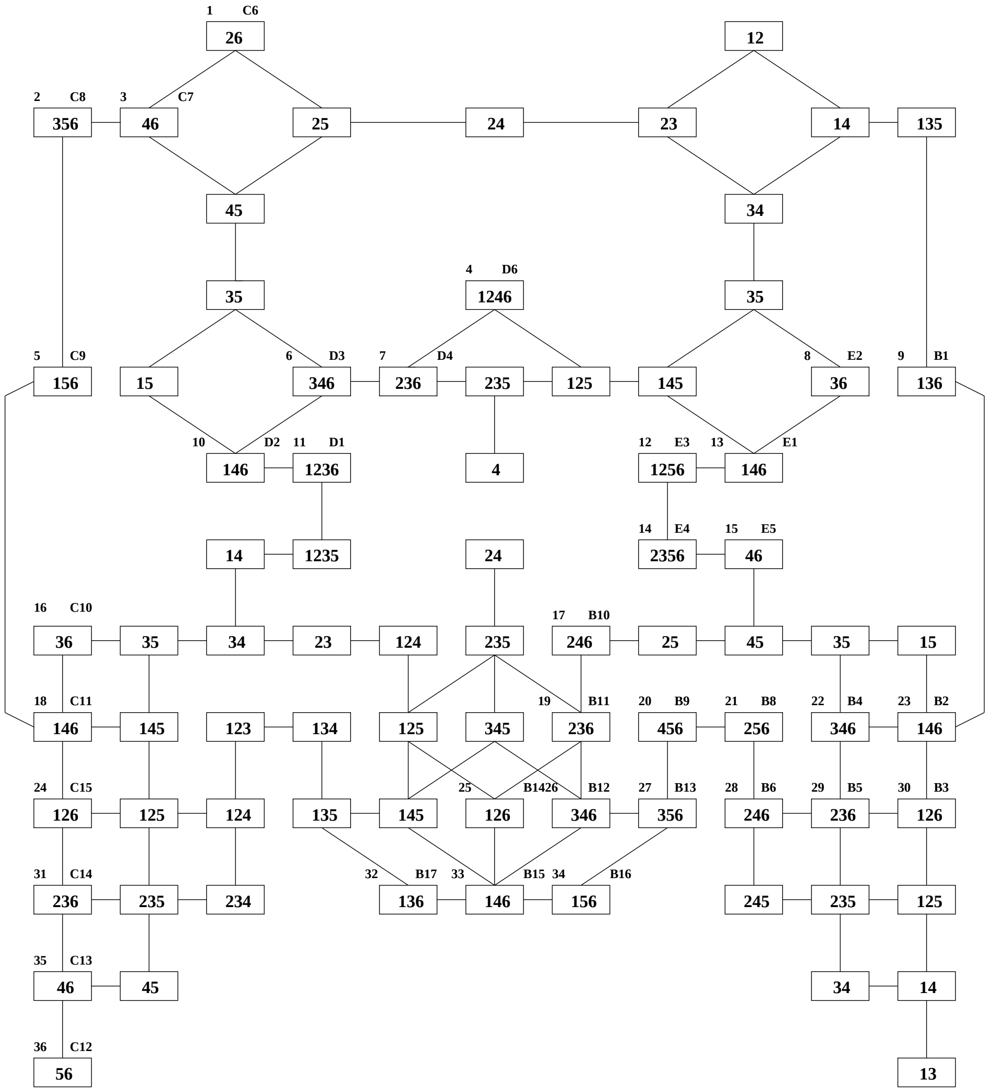
$\mathfrak{M}_{8,D}$



$\mathfrak{M}_{8,E}$



$\mathfrak{M}_9$



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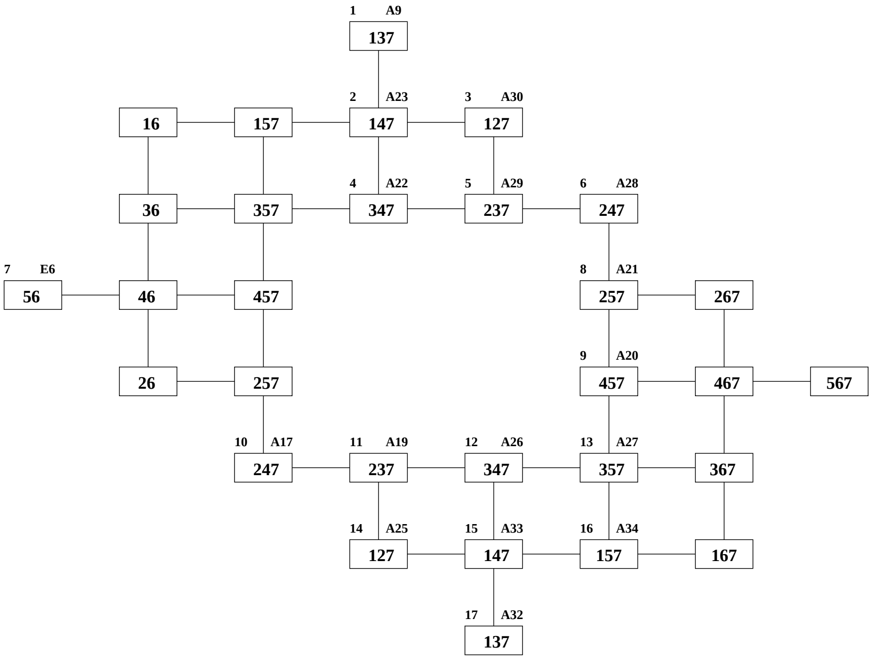
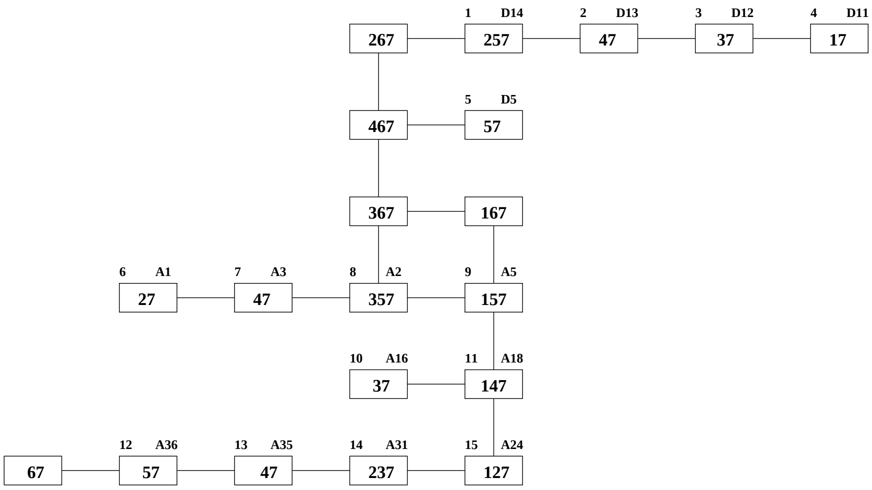
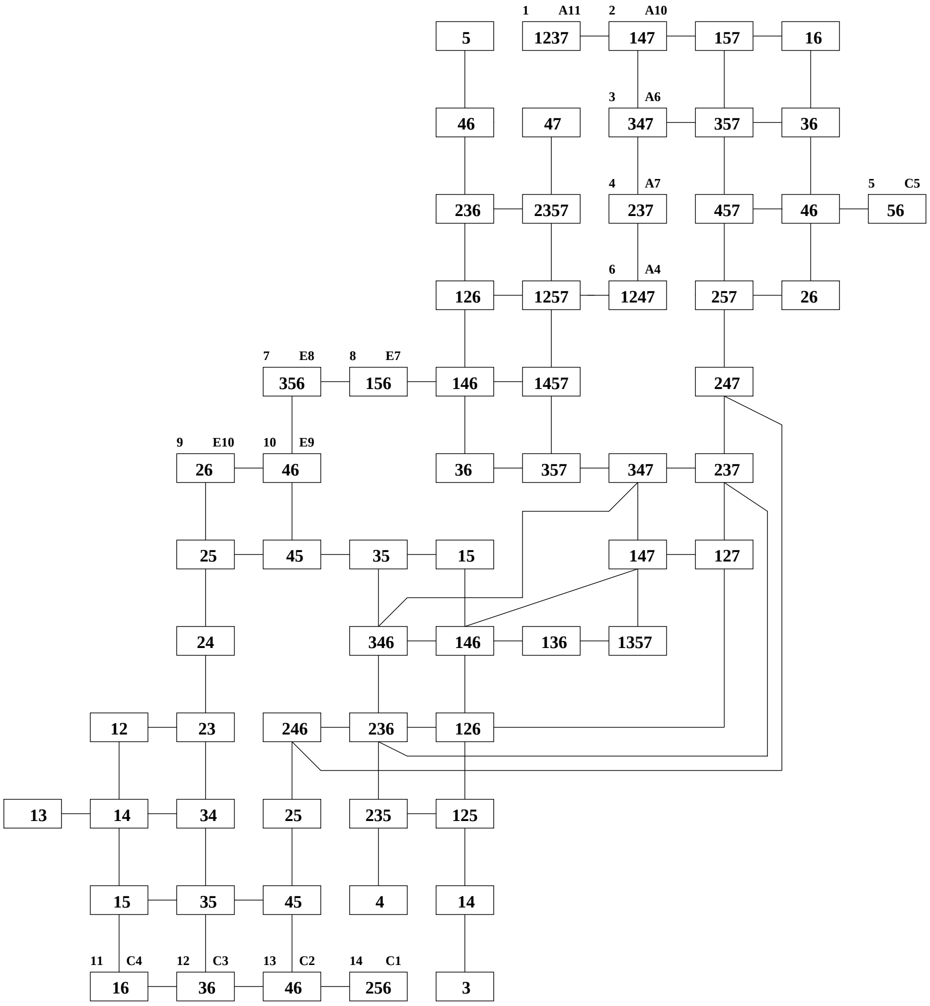


$\mathfrak{M}_{10,B}$



$\mathfrak{M}_{10,C}$



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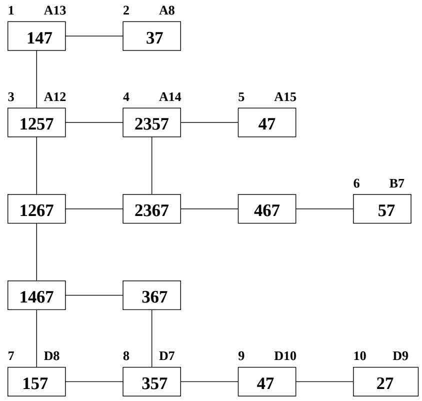

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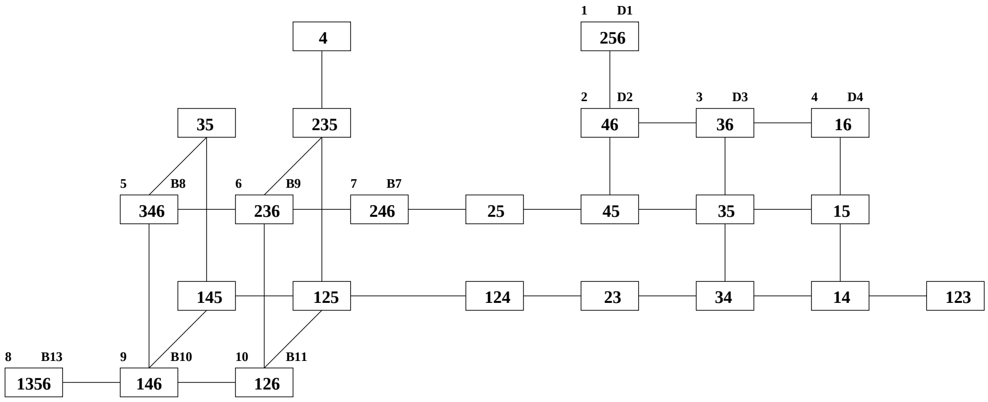

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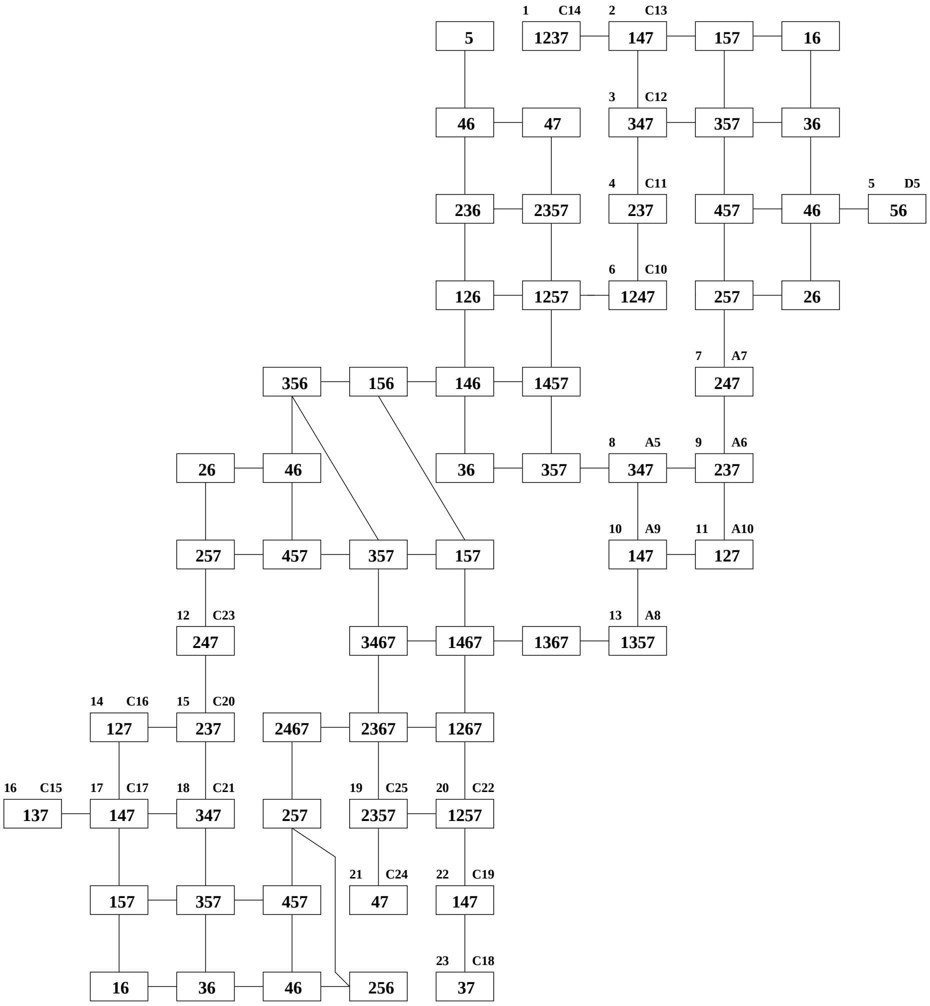

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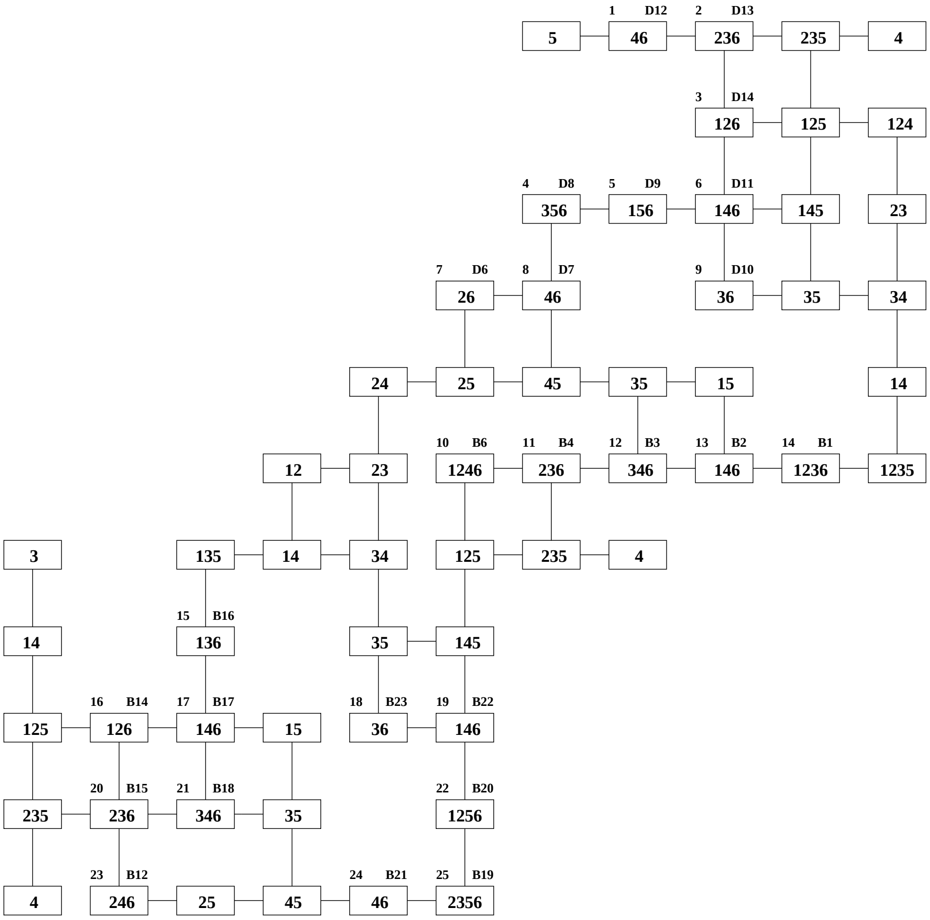

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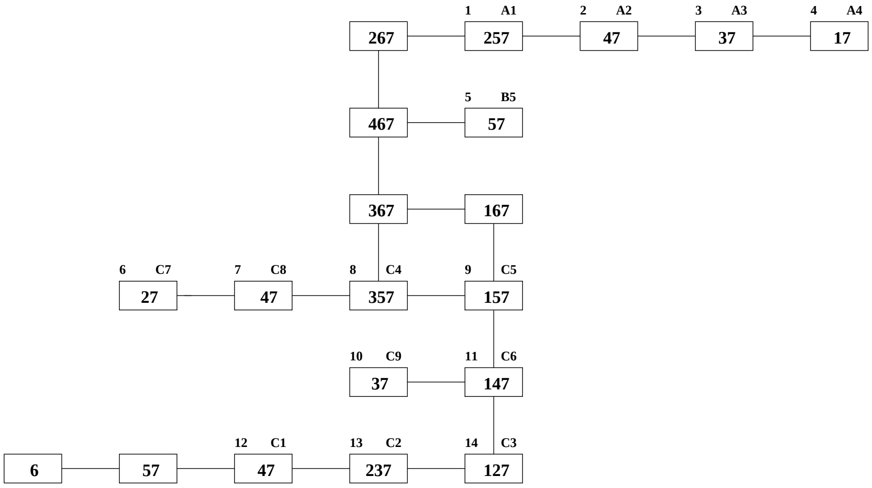

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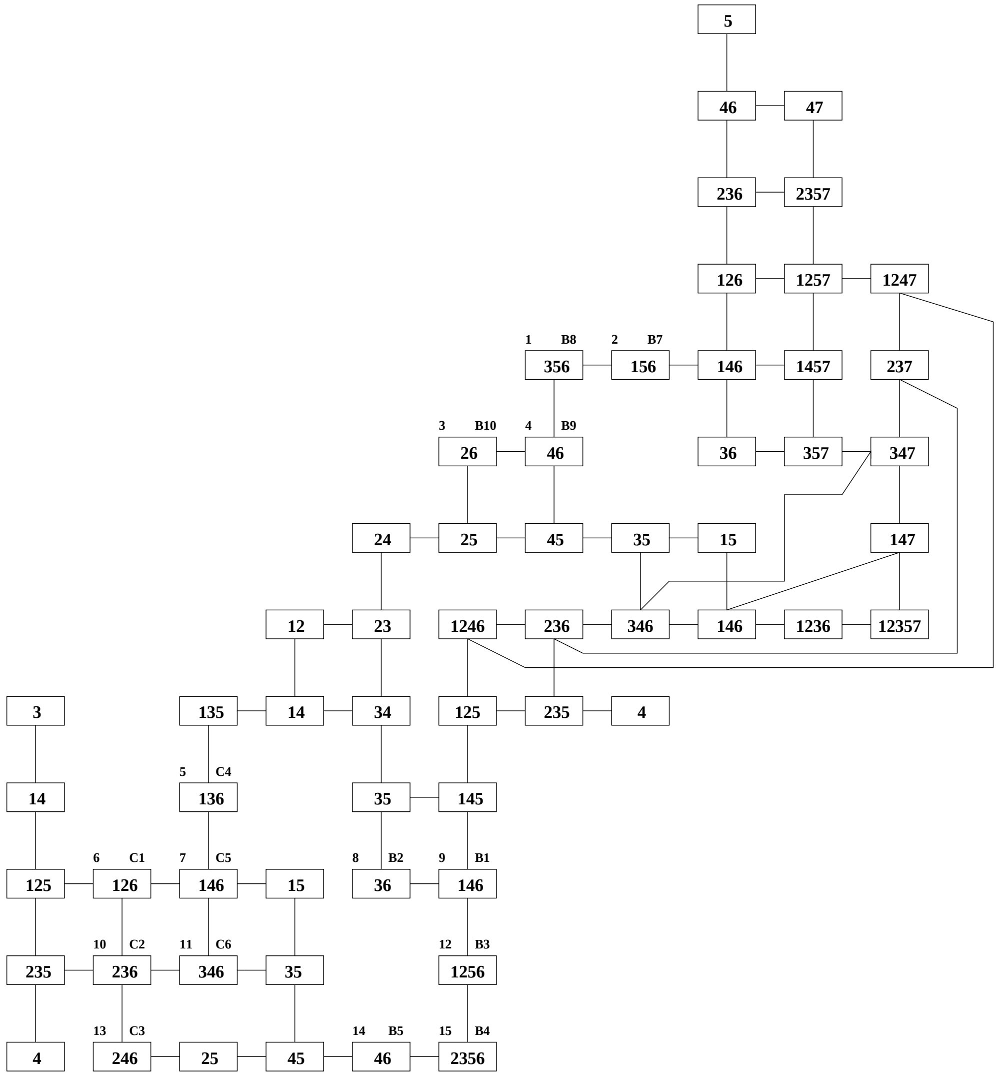

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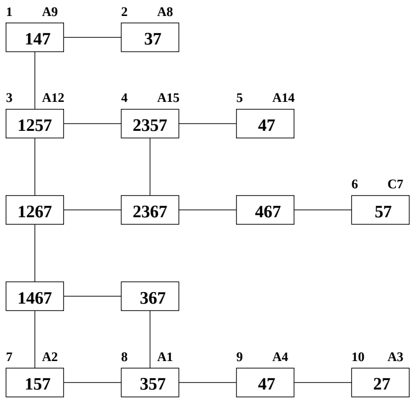

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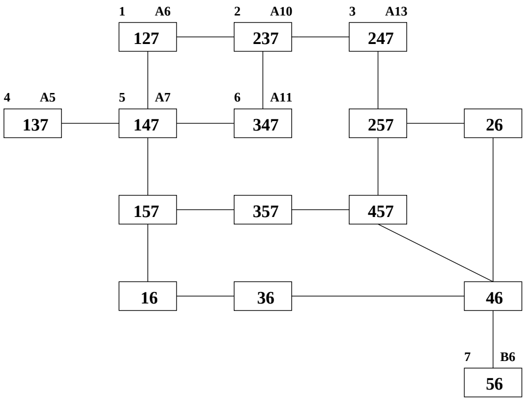

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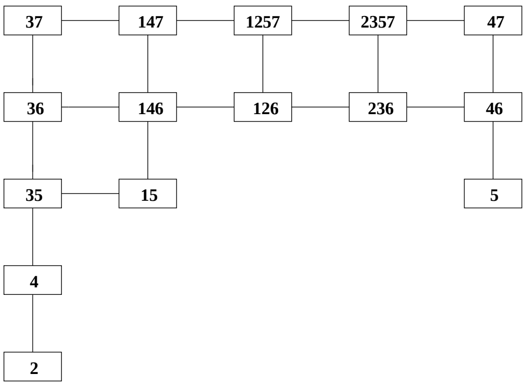

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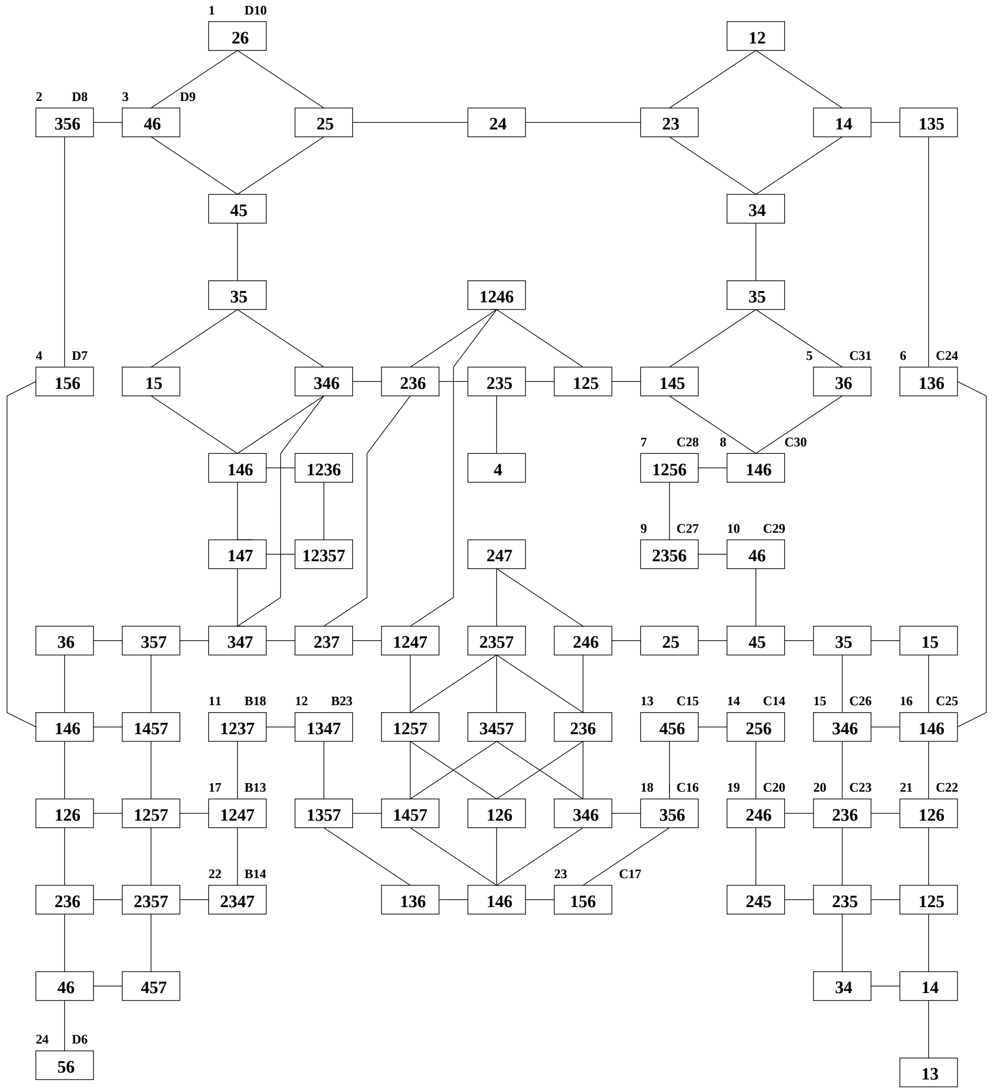

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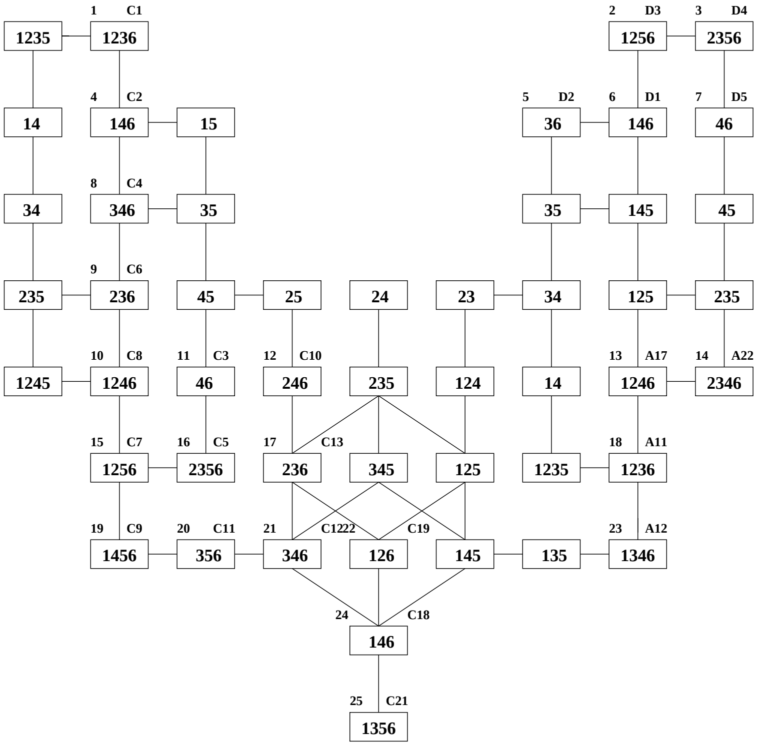

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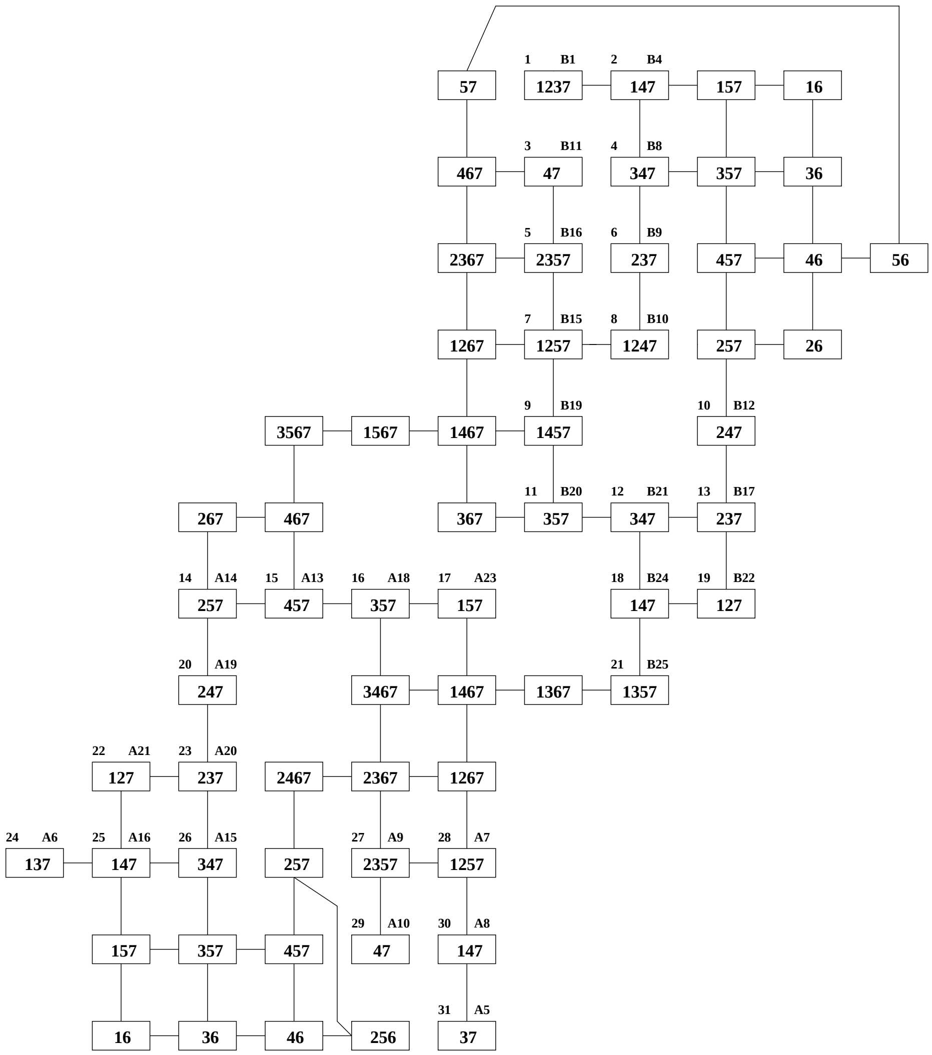

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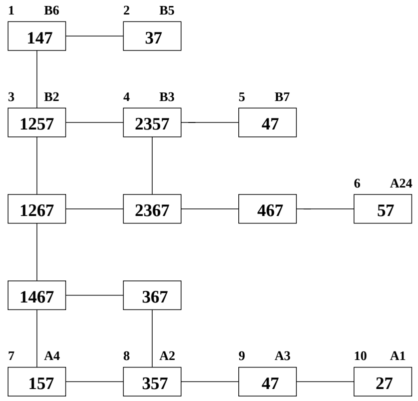

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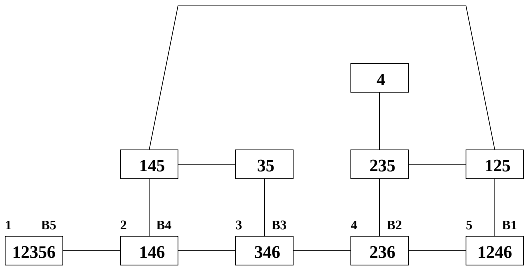

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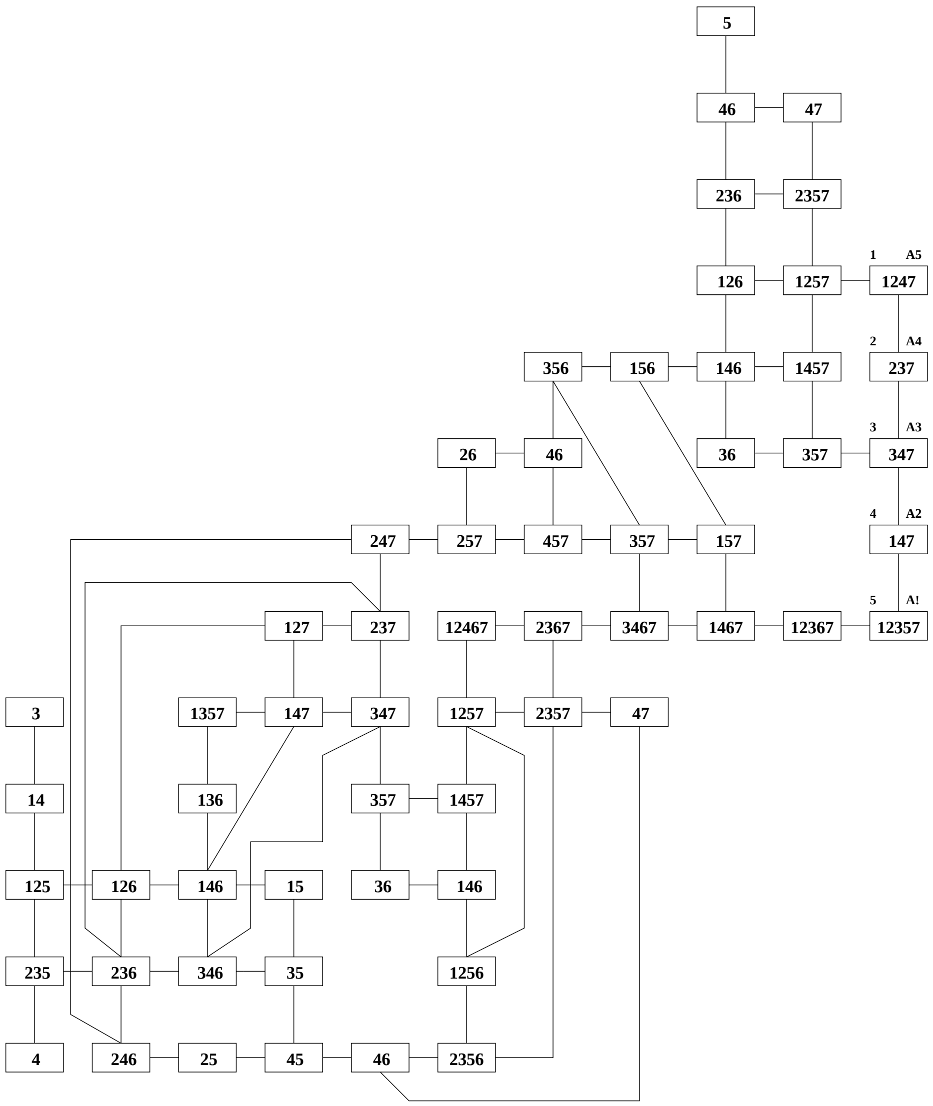

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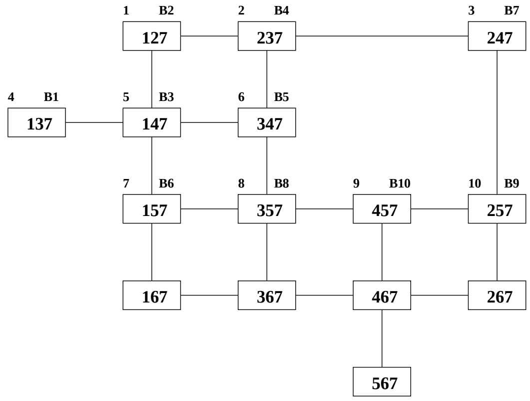

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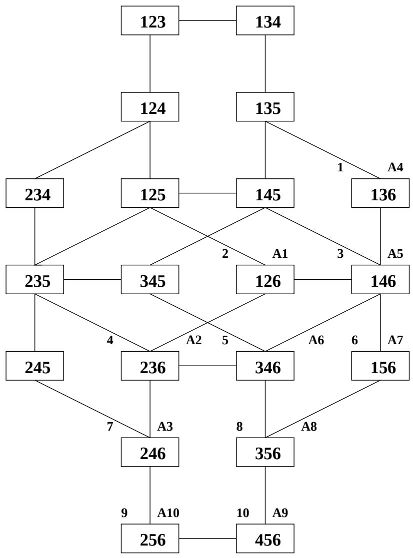

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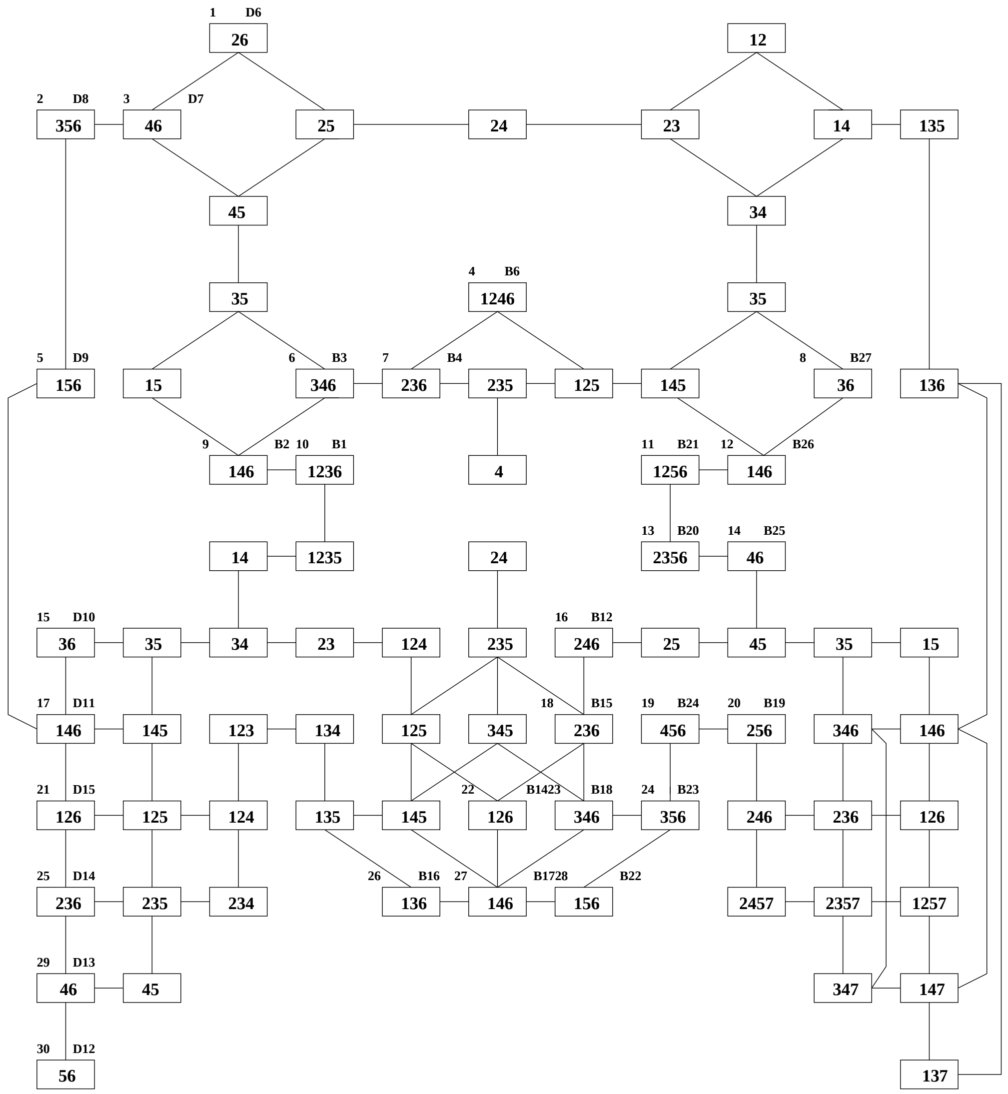

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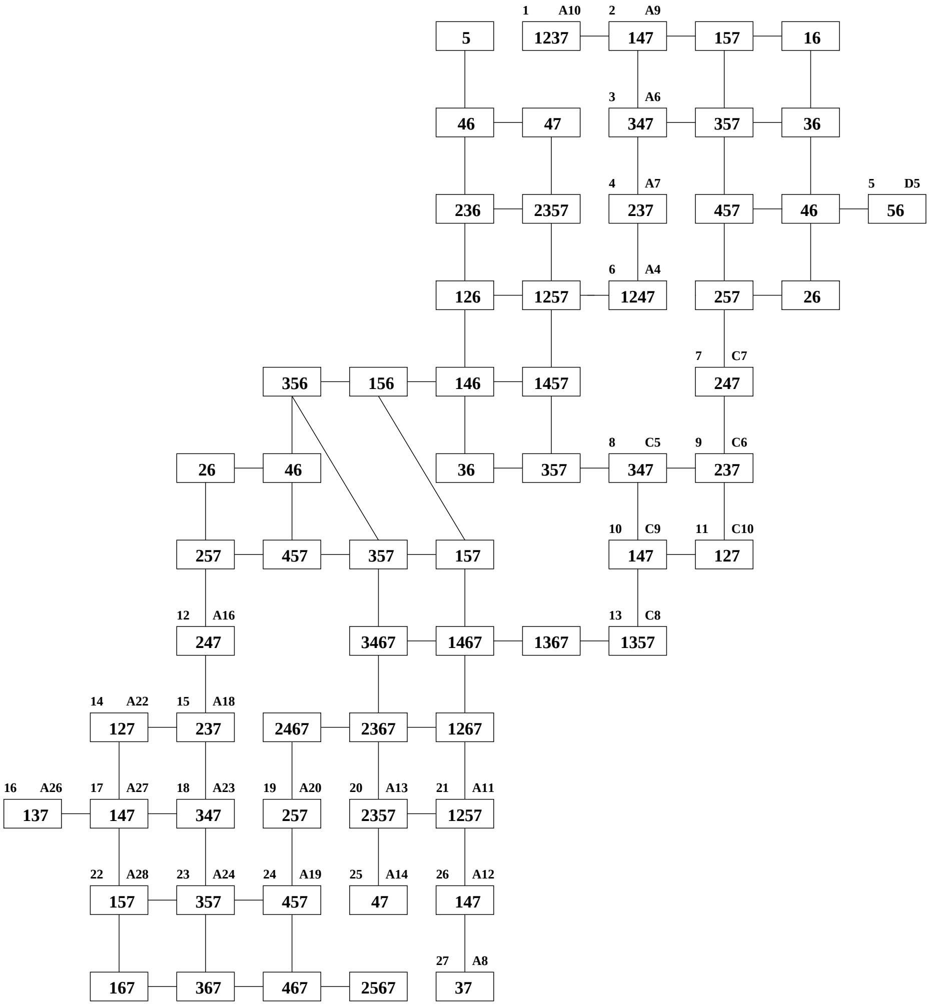

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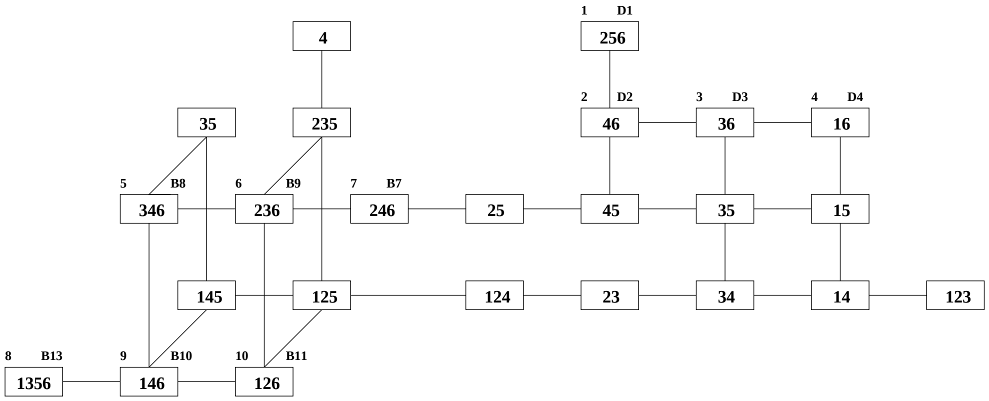

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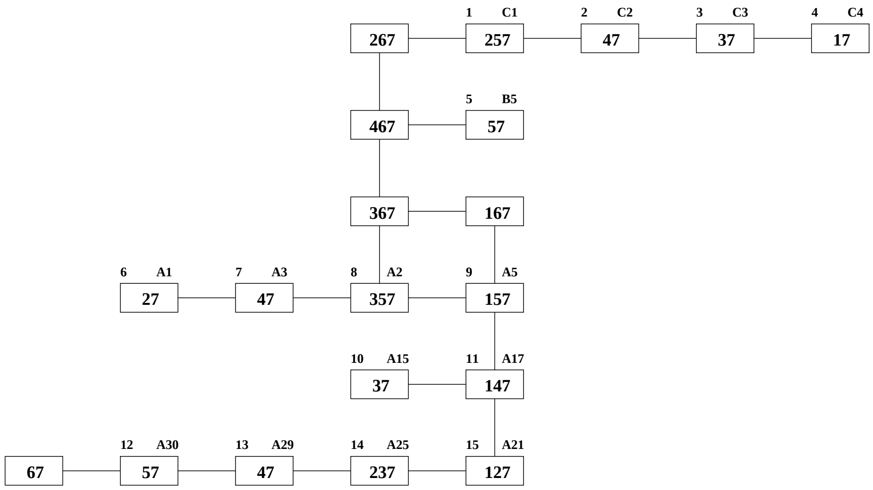

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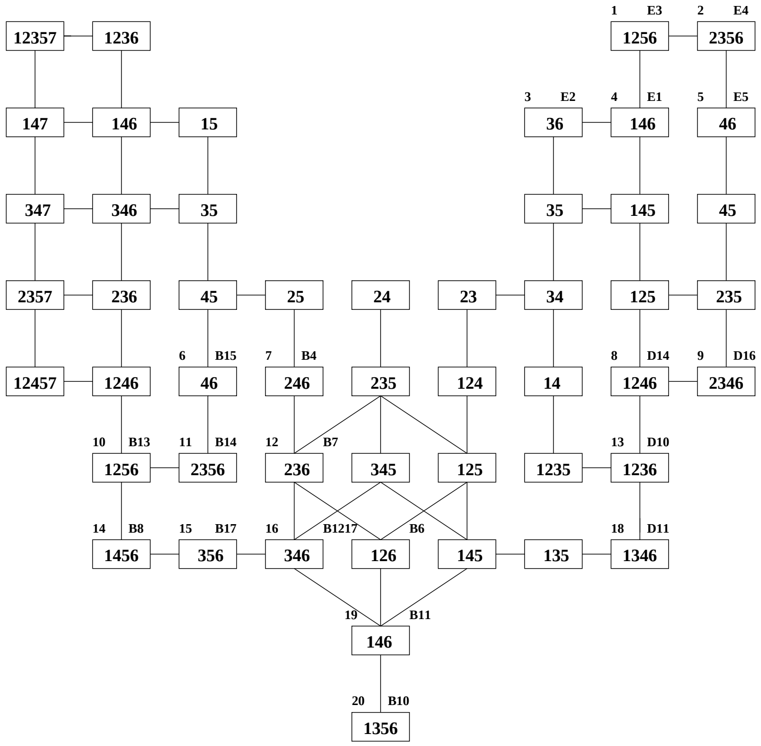

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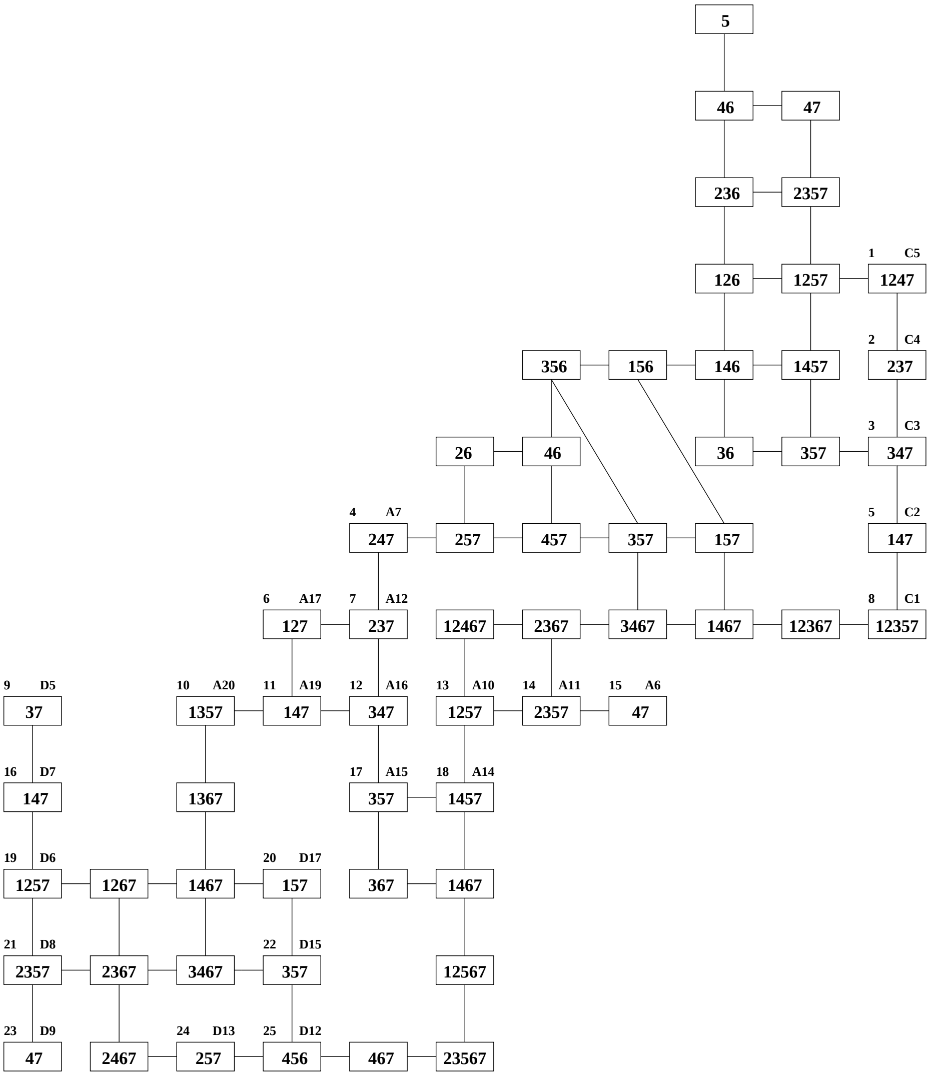

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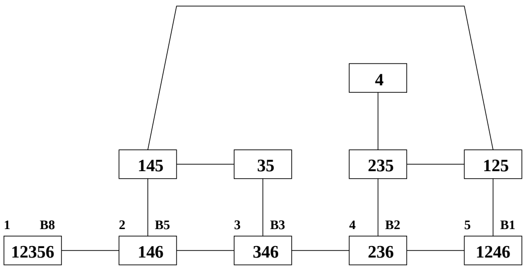

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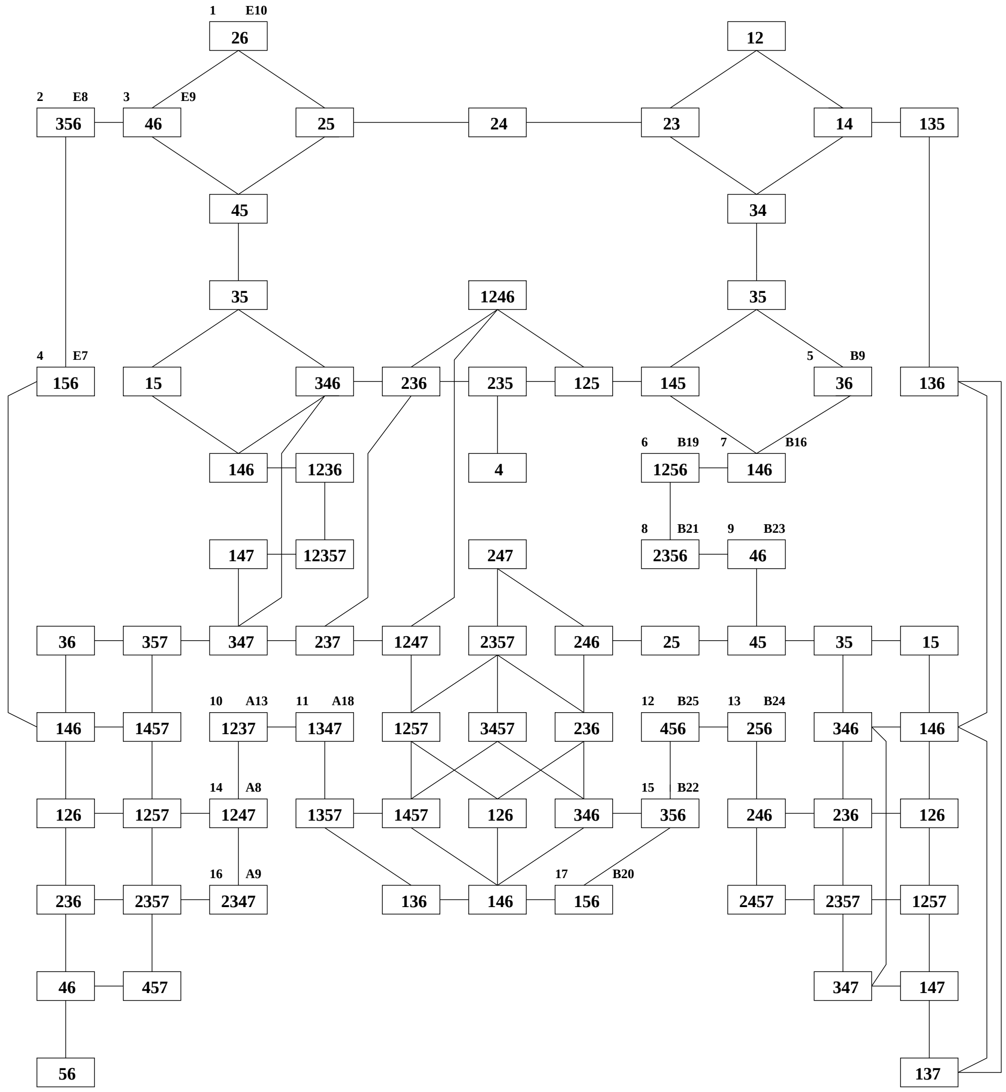

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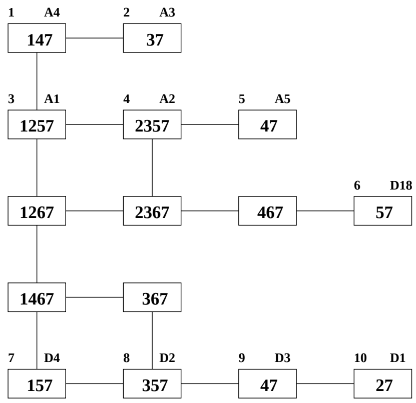

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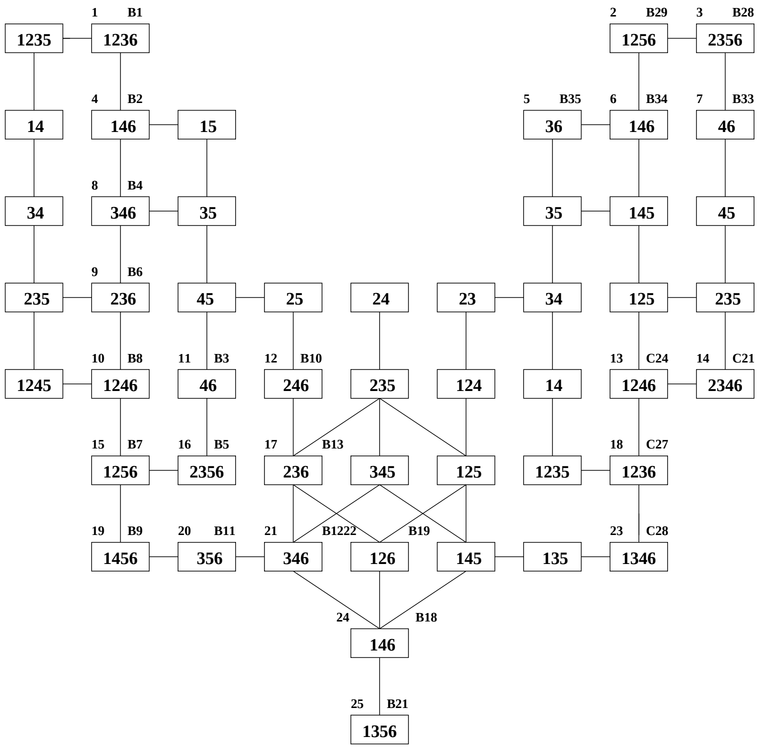

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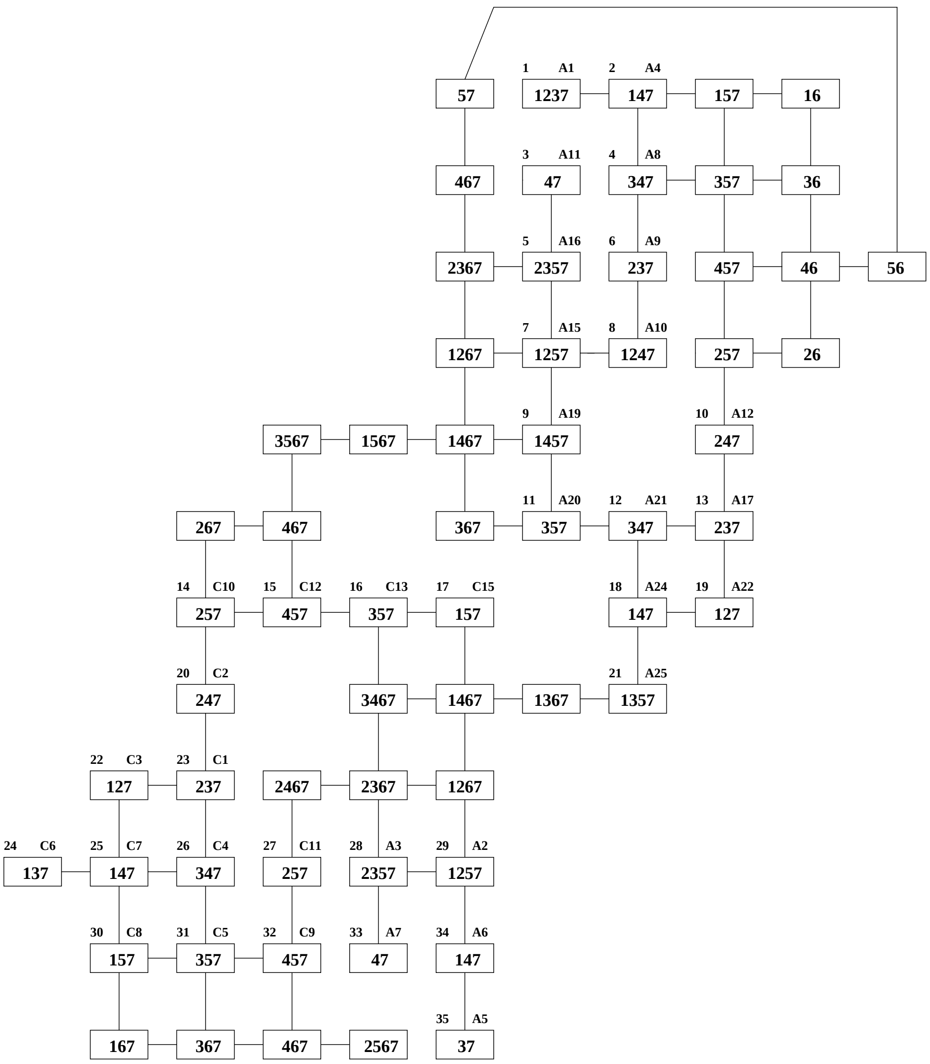

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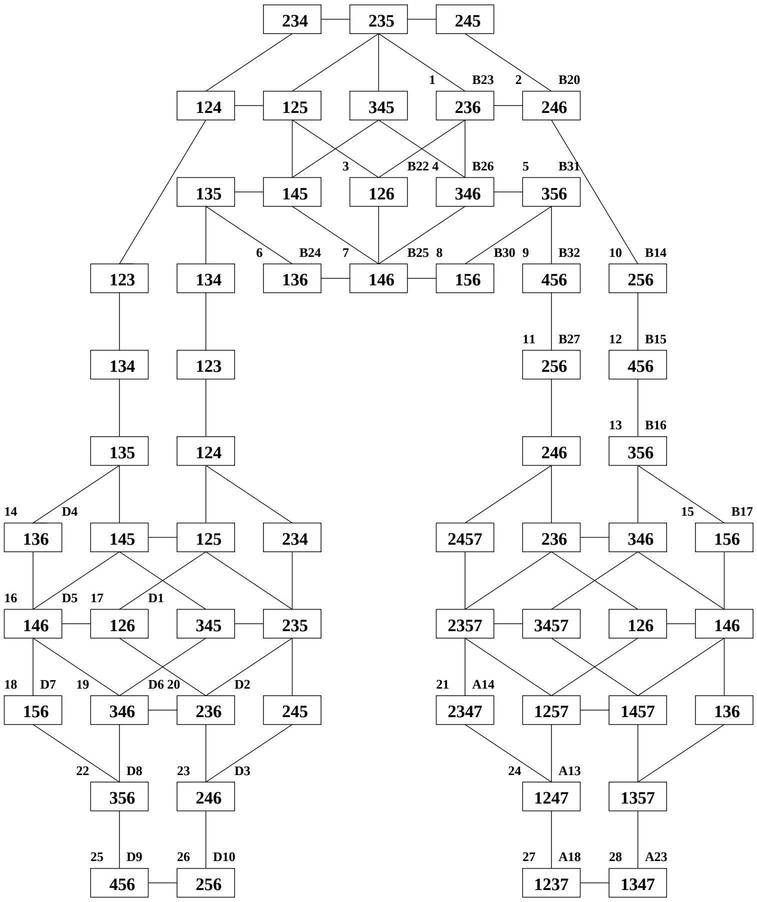

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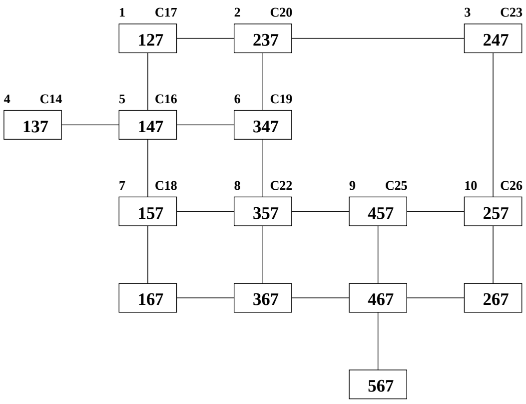

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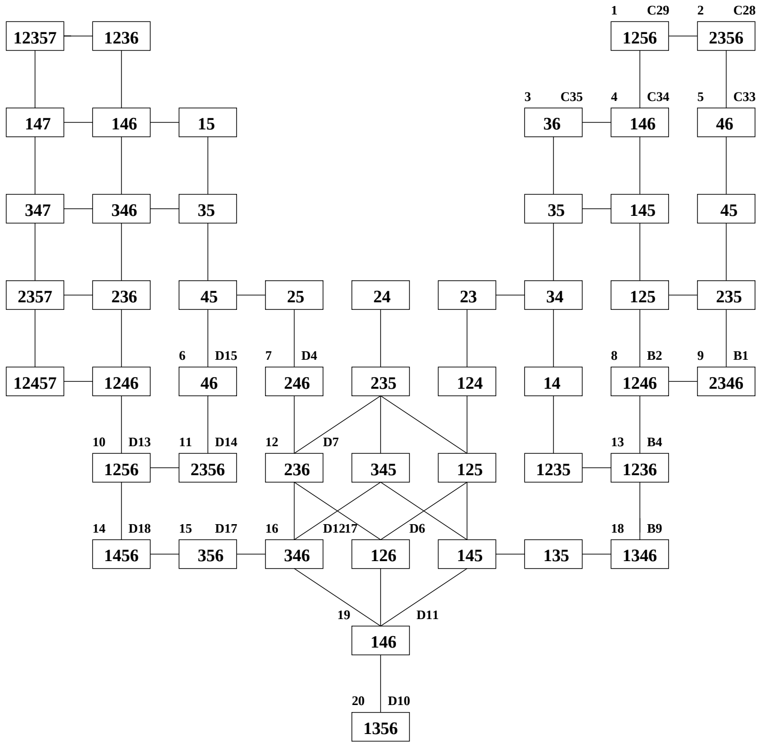

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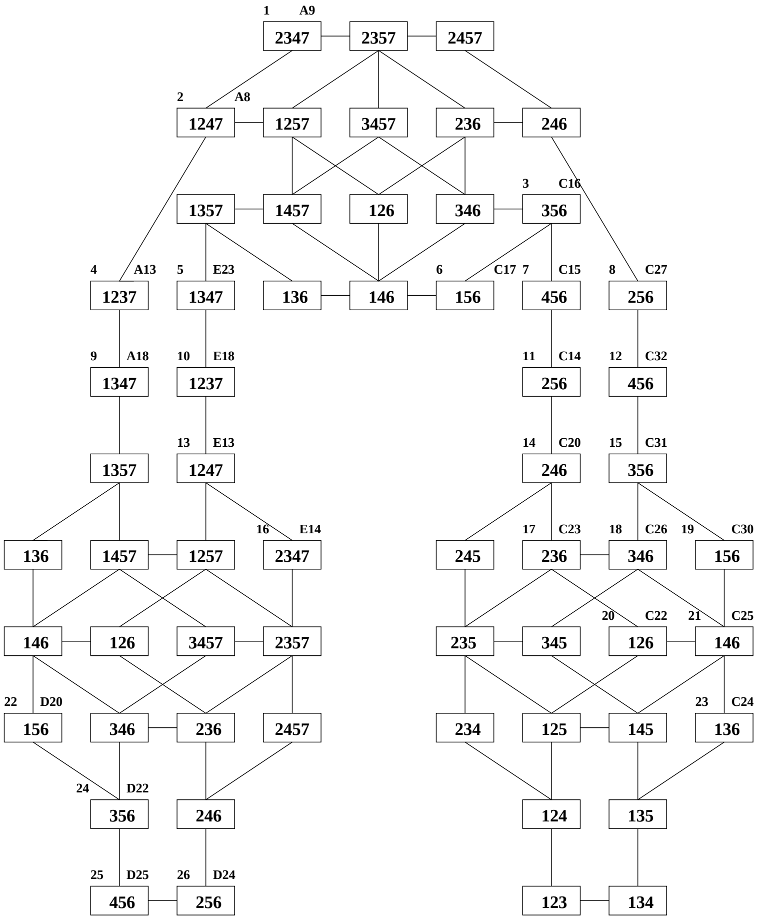

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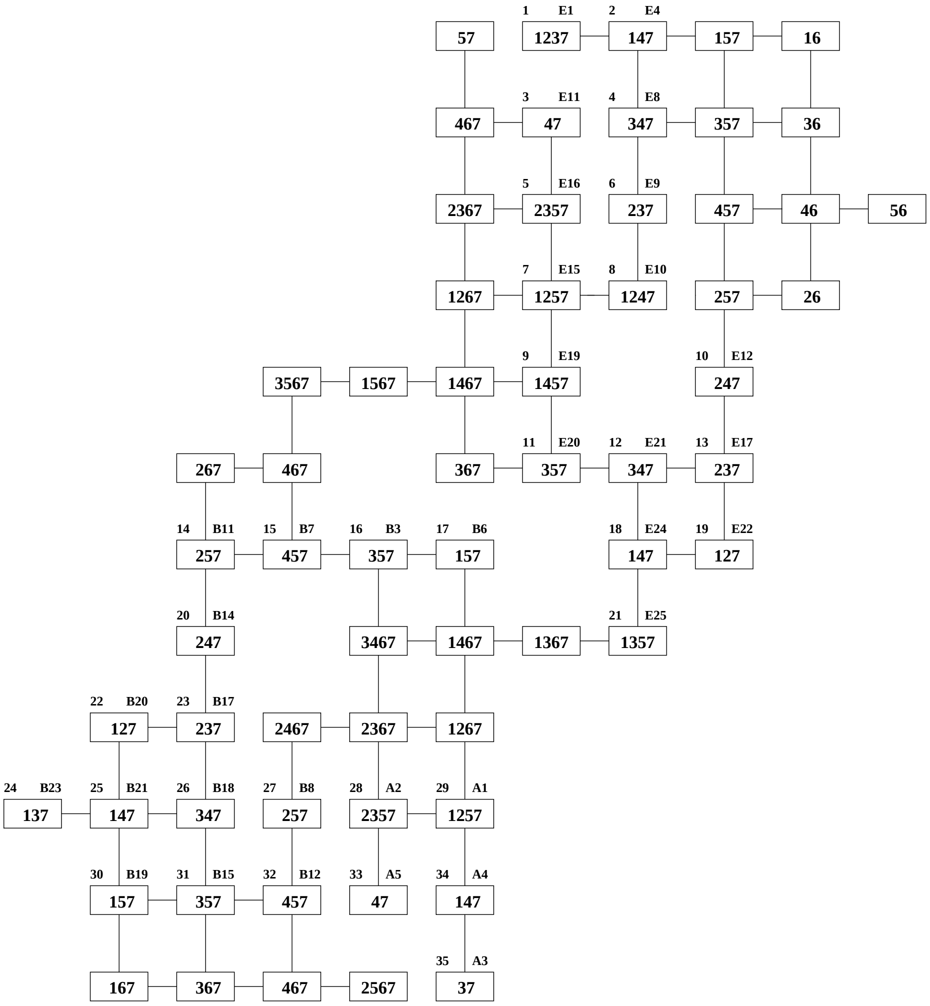

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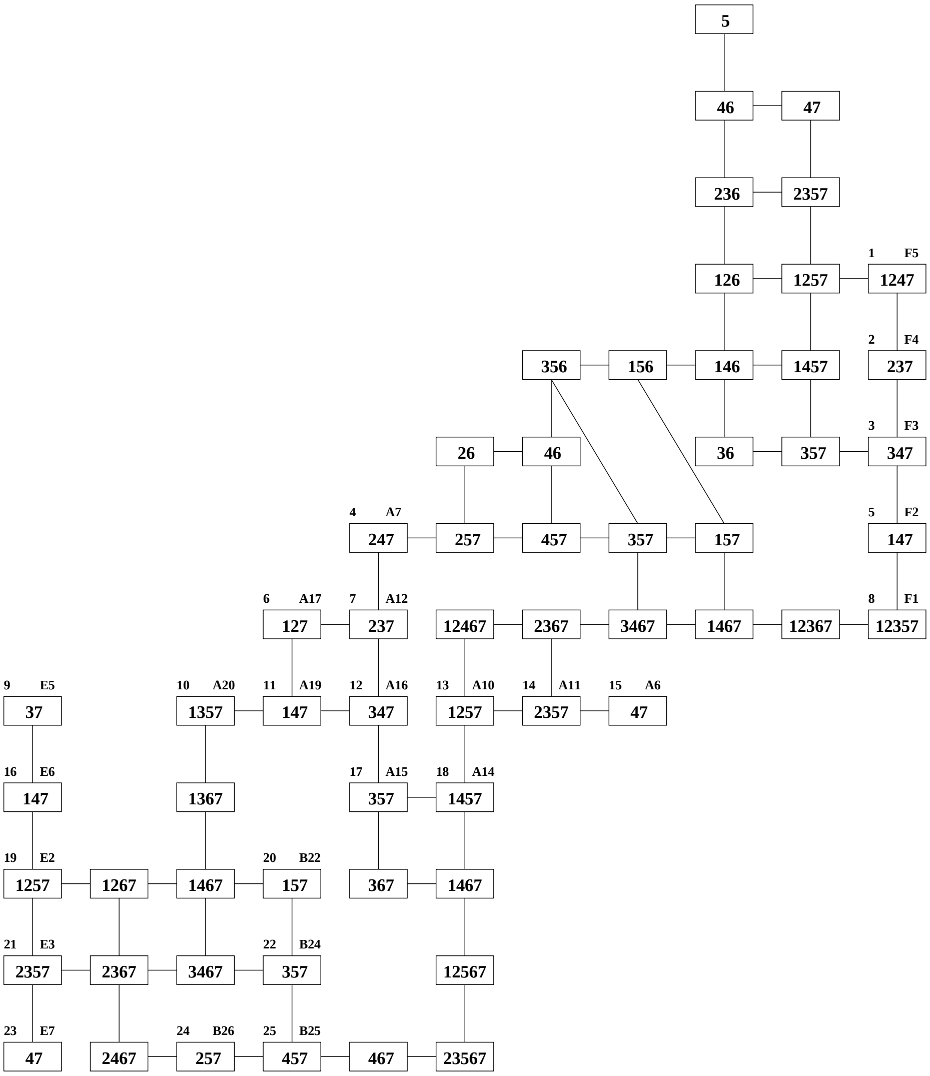

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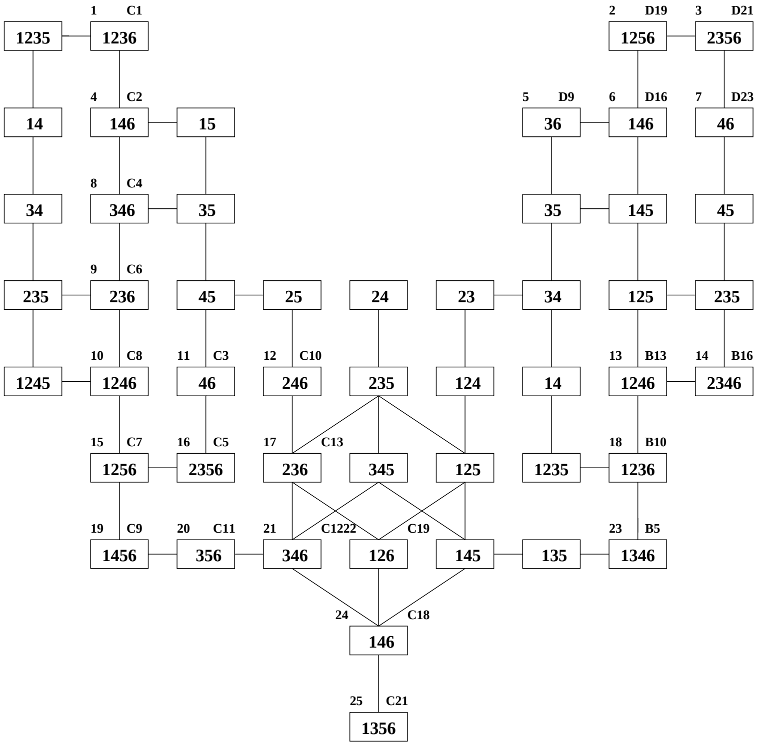

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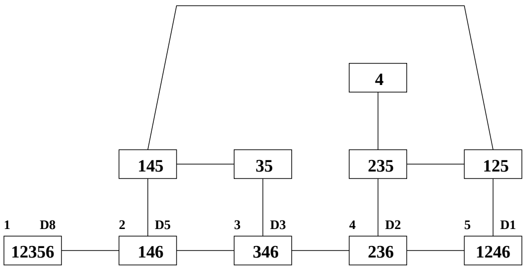

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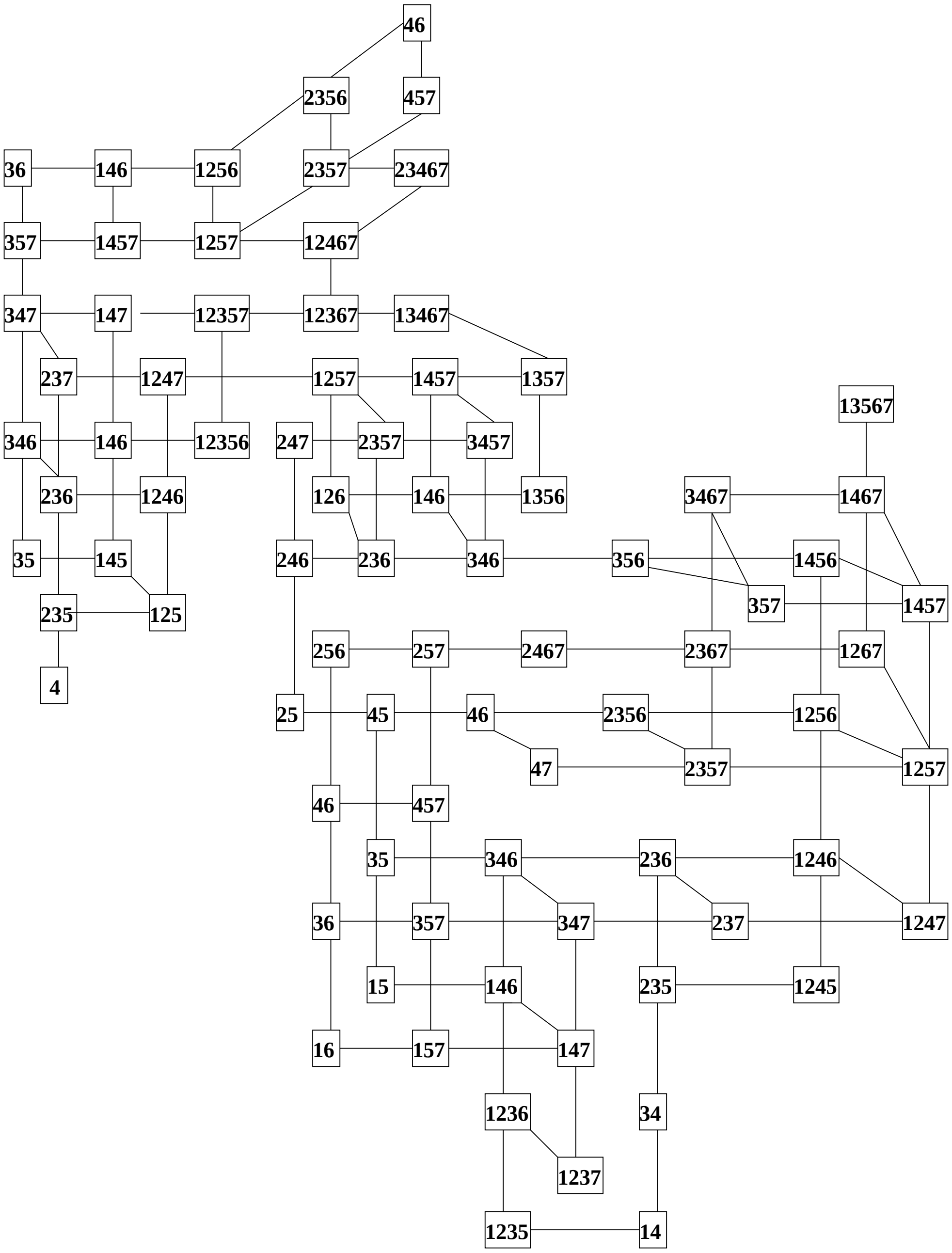

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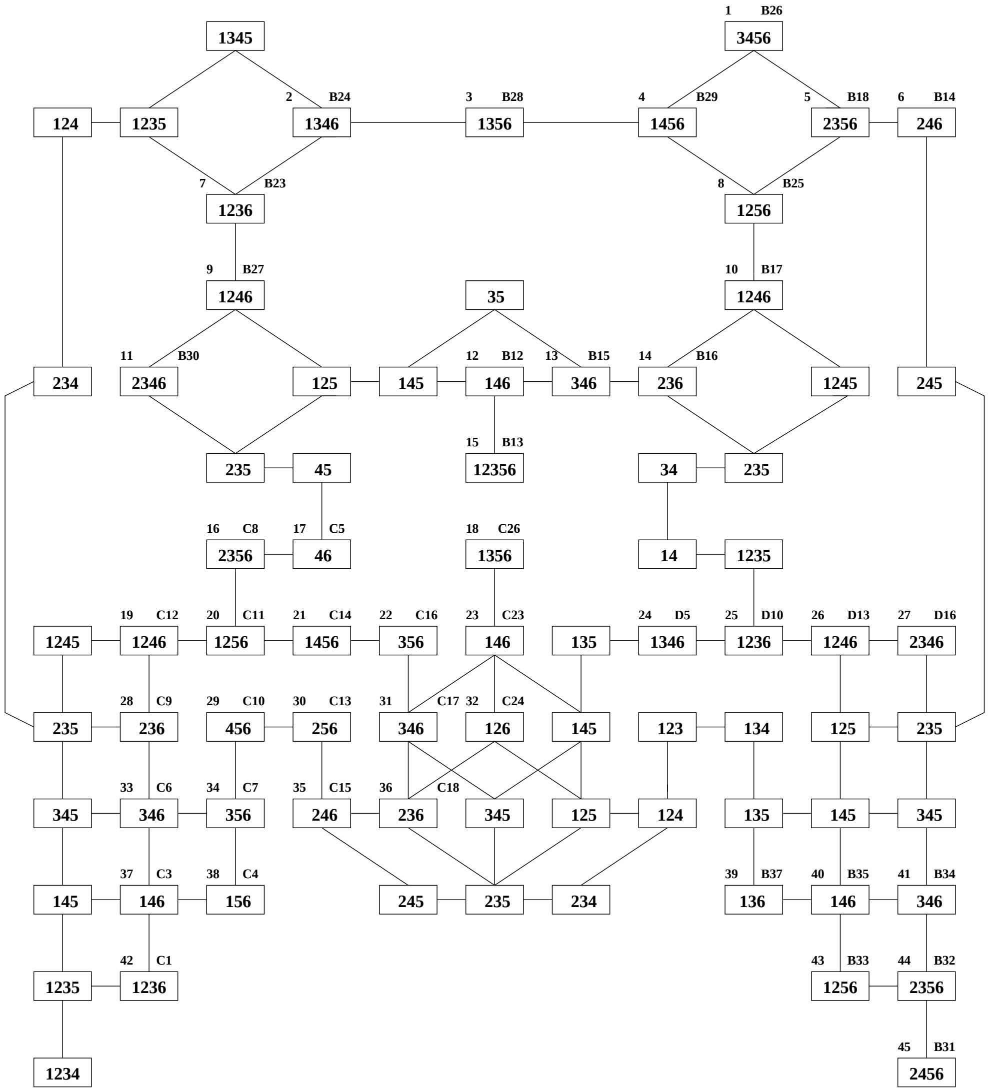

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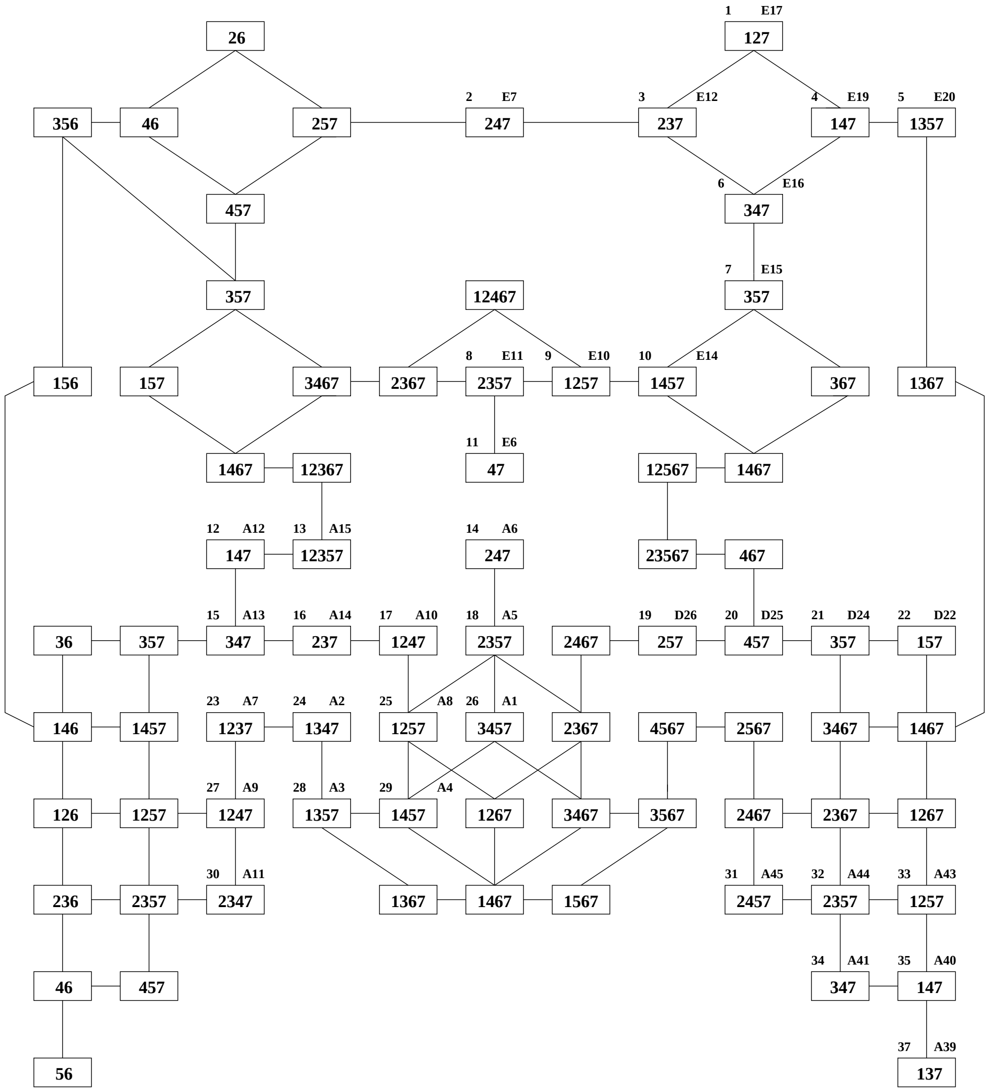

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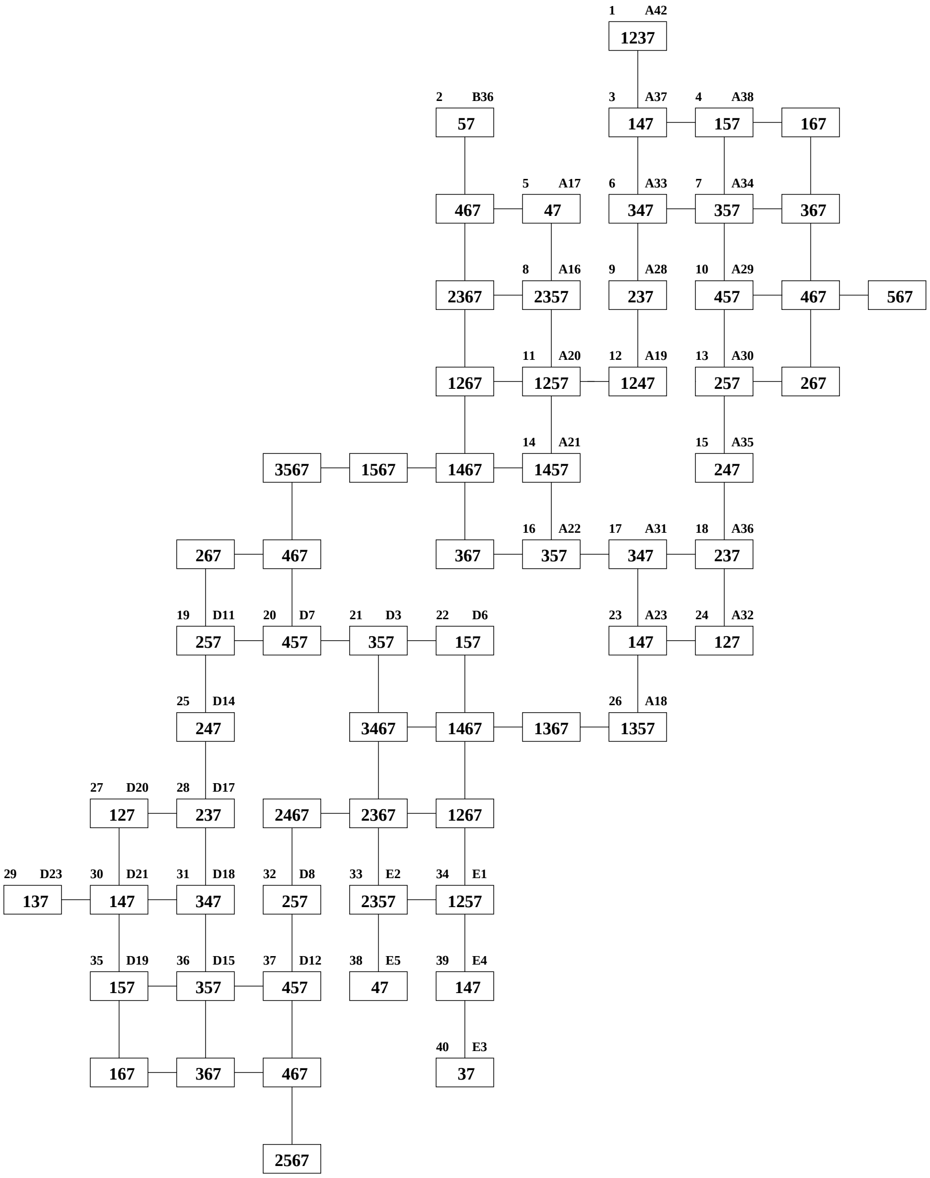

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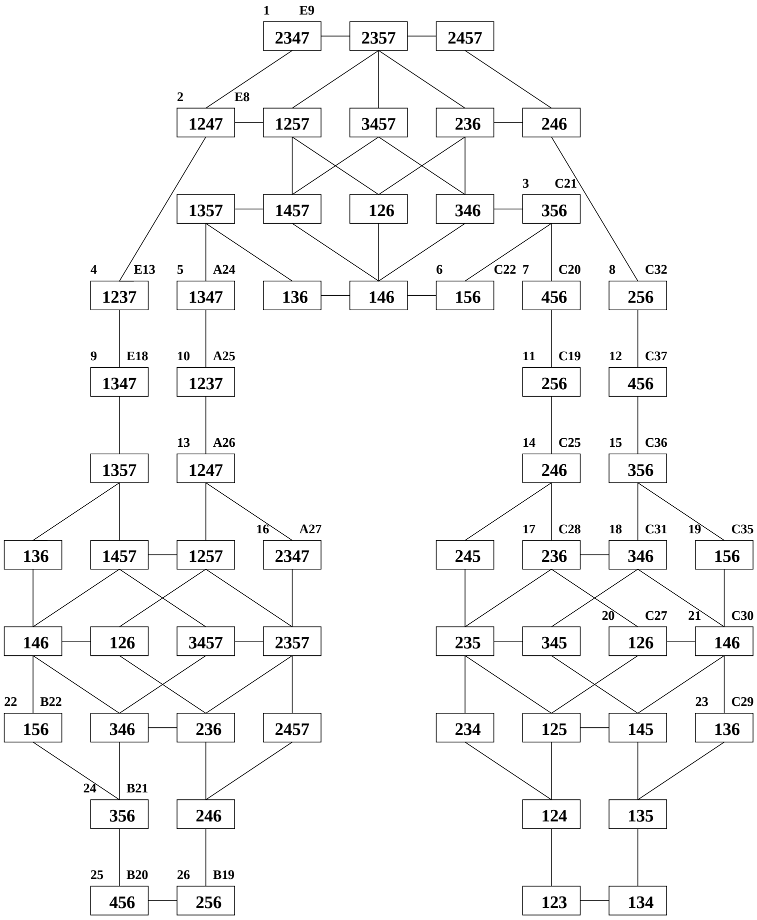
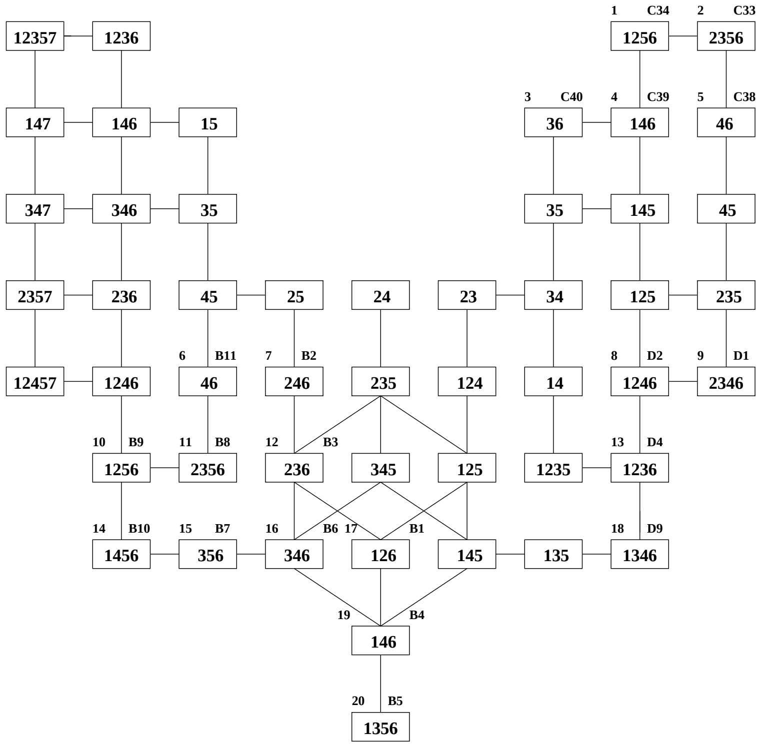


$\mathfrak{M}_{22,D}$



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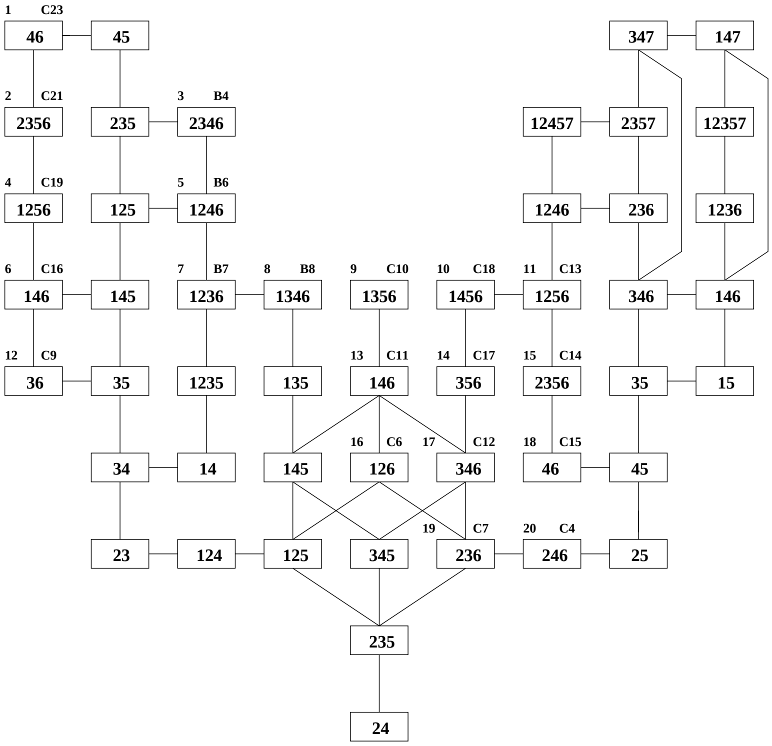

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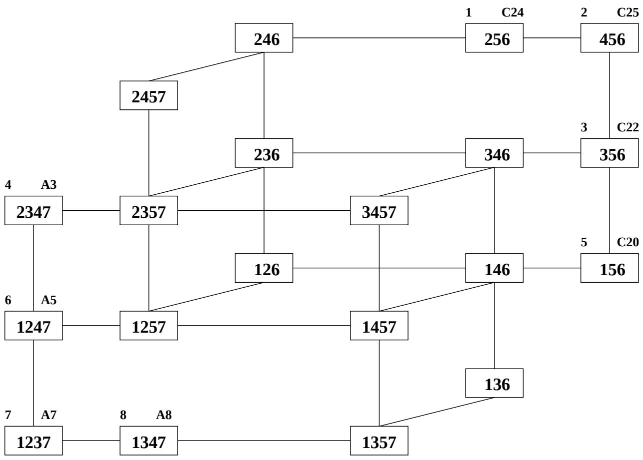

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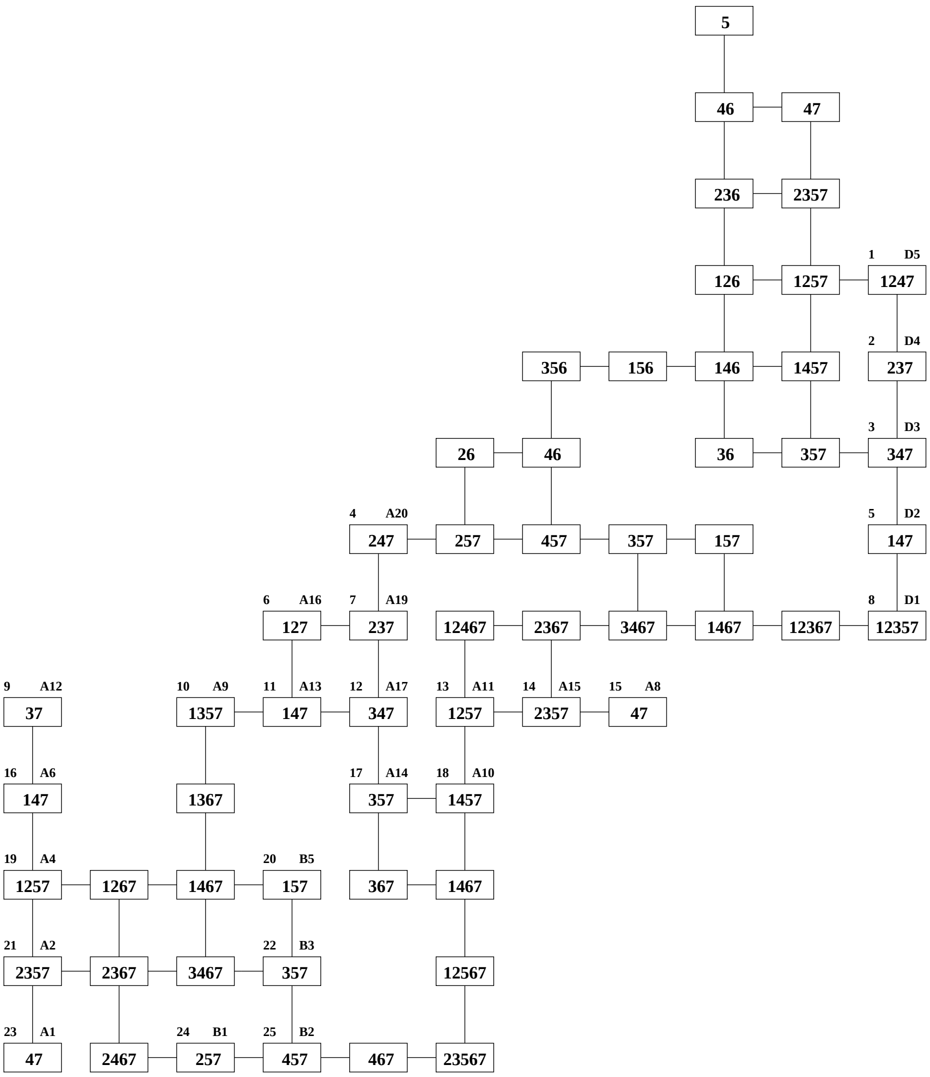

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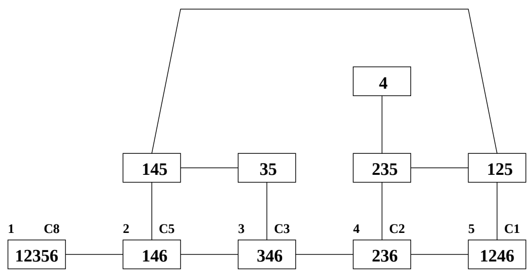

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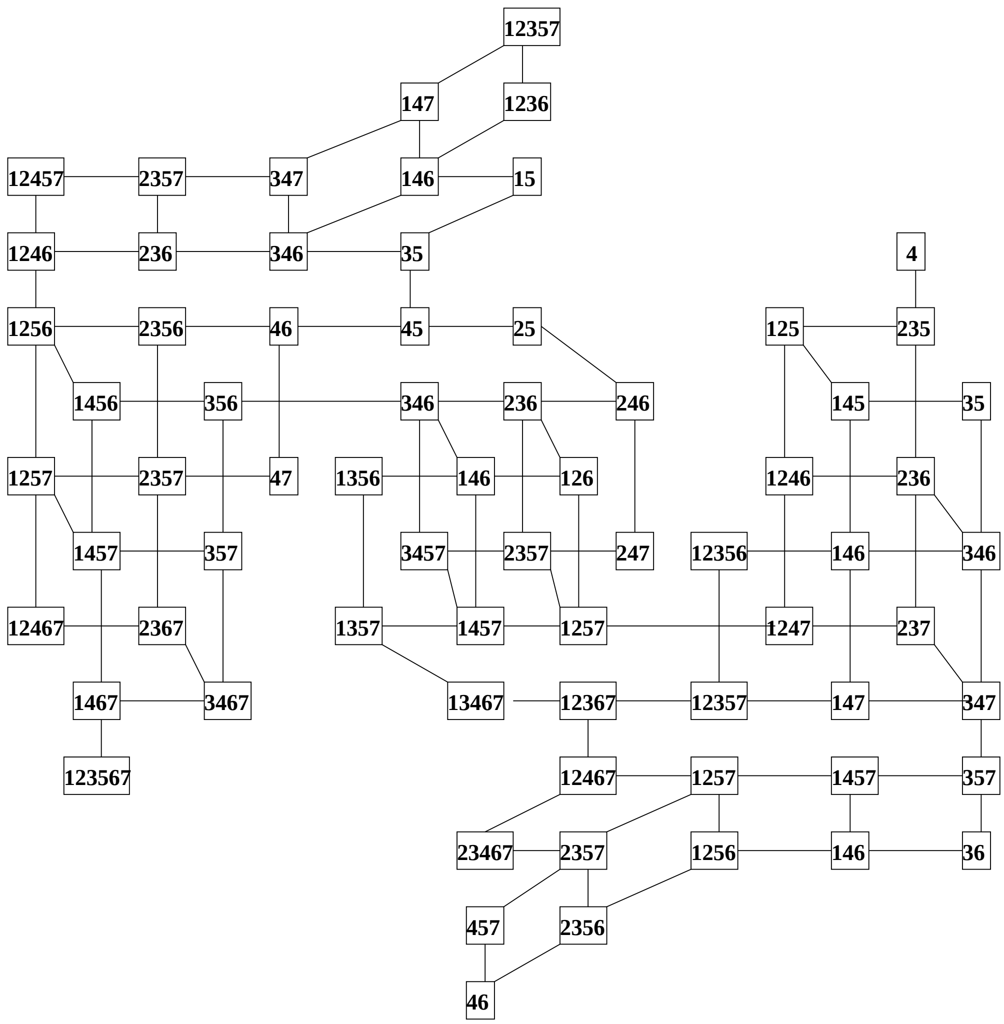
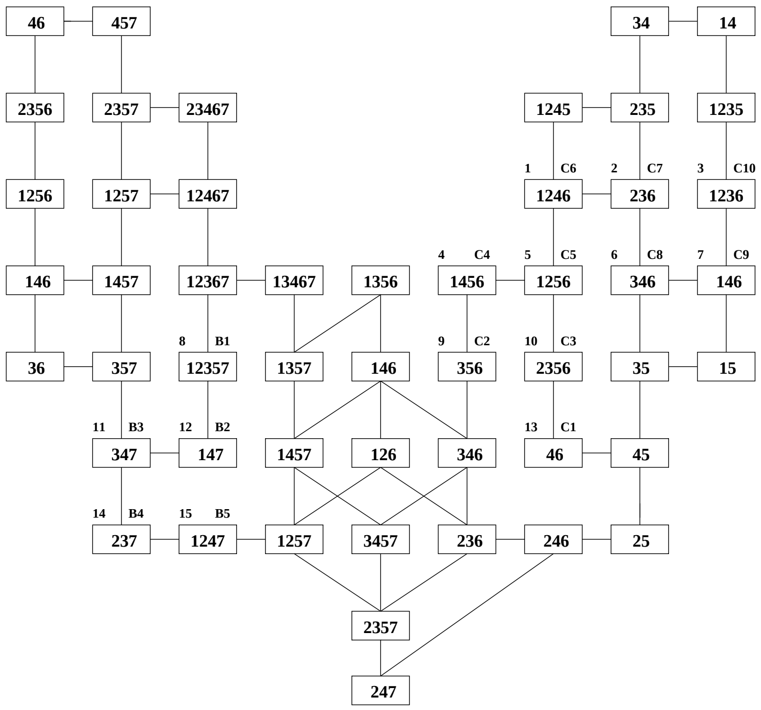
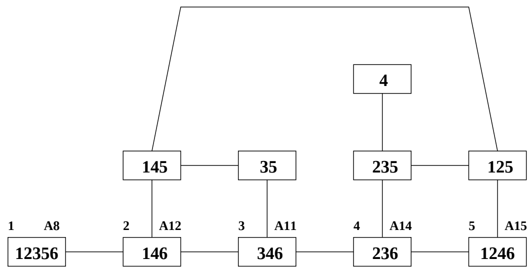


$\mathfrak{M}_{24}$



$\mathfrak{M}_{25,A}$



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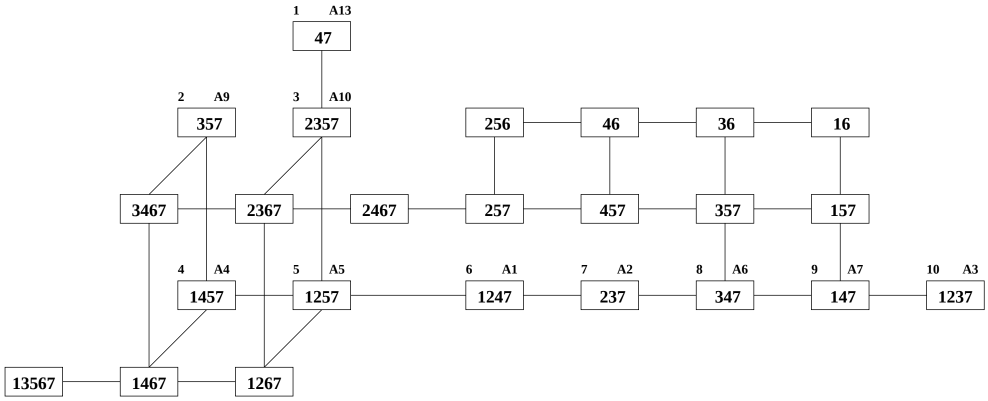
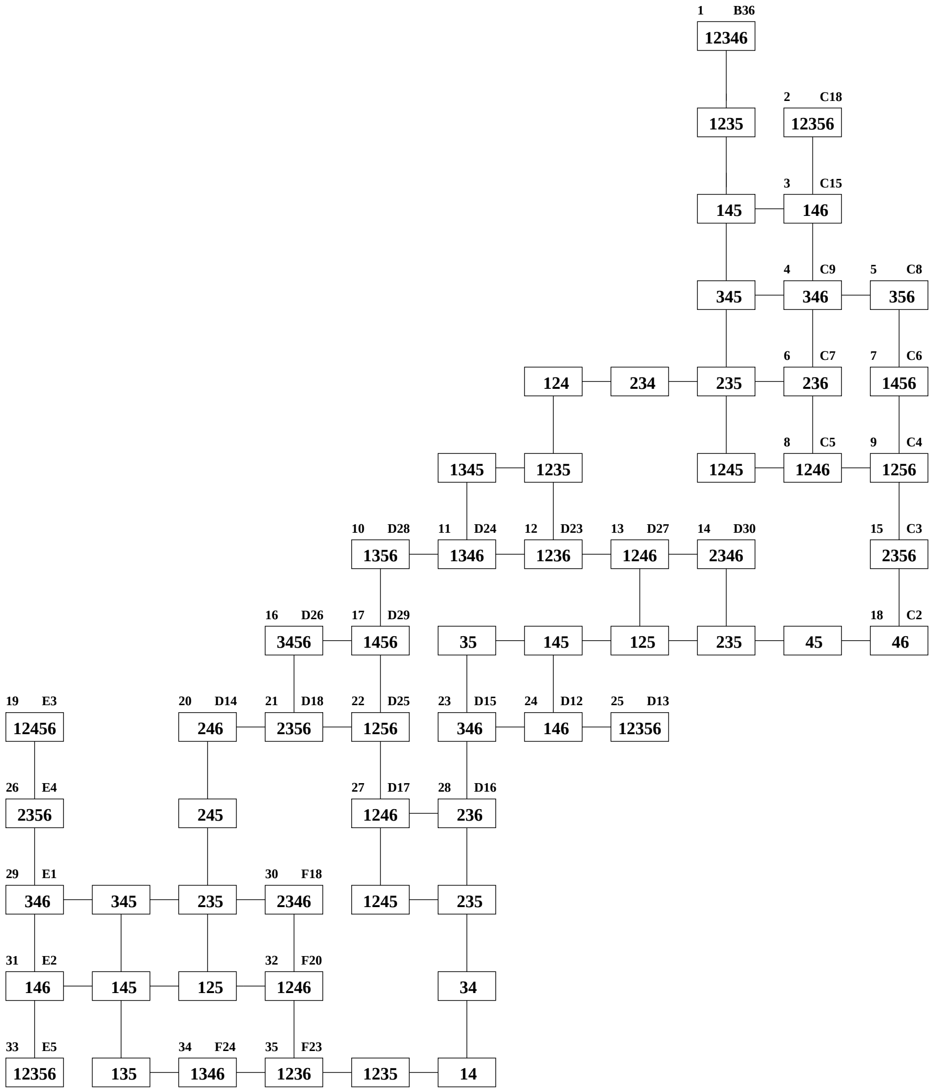
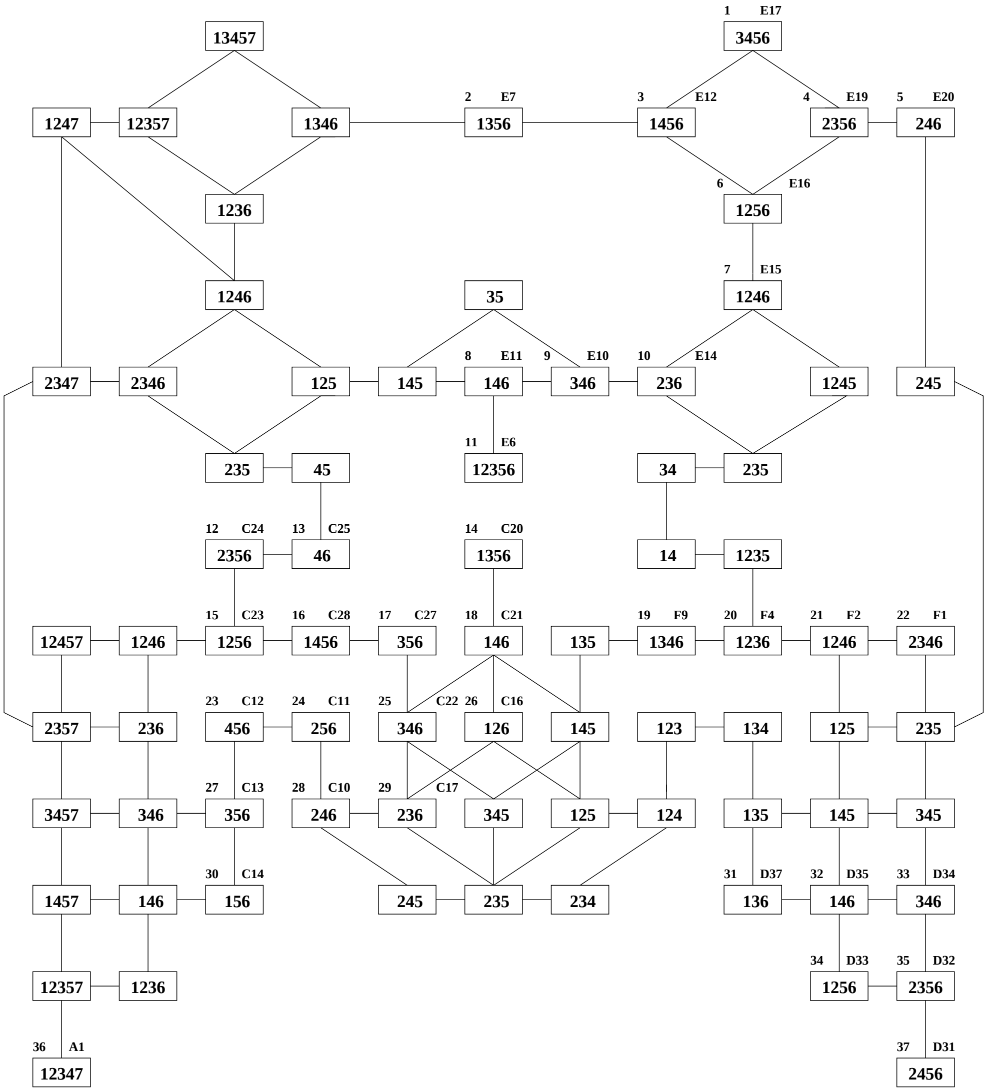


$\mathfrak{M}_{25,C}$



$\mathfrak{M}_{26,A}$



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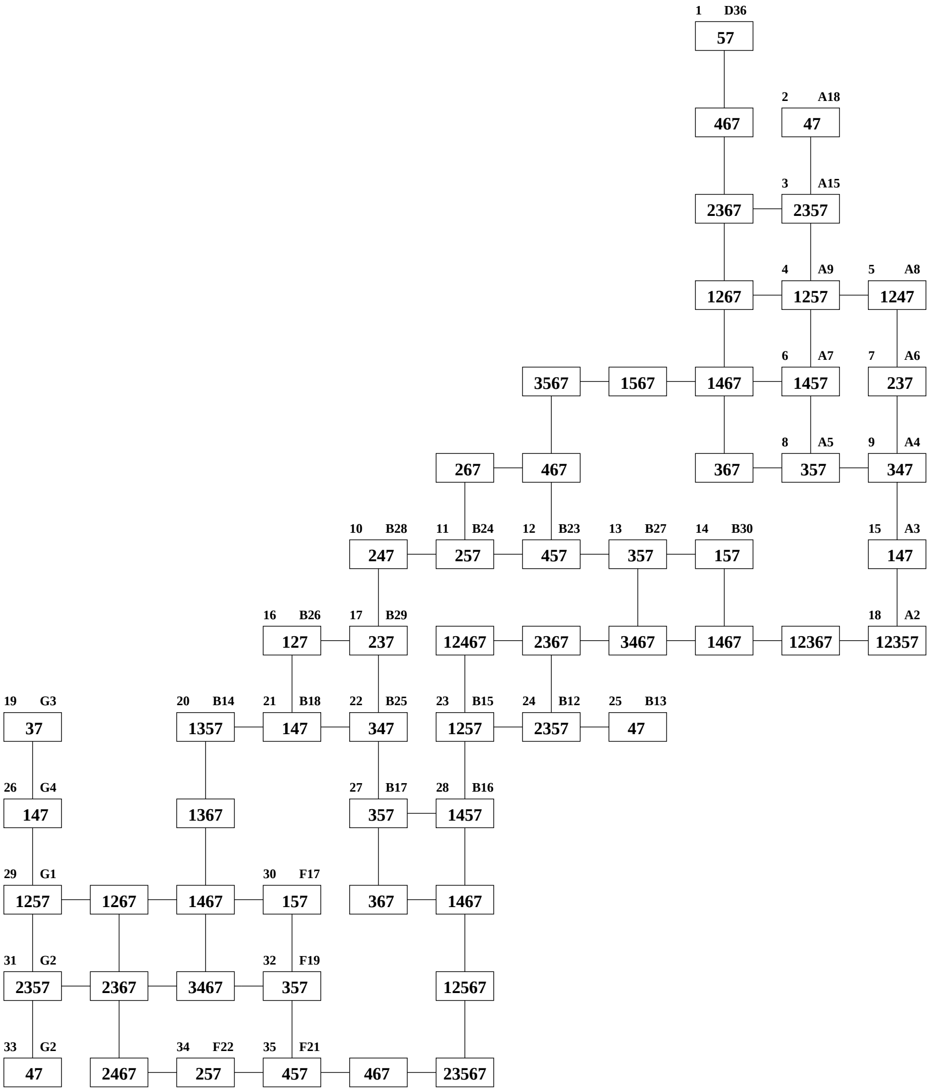

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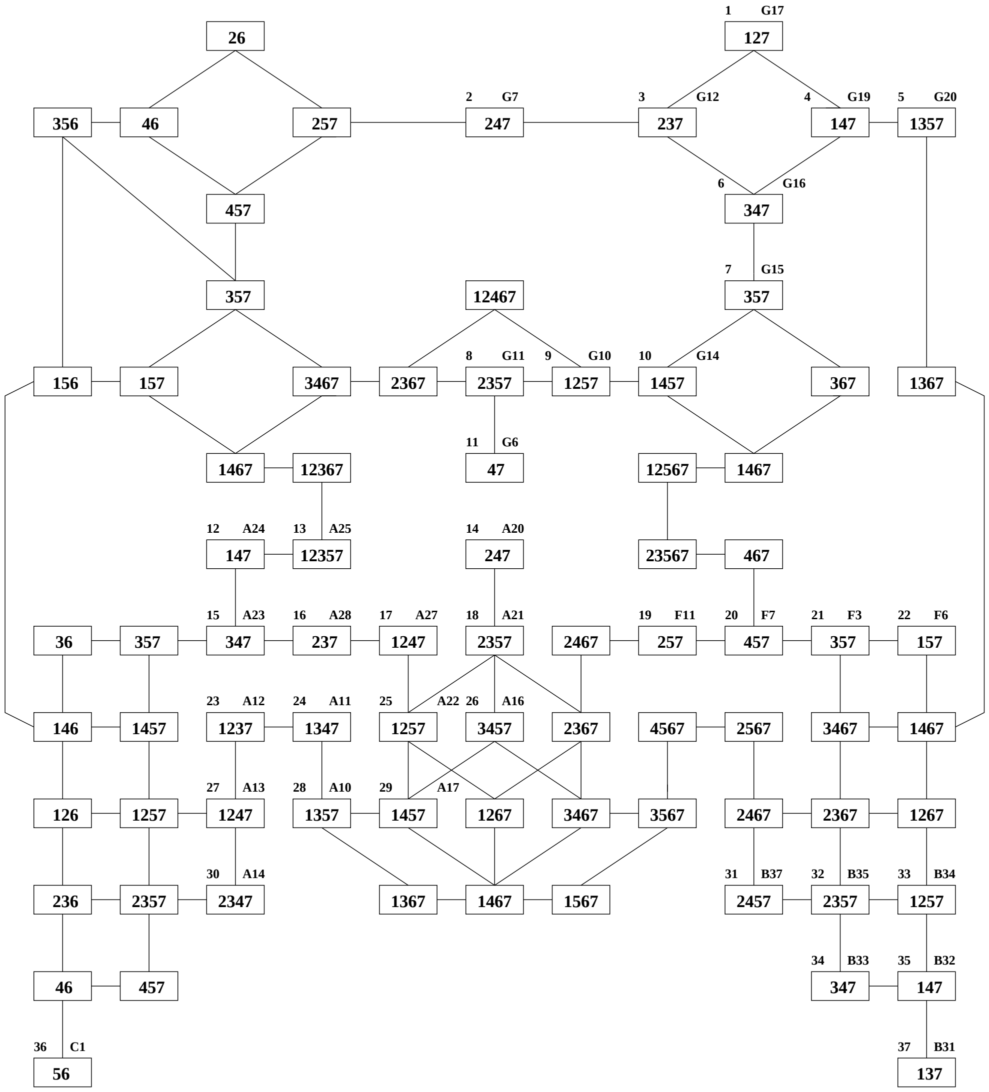
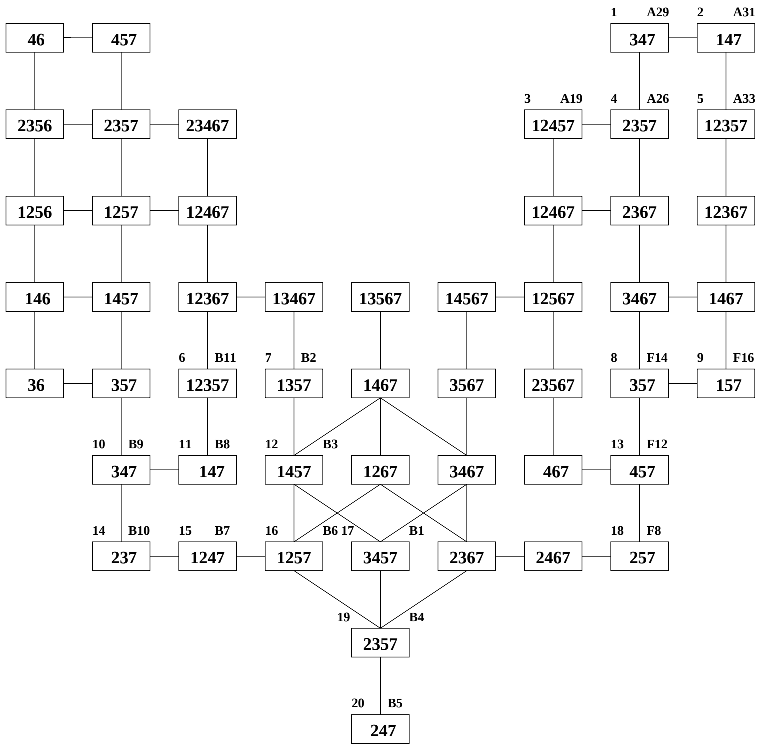


$\mathfrak{M}_{26,D}$



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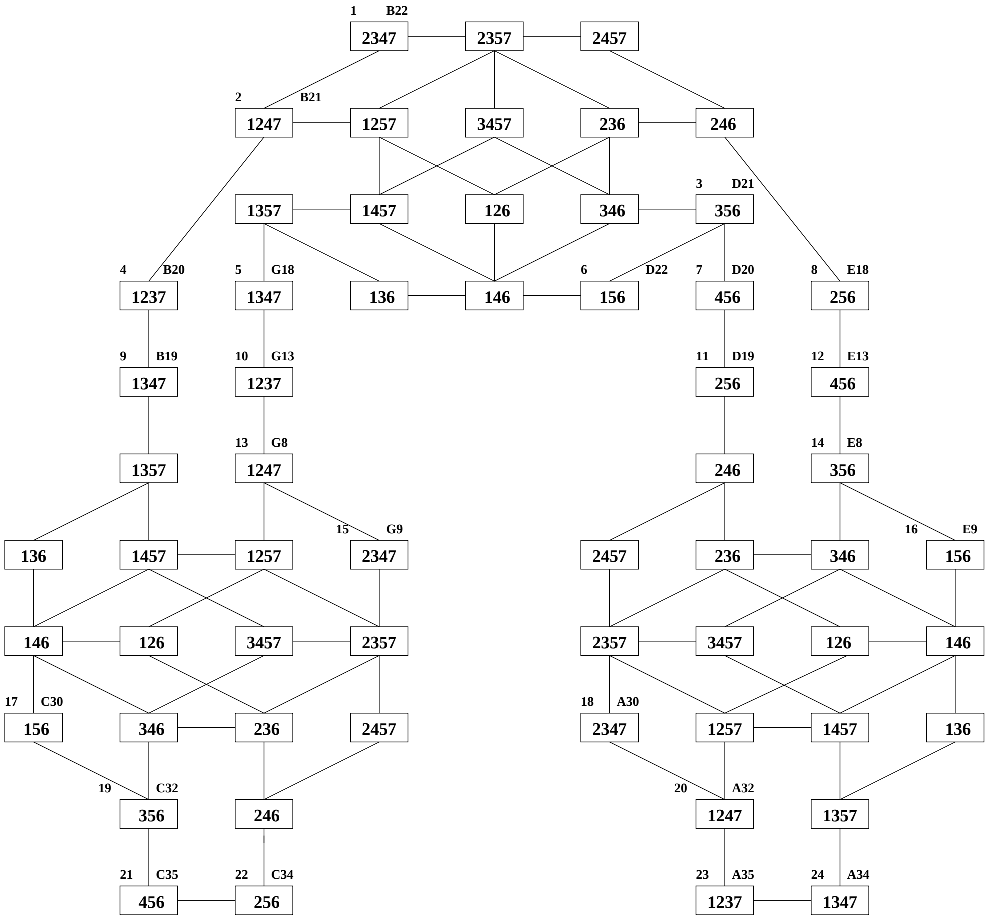
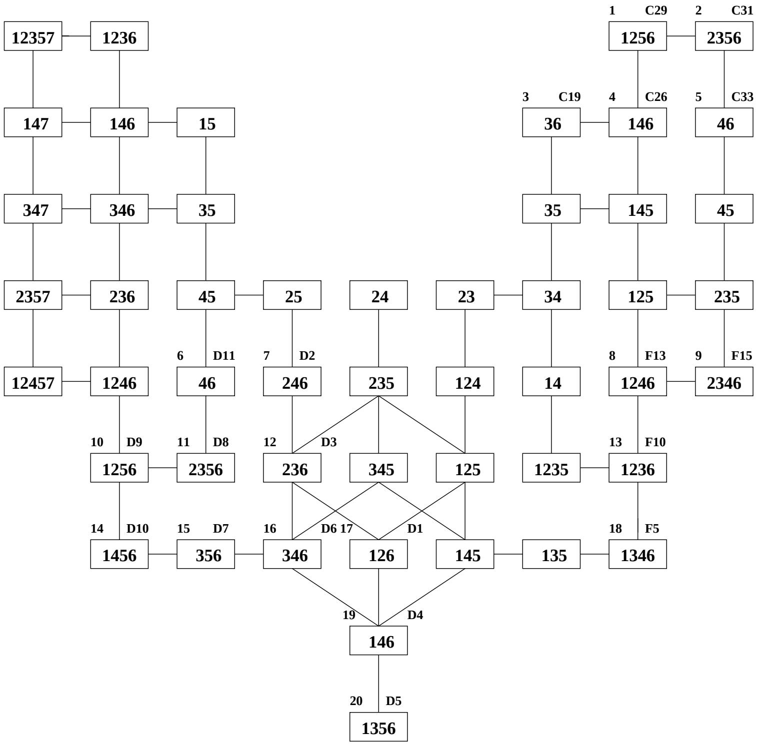
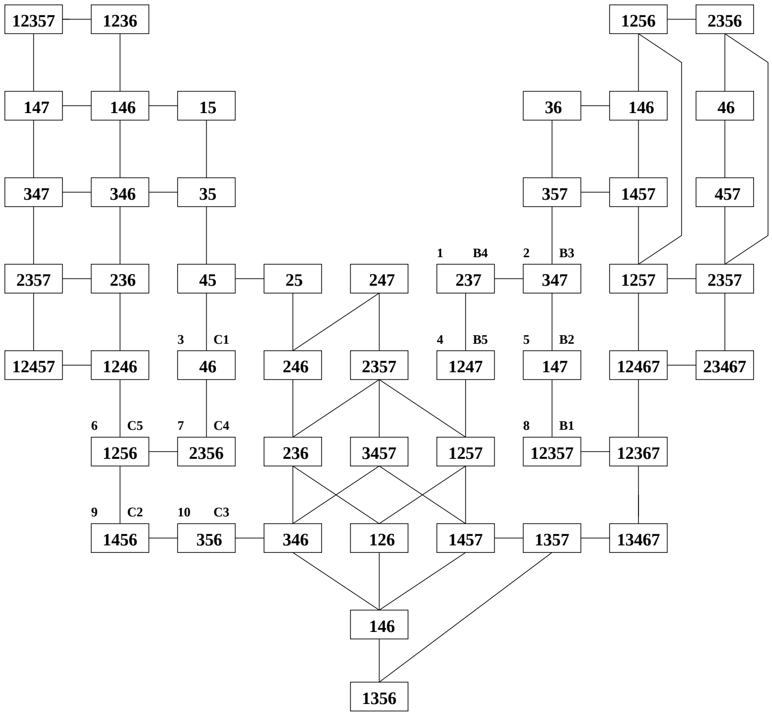


$\mathfrak{M}_{26,F}$



$\mathfrak{M}_{26,G}$



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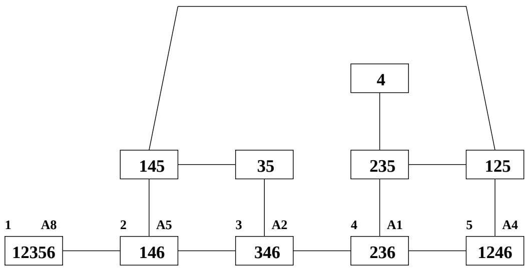
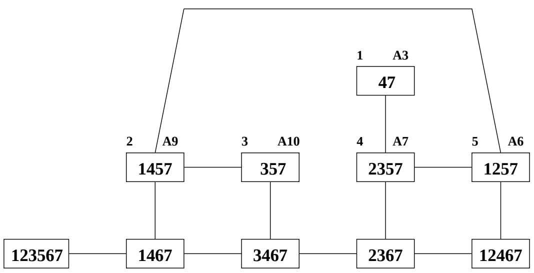
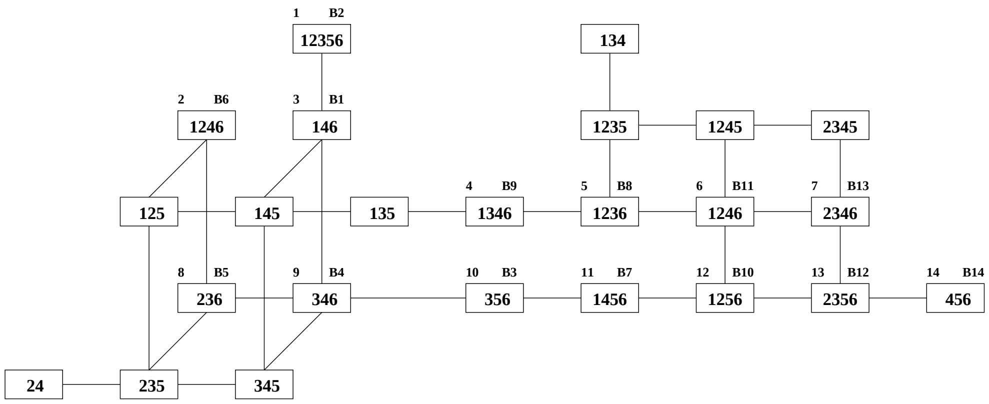


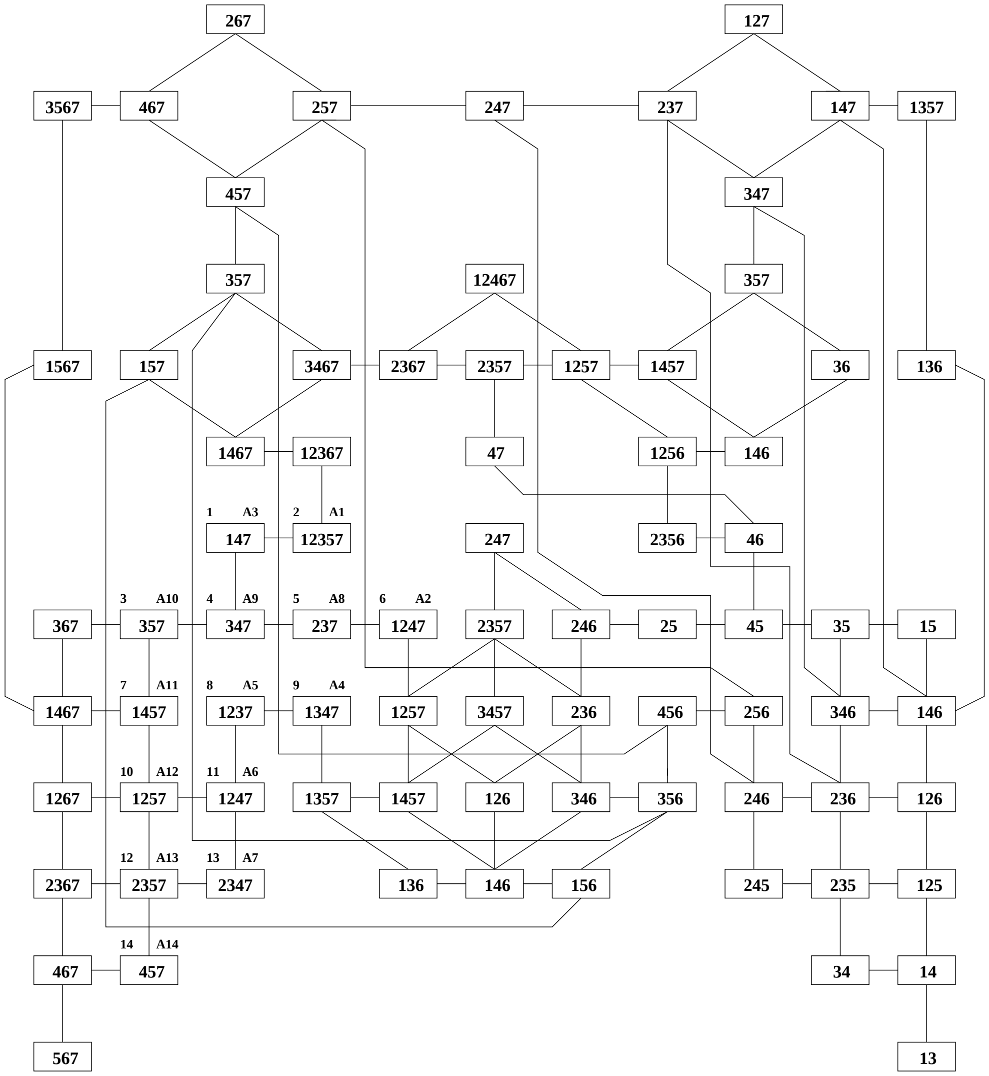
$\mathfrak{M}_{27,B}$



$\mathfrak{M}_{27,C}$



$\mathfrak{M}_{28,A}$



$\mathfrak{M}_{28,B}$