The set of dominance-minimal roots

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Abstract

If α and β are positive roots in the root system of a Coxeter group W, we say that α dominates β if $w \cdot \beta$ is negative whenever $w \cdot \alpha$ is negative for $w \in W$. We say that α is elementary or dominance-minimal, if it does not dominate any $\beta \neq \alpha$. It is shown in [BH] that the set $\mathcal{E}$ of dominance-minimal roots is finite if W has finite rank, and this is used to show that W is automatic. To limit the size of the relevant automata, and possibly facilitate other Coxeter group algorithms, we give an explicit description of the set of elementary roots.

§1 Introduction

The proof of Theorem (2.8) of [BH] yields that $|\mathcal{E}|$ is bounded by

$$\sum_{d=1}^{c^{|R|}(|R|+1)+1} |R|^d = \frac{|R|}{|R|-1}\big(|R|^{c^{|R|}(|R|+1)+1} - 1\big),$$

where c equals the cardinality of the set

$$\big\{ \cos(n\pi/m_{rs}) \mid r,\, s \in R,\, m_{rs} < \infty \text{ and } n \in \{1, \ldots, m_{rs}-1\} \big\}.$$

This bound, however, is rather large. For example, if

$$W = \langle\, r, s \mid r^2 = s^2 = (rs)^3 = 1 \,\rangle,$$

then $|\mathcal{E}| = 3$, but $|R| = 2$ and $c = 3$, and thus

$$\frac{|R|}{|R|-1}\big(|R|^{c^{|R|}(|R|+1)+1} - 1\big) = 2\big(2^{28} - 1\big).$$

Moreover, the algorithm given in [BH] to calculate the set of elementary roots is not very practical. In [C] (p.104, ll.41-43) Casselman asks for a more realistic algorithm and a better bound for $|\mathcal{E}|$. We will explicitly determine $\mathcal{E}$, thus making it possible to calculate $|\mathcal{E}|$ precisely.

§2 Summary of background material

Let (W, R) be a Coxeter system, as defined in [B] or [H], and for $r, s \in R$ let m_{rs} be the order of rs. Let $\Pi = \{a_r \mid r \in R\}$ be the set of *simple roots*, and for $a \in \Pi$ denote the element of R corresponding to a by r_a. Further, let V be the $\mathbb{R}$-vector space with basis Π, and let $\langle\, ,\, \rangle\colon V \times V \to \mathbb{R}$ be a symmetrical bilinear form which satisfies $\langle a_r, a_s \rangle = -\cos(\pi/m_{rs})$ for all r, $s \in R$ such that m_{rs} is finite, and $\langle a_r, a_s \rangle \leq -1$ if m_{rs} is infinite. Then $r \cdot v = v - 2\langle a_r, v\rangle a_r$ (for all $r \in R$ and $v \in V$) determines a faithful action of W on V which preserves $\langle\, ,\, \rangle$. The set $\Phi = \{w \cdot a_r \mid r \in R,\ w \in W\}$ is the *root system* of W in V. For $S \subseteq \Pi$, we denote the Coxeter group generated by r_a with $a \in S$ by W_S, and the root system of W_S on the subspace of V spanned by S by Φ_S.

The *Coxeter graph* of W has vertex set Π, and two vertices a_r and a_s are adjoined by an edge or *bond* of weight m_{rs} if $m_{rs} \geq 3$ or $m_{rs} = \infty$; if $m_{rs} = 3$, the bond adjoining a_r and a_s is said to be *simple*, and if $m_{rs} \geq 4$ or $m_{rs} = \infty$, the bond adjoining a_r and a_s is labelled by m_{rs}. We frequently identify $S \subseteq \Pi$ with the subgraph of the Coxeter graph consisting of the vertices in S and the edges adjoining them.

Each root α can be written uniquely as $\sum_{a \in \Pi} \lambda_a a$; we say that λ_a is the *coefficient* of a in α, and define the *support* $\mathrm{supp}(\alpha)$ of α to be the set of $a \in \Pi$ such that $\lambda_a \neq 0$. It can be easily seen that $\mathrm{supp}(\alpha)$ is a connected subgraph of the Coxeter graph. Further, α is said to be *positive* (*negative*), if all $\lambda_a \geq 0$ (all $\lambda_a \leq 0$), and the sets of positive and negative roots are denoted by Φ^+ and Φ^- respectively. It is a well known fact that $w \cdot a_r$ is positive for $w \in W$ and $r \in R$ if $l(wr) \geq l(w)$ (where $l(w)$ is the minimal integer n such that w equals $r_1 \cdots r_n$ for some $r_i \in R$). This yields that each root is either positive or negative. An extension of the proof of this fact yields the next proposition.

(2.1) PROPOSITION *Let λ be the coefficient of a simple root in $\alpha \in \Phi$.*

(i) If $\lambda > 0$, then $\lambda \geq 1$.

(ii) If $\langle a, b \rangle \geq -1$ for all a, $b \in \Pi$, then λ is a polynomial in

$$C = \left\{ \frac{\sin(l\pi/m)}{\sin(\pi/m)} \mid 4 \leq m = m_{rs} < \infty \text{ for } a_r,\, a_s \in \Pi \text{ and } l \in \mathbb{N} \text{ with } l \leq \tfrac{m}{2} \right\}$$

with coefficients in $\mathbb{N}_0$ (the non-negative integers), or coefficients in $-\mathbb{N}_0$. In particular, if $\lambda \in (0, 2)$, then $\lambda = 1$, or $\lambda = 2\cos(\pi/m_{st})$ for some s, $t \in R$ with $m_{st} \geq 4$.

Proof. Suppose that λ is the coefficient of x in α, and let $w \in W$ and $r \in R$ with $\alpha = w \cdot a_r$. If $l(w) = 0$, the assertion is trivially true; so suppose that $l(w) > 0$, and let $s \in R$ and $w' \in W$ such that $w = w's$ and $l(w') = l(w) - 1$. Choose $w'' \in wW_{\{a_r, a_s\}}$ of minimal length; then $l(w'') < l(w)$ as $l(ws) < l(w)$, and $l(w''r)$, $l(w''s) \geq l(w'')$ by minimality of w''. Hence the inductive hypothesis applies to $w'' \cdot a_r$ and $w'' \cdot a_s$, and the roots $w'' \cdot a_r$ and $w'' \cdot a_s$ are positive.

Next, let $u \in W_{\{a_r,a_s\}}$ with $w = w''u$. Then $u \cdot a_r = \mu a_r + \nu a_s$ for some μ, $\nu \in \mathbb{R}$; moreover, a straightforward calculation of the rank 2 case yields that μ, $\nu \leq 0$ or $\mu, \nu \geq 0$. Let μ_x, $\nu_x \geq 0$ be the coefficients of x in $w'' \cdot a_r$ and $w'' \cdot a_s$ respectively, so that $\lambda = \mu\mu_x + \nu\nu_x$.

If $\lambda > 0$, then μ, $\mu_x > 0$ (and thus $\nu \geq 0$), or ν, $\nu_x > 0$ (and thus $\mu \geq 0$). If $\mu > 0$, it can be easily verified that $\mu \geq 1$; further, if $\mu > 0$, then $\mu_x \geq 1$ by the inductive hypothesis. Hence $\lambda \geq \mu\mu_x \geq 1$ if μ, $\mu_x > 0$; symmetrical arguments apply if ν, $\nu_x > 0$, and this proves (i).

Suppose now that $\langle a, b \rangle \geq -1$ for all a, $b \in \Pi$. By the inductive hypothesis, μ_x and ν_x are polynomials in C with coefficients in $\mathbb{N}_0$. Further, either $\langle a_r, a_s \rangle = -1$, or m_{rs} is finite, and in both cases it can be easily verified that μ and ν are both polynomials in C with coefficients in $\mathbb{N}_0$, or both polynomials in C with coefficients in $-\mathbb{N}_0$, and (ii) follows. □

Following [S], we define for $\alpha \in \Phi^+$ the *depth* of α to be the minimal integer l such that $w \cdot \alpha$ is negative for some $w \in W$ of length l; it follows that α equals $u \cdot a_s$ for some $s \in R$ and $u \in W$ of length $\mathrm{dp}(\alpha) - 1$. In [BH], it is shown that if $r \in R$ and $\alpha \in \Phi^+ \setminus \{a_r\}$, then

$$\mathrm{dp}(r \cdot \alpha) = \begin{cases} \mathrm{dp}(\alpha) - 1 & \text{if } \langle \alpha, a_r \rangle > 0, \\ \mathrm{dp}(\alpha) & \text{if } \langle \alpha, a_r \rangle = 0, \\ \mathrm{dp}(\alpha) + 1 & \text{if } \langle \alpha, a_r \rangle < 0. \end{cases} \qquad (*)$$

For α, $\beta \in \Phi^+$ we say that β *precedes* α (we write $\beta \preceq \alpha$) if and only if there exists a $w \in W$ such that $\alpha = w \cdot \beta$ and $\mathrm{dp}(\alpha) - \mathrm{dp}(\beta) = l(w)$; we write $\beta \prec \alpha$ if $\beta \preceq \alpha$ and $\beta \neq \alpha$. It is shown in [BH] that $\preceq$ defines a partial order on the set of positive roots. Note that $(*)$ yields for $r \in R$ and $\alpha \in \Phi^+ \setminus \{a_r\}$ that $r \cdot \alpha \prec \alpha$ if and only if $\langle \alpha, a_r \rangle > 0$. It follows that each positive root is preceded by a simple root. If $\alpha \preceq \beta$, an easy induction on $\mathrm{dp}(\beta) - \mathrm{dp}(\alpha)$ yields that the coefficients of simple roots in α must be less than or equal to the corresponding coefficients in β.

For α, $\beta \in \Phi^+$ we say that α *dominates* β with respect to W (we write $\alpha \;\mathsf{dom}\; \beta$) if and only if $w \cdot \beta \in \Phi^-$ whenever $w \cdot \alpha \in \Phi^-$ for $w \in W$. It is shown in [BH] that dom also defines a partial order on the set of positive roots. The dominance-minimal roots are called *elementary*, and the set of elementary roots is denoted by $\mathcal{E}$; that is, $\mathcal{E}$ is the set of positive roots that do not dominate any root other than themselves. From here on, when we refer to a minimal element of a set of roots, we mean precedence-minimal rather than dominance-minimal.

The next proposition summarizes some basic properties of dominance from [BH].

(2.2) PROPOSITION *Let $\alpha, \beta \in \Phi^+$ and $r \in R$. Then*

(i) $\alpha \;\mathsf{dom}\; \beta$ *if and only if* $\langle \alpha, \beta \rangle \geq 1$ *and* $\mathrm{dp}(\alpha) \geq \mathrm{dp}(\beta)$;

(ii) if $\alpha \;\mathsf{dom}\; \beta$ *and* $w \cdot \beta \in \Phi^+$ *for some* $w \in W$, *then* $(w \cdot \alpha) \;\mathsf{dom}\; (w \cdot \beta)$;

(iii) if $\beta \notin \mathcal{E}$ *and* $\beta \preceq \alpha$, *then* $\alpha \notin \mathcal{E}$;

(iv) if $\alpha \in \mathcal{E}$, *then* $r \cdot \alpha \in \mathcal{E}$ *if and only if* $\langle \alpha, a_r \rangle \in (-1, 1)$.

§3 Initial reduction

For $S \subseteq \Pi$, let $\Phi_S^{\gneq 0}$ denote the set of positive roots with support S, and define $\mathcal{E}_S = \Phi_S^{\gneq 0} \cap \mathcal{E}$. Then Φ is the disjoint union of all $\Phi_S^{\gneq 0}$ with $S \subseteq \Pi$. It is clear that $\Phi_S^{\gneq 0}$ is empty if S is infinite, or if the Coxeter graph corresponding to S has more than one connected component. So $\mathcal{E}$ is the disjoint union of all $\mathcal{E}_S$ with $S \subseteq \Pi$ finite and connected.

It follows readily from (2.2)(i) that a root $\alpha \in \Phi_S^+$ is elementary in Φ_S^+ (with respect to W_S) if and only if it is elementary in Φ^+. Hence we may restrict our attention to Φ_S and W_S when investigating $\mathcal{E}_S$. The root system Φ_S is well known if W_S is finite. Furthermore, since the subspace spanned by Φ_S is Euclidean in this case, it follows from (2.2)(i) that every root in Φ_S^+ is elementary (since $\langle \alpha, \beta \rangle = \cos(\theta)$ for some $\theta \in (0, \pi)$ if $\alpha \neq \beta \in \Phi_S$); that is, $\mathcal{E}_S = \Phi_S^{\gneq 0}$. It remains to determine $\mathcal{E}_S$ for $S \subseteq \Pi$ finite and connected such that W_S infinite.

(3.1) PROPOSITION *$\mathcal{E}_S$ is empty if S contains a circuit or an infinite bond.*

Proof. Let α_0 be a root whose support contains a circuit, and let $\alpha \preceq \alpha_0$ be minimal such that $\mathrm{supp}(\alpha)$ contains a circuit; we will show that $\alpha \notin \mathcal{E}$, and thus $\alpha_0 \notin \mathcal{E}$ by (2.2)(iii).

Let $r \in R$ with $r \cdot \alpha \prec \alpha$; by minimality of α it follows that the support of $r \cdot \alpha$ does not contain any circuits. If $\alpha = \sum_{x \in \Pi} \lambda_x x$, then

$$r \cdot \alpha = \sum_{x \neq a_r} \lambda_x x + (\lambda_{a_r} - 2\langle \beta, a_r \rangle) a_r,$$

and we deduce that the coefficient of a_r in $r \cdot \alpha$ equals 0, and that a_r is part of a circuit in $\mathrm{supp}(\alpha)$. Hence there exist at least two elements b, c of $\mathrm{supp}(\alpha) \setminus \{a_r\}$ adjoined to a_r. By definition of $\langle\ ,\ \rangle$, it follows that $\langle a_r, b \rangle$ and $\langle a_r, c \rangle$ are both at most $-\cos(\frac{\pi}{3}) = -\frac{1}{2}$, while $\langle a_r, x \rangle \leq 0$ for all other $x \in \Pi \setminus \{a_r\}$. Furthermore, λ_b, $\lambda_c \geq 1$ by (2.1)(i), and thus

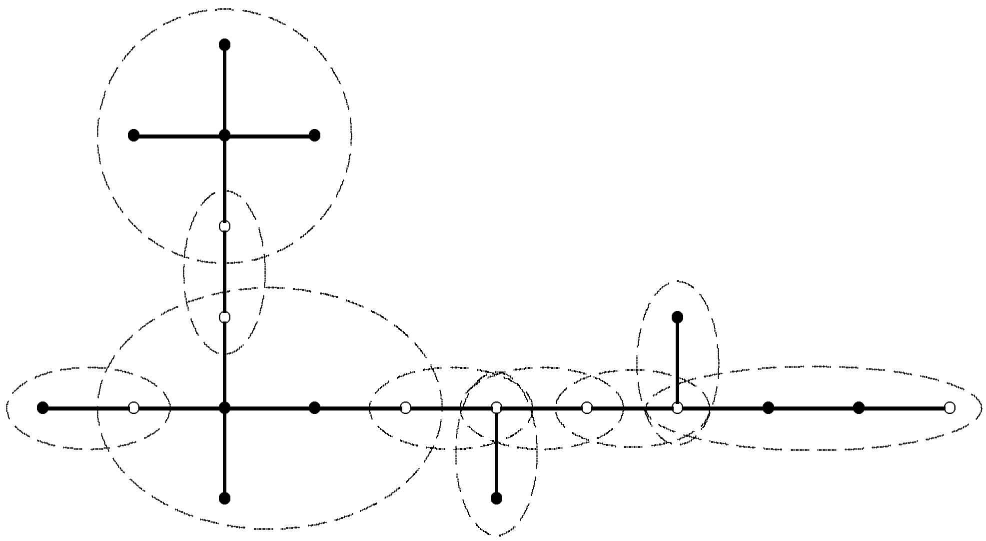
$$\langle r \cdot \alpha, a_r \rangle = \sum_{x \neq a_r} \lambda_x \langle x, a_r \rangle \leq \lambda_b \langle b, a_r \rangle + \lambda_c \langle c, a_r \rangle \leq -1.$$

So $\langle \alpha, a_r \rangle = -\langle r \cdot \alpha, a_r \rangle \geq 1$, and thus α dom a_r by (2.2)(i) (as α is certainly of depth greater than $\mathrm{dp}(a_r) = 1$). Hence $\alpha \notin \mathcal{E}$, and $\alpha_0 \notin \mathcal{E}$, as required. A similar argument applies when $\mathrm{supp}(\alpha)$ contains an infinite bond. □

We will now determine $\mathcal{E}_S$ for $S \subseteq \Pi$ finite and connected, and not containing any circuits (that is, S is a tree); moreover, we can assume that S does not contain any infinite bonds and that W_S is infinite. To facilitate the relevant statements, we make the following definition.

(3.2) Definition Let $x \in S$ and $T \subseteq S$. We say that x is a *disconnector* of S if the subgraph of the Coxeter graph corresponding to $S \setminus \{x\}$ is not connected. Define $\mathcal{D}(S)$ to be the set of disconnectors of S. A *T-component* of S is a maximal connected subset C of S such that $T \cap \mathcal{D}(C) = \emptyset$.

For example, if S is the graph shown below, and the subset T corresponds to the white vertices, then the T-components of S are as indicated:



Note that in this example, each $y \in S \setminus T$ is contained in exactly one T-component $C(y)$ of S; moreover, $C(y)$ is the union of the connected component of $S \setminus T$ containing y, and the set of $x \in T$ which are adjoined to some element of this connected component. Each $x \in T$ is contained in as many T-components of S as there are $z \in S$ adjoined to x, and we denote this number by $\mathrm{val}_S(x)$. Finally, observe that each T-component C of S equals $\bigcap_{x \in T} C_x$, where C_x is the $\{x\}$-component of S containing C. It can be easily shown that the above observations are true in general (given that S is a tree).

(3.3) Definition For $\alpha \in \Phi$, let $\mathbf{C}_1(\alpha)$ be the set of $a \in \Pi$ with coefficient 1 in α. We say that α is *indecomposable* if $\mathbf{C}_1(\alpha) \cap \mathcal{D}\big(\mathrm{supp}(\alpha)\big) = \emptyset$, otherwise α is said to be *decomposable.* For $S \subseteq \Pi$, we denote the set of indecomposable elementary roots with support S by $\mathcal{E}_S^{\mathrm{indec}}$.

If α is a decomposable root and $x \in \mathbf{C}_1(\alpha) \cap \mathcal{D}\big(\mathrm{supp}(\alpha)\big)$, the following assertion yields that α can be written as the 'sum' (minus a correction factor) of roots, each of which has support an $\{x\}$-component of $\mathrm{supp}(\alpha)$ (and is thus of cardinality less than $|\,\mathrm{supp}(\alpha)|$). Moreover, α is elementary if and only if each of the summands is elementary.

(3.4) Proposition *Let $x \in S$ and denote the $\{x\}$-components of S by $S_1, \ldots, S_n$. Then*

$$\phi\colon (\beta_1, \ldots, \beta_n) \mapsto \beta_1 + \cdots + \beta_n - (n-1)x$$

defines a one-one correspondence between the set of n-tuples in $\Phi^+_{S_1} \times \cdots \times \Phi^+_{S_n}$ such that the coefficient of x in each component equals 1, and the set of roots in Φ^+_S with coefficient 1 for x. Moreover, this map restricts to a one-one correspondence between the set of n-tuples in $\mathcal{E}_{S_1} \times \cdots \times \mathcal{E}_{S_n}$ such that x has coefficient 1 in each component, and the set of roots in $\mathcal{E}_S$ with coefficient 1 for x.

Before we prove (3.4), we establish the following lemma.

(3.5) Lemma *Let $\alpha \in \Phi^+$ and suppose that $x \in \Pi$ has coefficient 1 in α. Then $x \preceq \alpha$, and there exists a $w \in W_{\mathrm{supp}(\alpha)\setminus\{x\}}$ such that $\alpha = w \cdot x$ and $l(w) = \mathrm{dp}(\alpha) - 1$.*

Proof. If α is of depth 1, the assertion is certainly true. Suppose now that $\mathrm{dp}(\alpha) > 1$, and let $y \in S$ with $r_y \cdot \alpha \prec \alpha$. If $y \neq x$, the assertion follows by induction, so suppose that $y = x$. Then the coefficient of x in $r_x \cdot \alpha$ is less than 1, and thus equal to 0 by (2.1)(i). Hence $\langle \alpha, x\rangle = \frac{1}{2}$, and it follows that x is adjacent to exactly one element z of $\mathrm{supp}(r_x \cdot \alpha)$; and, moreover, that x and z are adjoined by a simple bond, and the coefficient of z in $r_x \cdot \alpha$ equals 1. By induction, there exists a $u \in W_{\mathrm{supp}(r_x\cdot\alpha)\setminus\{z\}}$ of length $\mathrm{dp}(r_x \cdot \alpha) - 1$ such that $r_x \cdot \alpha = u \cdot z$. Since x is not adjoined to $\mathrm{supp}(r_x \cdot \alpha) \setminus \{z\}$, we deduce that $r_x u = u r_x$, and thus

$$\alpha = r_x \cdot (u \cdot z) = u \cdot (r_x \cdot z) = u \cdot (z + x) = u \cdot (r_z \cdot x) = (u r_z) \cdot x.$$

As $l(ur_z) \leq l(u) + 1 = \mathrm{dp}(r_x \cdot \alpha) = \mathrm{dp}(\alpha) - 1$, we find that α is preceded by x, as required. □

Proof of (3.4). It can be easily verified using Lemma (3.5) that ϕ is well defined and onto; moreover, $\phi(w_1 \cdot x, \ldots, w_n \cdot x) = (w_1 \cdots w_n) \cdot x$ for $w_i \in W_{S_i\setminus\{x\}}$. Since ϕ is certainly one-one, it follows that ϕ is a one-one correspondence.

In order to show that $\phi(\beta_1, \ldots, \beta_n) \in \mathcal{E}_S$ if and only if $\beta_i \in \mathcal{E}_{S_i}$ for all i, it suffices to show that if $w = w_1 \cdots w_n$ with $w_i \in W_{S_i\setminus\{x\}}$, then $w \cdot x \notin \mathcal{E}$ if and only if $w_i \cdot x \notin \mathcal{E}$ for some $i \in \{1, \ldots, n\}$. Since it can be easily seen that $w_i \cdot x \preceq w \cdot x$ for $i \in \{1, \ldots, n\}$, Proposition (2.2)(iii) implies that $w \cdot x \notin \mathcal{E}$ if $w_i \cdot x \notin \mathcal{E}$ for some i; hence it remains to show the converse.

Suppose that $w \cdot x$ dominates some $\gamma \in \Phi^+ \setminus \{w \cdot x\}$. Since $x \in \mathcal{E}$, (2.2)(ii) yields that $w^{-1} \cdot \gamma \in \Phi^-$. As $w \in W_{S\setminus\{x\}}$, in particular $x \notin \mathrm{supp}(\gamma)$; since the support of γ has to be connected, we may assume without loss of generality that $\gamma \in \Phi_{S_1\setminus\{x\}}$. Then $(w_2 \cdots w_n)^{-1} \cdot \gamma$ is positive, and hence $(w_1 \cdot x)$ dom $(w_2 \cdots w_n)^{-1} \cdot \gamma$ by (2.2)(ii). Since $w \cdot x \neq \gamma$, it follows that $w_1 \cdot x \neq (w_2 \cdots w_n)^{-1} \cdot \gamma$, and thus $w_1 \cdot x \notin \mathcal{E}$, as required. □

(3.6) Corollary *Let $\alpha \in \Phi$, and suppose that $\mathrm{supp}(\alpha)$ is a tree which does not contain any infinite bonds. Further, let x and y be in the support of α, and suppose that x has coefficient 1 in α, and y has coefficient μ in α.*

(i) If x and y are adjoined by a simple bond, then $\mu = 1$, or $\mu \geq 2$.

(ii) If x and y are adjoined by a non-simple bond, then $\mu > 1$.

Proof. Suppose that $\alpha = \beta_1 + \cdots + \beta_n - nx$ in accordance with (3.4), where $\beta_i \in \Phi_{S_i}$ and S_i is an $\{x\}$-component of $\mathrm{supp}(\alpha)$ for all i. If $y \in S_i$, the coefficient of x in $r_x \cdot \beta_i$ equals $-1 - 2\langle x, y\rangle\mu$. By (2.1)(i), we deduce that this equals 0, or is greater than or equal to 1, and the assertion follows. $\square$

For $S \subseteq \Pi$ and $T \subseteq S$, we define Φ_S^T to be the set of roots $\alpha \in \Phi_S^{\geq 0}$ such that $\mathbf{C}_1(\alpha) = T$; that is, Φ_S^T is the set of roots in with support S and coefficient 1 for $x \in S$ if and only if $x \in T$. Define $\mathcal{E}_S^T$ to be $\Phi_S^T \cap \mathcal{E}$.

It is clear that $\Phi_S^{\geq 0}$ is the disjoint union of all Φ_S^T with $T \subseteq S$. If $T \cap \mathcal{D}(S) = \emptyset$, every element in $\mathcal{E}_S^T$ is indecomposable. If $T \cap \mathcal{D}(S) \neq \emptyset$ and x is an element of this intersection, let $S_1, \ldots, S_n$ be the $\{x\}$-compoments of S. By (3.4), each element of $\mathcal{E}_S^T$ can be written as

$$\alpha_1 + \cdots + \alpha_n - (n-1)x,$$

where $\alpha_i \in \mathcal{E}_{S_i}$ with coefficient 1 for x. Furthermore, if $y \in S_i$, the coefficient of y in α_i equals 1 if and only if $y \in T$; that is, $\alpha_i \in \mathcal{E}_{S_i}^{S_i \cap T}$. Conversely, it is clear that $\beta_1 + \cdots + \beta_n - (n-1)x$ is in $\mathcal{E}_S^T$ if $\beta_i \in \mathcal{E}_{S_i}^{S_i \cap T}$ for all i.

An iteration of this yields the following assertion, which reduces our quest to finding $\mathcal{E}_S^{\mathrm{indec}}$ for $S \subseteq \Pi$.

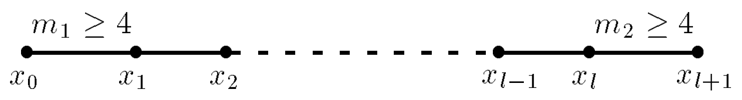
(3.7) Proposition *Suppose that $S \subseteq \Pi$ is a tree, and let $T \subseteq S$ and $S_1, \ldots, S_n$ be the T-components of S. Then the elements of $\mathcal{E}_S^T$ are:*

$$\sum_{i=1}^{n} \alpha_i - \sum_{x \in T} \bigl(\mathrm{val}_S(x) - 1\bigr)x,$$

where $\alpha_i \in \mathcal{E}_{S_i}^{S_i \cap T}$ for $i \in \{1, \ldots, n\}$. (Note that since $T \cap \mathcal{D}(S_i) = \emptyset$, clearly $\mathcal{E}_{S_i}^{S_i \cap T} \subseteq \mathcal{E}_{S_i}^{\mathrm{indec}}$).

(3.8) Proposition *$\mathcal{E}_S^{\mathrm{indec}}$ is empty if S contains two or more non-simple bonds.*

Proof. As in the proof of (3.1), we assume that $\alpha \in \Phi_S$ is minimal with coefficients greater than 1 for $x_1, \ldots, x_l$, and x_0, $x_l \in \mathrm{supp}(\alpha)$, where the graph of $\{x_0, \ldots, x_{l+1}\}$ is as follows:



Let $x \in \Pi$ with $r_x \cdot \alpha \prec \alpha$. Again, it suffices to show that $\langle r_x \cdot \alpha, x\rangle \leq -1$; for then α and all the roots preceded by α are non-elementary. By (3.1), we may assume without loss of generality that $\mathrm{supp}(\alpha)$ does not contain any circuits.

Denote the coefficient of $y \in S$ in α by λ_y; then $\lambda_{x_i} \geq \sqrt{2}$ for $i = 1, \ldots, l$. Minimality of α forces $x = x_i$ for some $i \in \{0, \ldots, l+1\}$. If $i = 0$, minimality of α yields further that $x_0 \notin \mathrm{supp}(r_{x_0} \cdot \alpha)$, and thus

$$\langle r_{x_0} \cdot \alpha, x_0 \rangle = \sum_{y \in S} \lambda_y \langle y, x_0 \rangle = 0 \langle x_0, x_0 \rangle + \sum_{y \neq x_0} \lambda_y \langle y, x_0 \rangle \leq \lambda_{x_1} \langle x_1, x_0 \rangle.$$

Since $\lambda_{x_1} \geq \sqrt{2}$ and $\langle x_1, x_0 \rangle = \cos(\pi/m_1) \leq -\cos(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$, this is less than or equal to -1, as required. Similar arguments apply if $i = l+1$.

Suppose next that $i \in \{1, \ldots, l\}$. Then the coefficient of x_i in $r_{x_i} \cdot \alpha$ is less than or equal to 1 by minimality of α. Since $\mathrm{supp}(\alpha)$ does not contain any circuits, and the support of $r_{x_i} \cdot \alpha$ must be connected, Proposition (2.1)(i) forces the coefficient of x_i in $r_{x_i} \cdot \alpha$ to be equal to 1. If $i = 1$, then $\lambda_{x_0} \geq \sqrt{2}$ by Corollary (3.6)(ii) together with (2.1)(ii); hence

$$\lambda_{x_{i-1}} \langle x_{i-1}, x_i \rangle = \lambda_{x_0} \langle x_0, x_1 \rangle \leq \sqrt{2}\big(-\cos(\pi/m_1)\big) \leq -1,$$

as $\cos(\pi/m_1) \geq \frac{1}{\sqrt{2}}$. If $i > 1$, Corollary (3.6)(i) implies that $\lambda_{x_{i-1}} \geq 2$, and thus again $\lambda_{x_{i-1}} \langle x_{i-1}, x_i \rangle \leq -1$. Symmetrical arguments also yield $\lambda_{x_{i+1}} \langle x_{i+1}, x_i \rangle \leq -1$ and therefore

$$\langle r_{x_i} \cdot \alpha, x_i \rangle \leq 1 + \lambda_{x_{i-1}} \langle x_{i-1}, x_i \rangle + \lambda_{x_{i+1}} \langle x_{i+1}, x_i \rangle \leq -1,$$

as required.

□

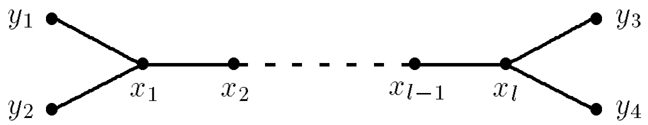
This leaves us to determine $\mathcal{E}_S^{\mathrm{indec}}$ for $S \subseteq \Pi$ a tree, containing at most one non-simple bond (which has finite weight) such that W_S is infinite.

§4 Simple bonds only *

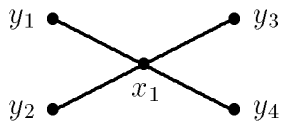
For the duration of this section suppose that $S \subseteq \Pi$ is a tree and contains only simple bonds. Note that if $\alpha \in \Phi_S$, all the coefficients in α are integers; hence it is clear that $\langle \alpha, x \rangle$ is an integer multiple of $\frac{1}{2}$ for $x \in S$. In particular, if α is elementary and $\alpha \neq x$, then $\langle \alpha, x \rangle \leq \frac{1}{2}$; moreover, if $r_x \cdot \alpha \prec \alpha$, then $\langle \alpha, x \rangle = \frac{1}{2}$.

* A description of the set of elementary roots in this case has been obtained in [He]

(4.1) Proposition $\mathcal{E}_S^{\mathrm{indec}}$ *is empty if* S *contains*



or



(Of course, the right hand case is just a special case of the left hand one).

Proof. As in the proof of (3.1), we let α be minimal with y_1, y_2, y_3, $y_4 \in \operatorname{supp}(\alpha)$ and coefficient $\lambda_i > 1$ for x_i for all $i \in \{1, \ldots, n\}$. Note that since λ_i is an integer, it follows that $\lambda_i \geq 2$. Further, let $z \in S$ with $r_z \cdot \alpha \prec \alpha$; it suffices to show that $\langle r_z \cdot \alpha, z\rangle \leq -1$.

Minimality of α forces $z = x_i$ for some $i \in \{1, \ldots, l\}$ or $z = y_j$ for some $j \in \{1, 2, 3, 4\}$. If $z = x_i$, minimality of α yields that the coefficient of x_i in $r_{x_i} \cdot \alpha$ is less than or equal to 1. If $1 < i < l$, we deduce that

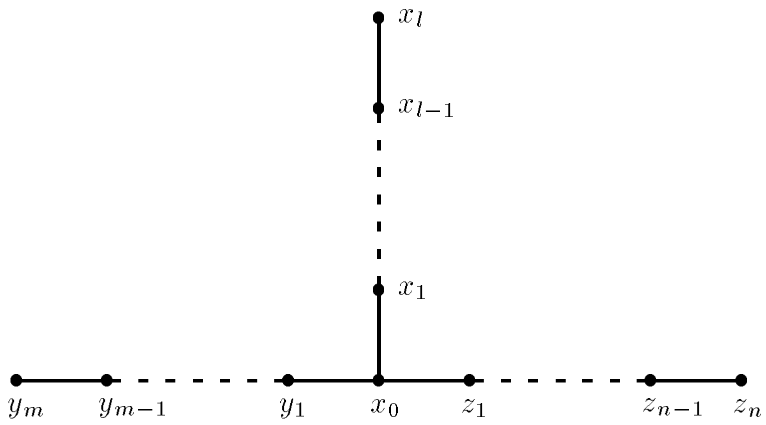
$$\langle r_{x_i} \cdot \alpha, x_i\rangle \leq 1 + \lambda_{i-1}\langle x_{i-1}, x_i\rangle + \lambda_{i+1}\langle x_{i+1}, x_i\rangle \leq 1 + 2\Big(-\frac{1}{2}\Big) + 2\Big(-\frac{1}{2}\Big) \leq -1,$$

as required. If $1 = i < l$, the coefficients of y_1 and y_2 in α are greater than or equal to 1, while $\lambda_{i+1} \geq 2$, and thus again $\langle r_{x_i} \cdot \alpha, x_i\rangle \leq -1$. Similarly, $\langle r_{x_i} \cdot \alpha, x_i\rangle \leq -1$ if $1 < i = l$. Finally, if $1 = i = l$, the coefficients of y_1, y_2, y_3 and y_4 in α are greater than or equal to 1, and again $\langle r_{x_i} \cdot \alpha, x_i\rangle \leq -1$.

If $z = y_1$, minimality of α yields that $y_1 \notin \operatorname{supp}(r_{y_1} \cdot \alpha)$. Since $\lambda_1 \geq 2$, we deduce that $\langle r_{y_1} \cdot \alpha, y_1\rangle \leq -1$. Symmetrical arguments apply if $z \in \{y_2, y_3, y_4\}$. □

This leaves us to determine $\mathcal{E}_S^{\mathrm{indec}}$ for S of the shape

(0)



with l, m, $n \geq 1$. We show now that each root in $\mathcal{E}_S^{\mathrm{indec}}$ is preceded by

$$x_l + 2(x_{l-1} + \cdots + x_1) + y_m + 2\big(y_{m-1} + \cdots + y_1\big) + 2x_0 + 2\big(z_1 + \cdots + z_{n-1}\big) + z_n,$$

by proving that if $\alpha = \sum_{x\in S} \lambda_x x$ and $\beta = \sum_{x\in S} \mu_x x$ are elementary roots with $\mu_x \le \lambda_x$ for all $x \in S$, then $\beta \preceq \alpha$.

If $\mathrm{dp}(\alpha) = 1$, this is certainly true. Suppose next that $\mathrm{dp}(\alpha) > 1$, and let $y \in S$ with $r_y \cdot \alpha \prec \alpha$; then $\langle \alpha, y\rangle = \frac{1}{2}$, and the coefficient of y in $r_y \cdot \alpha$ equals $\lambda_y - 1$. If $\mu_y \le \lambda_y - 1$, induction yields that $\beta \preceq r_y \cdot \alpha$, and thus $\beta \preceq \alpha$ by transitivity of $\preceq$. Suppose next that $\mu_y > \lambda_y - 1$; that is, $\mu_y = \lambda_y$. If $\beta = y$, then $\lambda_y = 1$, and thus $y \preceq \alpha$ by (3.5), as required. If $\beta \ne y$, then $\langle \beta, y\rangle \le \frac{1}{2}$. Further,

$$\langle \beta, y\rangle = \sum_{x\in S} \mu_x \langle x, y\rangle = \mu_y\langle y, y\rangle + \sum_{x\ne y} \mu_x\langle x, y\rangle \ge \lambda_y\langle y, y\rangle + \sum_{x\ne y} \lambda_x\langle x, y\rangle = \langle \alpha, y\rangle = \frac{1}{2},$$

and so we must have equality everywhere. Hence $\langle \beta, y\rangle = \frac{1}{2}$, and if $x \in S$ with $\langle x, y\rangle \ne 0$, then $\lambda_x = \mu_x$. Since the coefficient of y in $r_y \cdot \beta$ equals $\mu_y - 1 = \lambda_y - 1$, induction yields that $r_y \cdot \beta \preceq r_y \cdot \alpha$. Let $w \in W_S$ be of length $\mathrm{dp}(r_y \cdot \alpha) - \mathrm{dp}(r_y \cdot \beta)$ with $r_y \cdot \alpha = w \cdot (r_y \cdot \beta)$. Since for $x \in S$ with $\langle x, y\rangle \ne 0$, the coefficients of x in $r_y \cdot \beta$ and $r_y \cdot \alpha$ coincide, we deduce that $w \in W_{S_0}$, where S_0 consists of all $x \in S$ with $\langle x, y\rangle = 0$. Therefore $r_y w = w r_y$, and thus $\beta = r_y \cdot \big(r_y \cdot \beta\big) = r_y \cdot \big(w \cdot (r_y \cdot \alpha)\big) = w \cdot \alpha$. Since $\mathrm{dp}(\alpha) - \mathrm{dp}(\beta) = \mathrm{dp}(r_y \cdot \alpha) - \mathrm{dp}(r_y \cdot \beta) = l(w)$, we conclude that $\beta \preceq \alpha$, as required.

To shorten the notation, we write $[a \cdot\cdot\, b]$ for the sum of the simple roots on the minimal path in S joining $a \in S$ to $b \in S$. A straightforward induction on depth yields the following assertion.

(4.2) Proposition *Suppose that S is of shape (0), and set $x_0 = y_0 = z_0$.*

(1) *If $l, m, n \ge 2$, then $\mathcal{E}_S^{\mathrm{indec}}$ has exactly one element, namely*
$x_l + 2(x_{l-1} + \cdots + x_1) + y_m + 2\big(y_{m-1} + \cdots + y_1\big) + 2x_0 + 2\big(z_1 + \cdots + z_{n-1}\big) + z_n.$

(2) *If $l = 1$ and $m, n \ge 3$, the elements of $\mathcal{E}_S^{\mathrm{indec}}$ are as follows:*

(i) $x_1 + y_m + 2[y_{m-1} \cdot\cdot\, y_{j(3)}] + 3[y_{j(3)-1} \cdot\cdot\, y_{j(4)}] + \cdots + (M-1)[y_{j(M-1)-1} \cdot\cdot\, y_{j(M)}]$
$+ M[y_{j(M)-1} \cdot\cdot\, z_{k(M)-1}]$
$+ (M-1)[z_{k(M)} \cdot\cdot\, z_{k(M-1)-1}] + \cdots + 3[z_{k(4)} \cdot\cdot\, z_{k(3)-1}] + 2[z_{k(3)} \cdot\cdot\, z_{n-1}] + z_n,$
with $M \in \{2, \ldots, \min(m, n)+1\}$, $m > j(3) > j(4) > \ldots > j(M-1) > j(M) > 0$ and $0 < k(M) < \ldots < k(4) < k(3) < n$,

(ii) $2x_1 + y_m + 2[y_{m-1} \cdot\cdot\, y_j] + 3[y_{j-1} \cdot\cdot\, z_{k-1}] + 2[z_k \cdot\cdot\, z_{n-1}] + z_n,$
where $m > j > 0$ and $0 < k < n$,

(iii) $2x_1 + y_m + 2[y_{m-1} \cdot\cdot\, y_1] + 4[x_0 \cdot\cdot\, z_{k(2)-1}] + 3[z_{k(2)} \cdot\cdot\, z_{k(1)-1}] + 2[z_{k(1)} \cdot\cdot\, z_{n-1}] + z_n,$
where $0 < k(2) < k(1) < n$, and

(iv) $2x_1 + y_m + 2[y_{m-1} \cdot\cdot\, y_{j(1)}] + 3[y_{j(1)-1} \cdot\cdot\, y_{j(2)}] + 4[y_{j(2)-1} \cdot\cdot\, x_0] + 2[z_1 \cdot\cdot\, z_{n-1}] + z_n,$
where $m > j(1) > j(2) > 0$.

(3) *If $l = 1$, $m = 2$ and $n \ge 5$, the elements of $\mathcal{E}_S^{\mathrm{indec}}$ are as follows:*

(i) $x_1 + y_2 + 2[y_1 \cdot\cdot\, z_{n-1}] + z_n$,

(ii) $\lambda x_1 + y_2 + 2y_1 + 3[x_0 \cdot\cdot\, z_{k-1}] + 2[z_k \cdot\cdot\, z_{n-1}] + z_n$ where $k \in \{1, \dots, n-1\}$, $\lambda \in \{1, 2\}$,

(iii) $\lambda x_1 + y_2 + \mu y_1 + M[x_0 \cdot\cdot\, z_{k(M-1)-1}]$
$+ (M-1)[z_{k(M-1)} \cdot\cdot\, z_{k(M-2)-1}] + \cdots + 3[z_{k(3)} \cdot\cdot\, z_{k(2)-1}] + 2[z_{k(2)} \cdot\cdot\, z_{n-1}] + z_n$,
where $M \in \{4, \dots, n+1\}$, $0 < k(M-1) < k(M-2) < \dots < k(3) < k(2) < n$ and $\lambda \in \{\frac{M-1}{2}, \frac{M+1}{2}\}$, $\mu = \frac{M+1}{2}$ if M is odd, and $\lambda = \frac{M}{2}$, $\mu \in \{\frac{M}{2}, \frac{M}{2} + 1\}$ if M is even,

(iv) $2x_1 + 2y_2 + 3y_1 + 4[x_0 \cdot\cdot\, z_{k(3)-1}] + 3[z_{k(3)} \cdot\cdot\, z_{k(2)-1}] + 2[z_{k(2)} \cdot\cdot\, z_{n-1}] + z_n$,
where $0 < k(3) < k(2) < n$,

(v) $\lambda x_1 + 2y_2 + \mu y_1 + 5[x_0 \cdot\cdot\, z_{k(4)-1}] + 4[z_{k(4)} \cdot\cdot\, z_{k(3)-1}] + \cdots + 2[z_{k(2)} \cdot\cdot\, z_{n-1}] + z_n$,
where $0 < k(4) < k(3) < k(2) < n$ and $\lambda \in \{2, 3\}$, $\mu \in \{3, 4\}$,

(vi) $\lambda x_1 + 2y_2 + \mu y_1 + 6[x_0 \cdot\cdot\, z_{k(5)-1}] + 5[z_{k(5)} \cdot\cdot\, z_{k(4)-1}] + \cdots + 2[z_{k(2)} \cdot\cdot\, z_{n-1}] + z_n$,
where $0 < k(5) < k(4) < k(3) < k(2) < n$ and $(\lambda, \mu) \in \{(3, 3), (2, 4)\}$,

(vii) $3x_1 + 2y_2 + 4y_1 + 6x_0 + 4[z_1 \cdot\cdot\, z_{k(2)-1}] + 3[z_{k(3)} \cdot\cdot\, z_{k(2)-1}] + 2[z_{k(2)} \cdot\cdot\, z_{n-1}] + z_n$,
where $1 < k(3) < k(2) < n$,

(viii) $3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 3[z_2 \cdot\cdot\, z_{k-1}] + 2[z_k \cdot\cdot\, z_{n-1}] + z_n$, where $2 < k < n$,

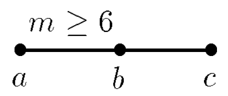
(ix) $3x_1 + 2y_2 + 4y_1 + 6x_0 + 5z_1 + 4z_2 + 2[z_3 \cdot\cdot\, z_{n-1}] + z_n$.

§5 One non-simple bond of weight $m \geq 6$

For the duration of this section assume that $S \subseteq \Pi$ is a tree and contains exactly one non-simple bond which has weight $m \geq 6$. Let a and b be the vertices of this non-simple bond, and denote $2\cos(\frac{\pi}{m})$ by c_m; then $c_m \geq \sqrt{3}$.

We show now that each $\alpha \in \mathcal{E}_S$ is preceded by $a + c_m b$ or $c_m a + b$ (this is true for all $m \geq 4$). Let $\beta \preceq \alpha$ be maximal such that $\{a, b\} \not\subseteq \operatorname{supp}(\beta)$; we may assume without loss of generality that $b \notin \operatorname{supp}(\beta)$. By maximality of β, we find that $\beta \prec r_b \cdot \beta \preceq \alpha$ and $a \in \operatorname{supp}(\beta)$. So $r_b \cdot \beta \in \mathcal{E}$ by (2.2)(iii), and thus $\langle \beta, b \rangle > -1$ by (2.2)(iv). We deduce that the coefficient of a in β is less than $\frac{2}{c_m} \leq \sqrt{2}$, and thus must be equal to 1 by (2.1)(ii). Lemma (3.5) yields that $a \preceq \beta$, and that there exists a $w \in W_{\operatorname{supp}(\beta) \setminus \{a\}}$ of length $\operatorname{dp}(\beta) - 1$ such that $\beta = w \cdot a$. As $b \notin \operatorname{supp}(\beta)$, and since S does not contain any circuits, we know that b cannot be adjacent to $\operatorname{supp}(\beta) \setminus \{a\}$, and so $r_b w = w r_b$. Now $r_b \cdot \beta = w \cdot (a + c_m b)$, and since $\operatorname{dp}(r_b \cdot \beta) - \operatorname{dp}(a + c_m b) = \operatorname{dp}(\beta) - \operatorname{dp}(a) = l(w)$, we find that $r_b \cdot \beta$ is preceded by $a + c_m b$. Transitivity of $\preceq$ yields that α is preceded by $a + c_m b$, as required.

(5.1) Lemma *Suppose that S contains the following subgraph*



and let $\alpha \in \Phi_S$ be preceded by $c_m a + b$, with a, $c \in \operatorname{supp}(\alpha)$, and coefficient for b in α greater than 1. Then $\alpha \notin \mathcal{E}$.

Proof. Let β be maximal with $c_m a + b \preceq \beta \preceq \alpha$ such that the coefficient of b in β equals 1; then $\beta \prec r_b \cdot \beta \preceq \alpha$ by maximality of β. If $c \in \operatorname{supp}(\beta)$, we find that

$$\langle \beta, b \rangle \leq 1 + \Big(-\frac{c_m}{2}\Big) c_m + \Big(-\frac{1}{2}\Big) 1 \leq -1,$$

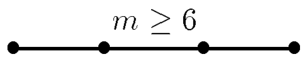
and thus $r_b \cdot \beta \notin \mathcal{E}$ by (2.2)(iv); hence $\alpha \notin \mathcal{E}$, as required. Assume now that $c \notin \operatorname{supp}(\beta)$, and let γ be of maximal depth with $r_b \cdot \beta \preceq \gamma \preceq \alpha$ such that $c \notin \operatorname{supp}(\gamma)$; then $\gamma \prec r_c \cdot \gamma \preceq \alpha$ by maximality of γ. The coefficient of b in γ must be greater than or equal to the coefficient of b in $r_b \cdot \beta$, and this is greater than or equal to

$$-1 + 2\Big(-\frac{c_m}{2}\Big) c_m \geq 2.$$

Hence clearly $\langle \gamma, c \rangle \leq -1$, and thus $\alpha \notin \mathcal{E}$ by (2.3)(iii), (iv). □

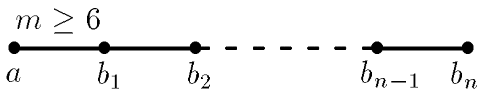
Since in an indecomposable root the coefficient of each disconnector must be greater than 1, the previous lemma has the following immediate consequence.

(5.2) Proposition $\mathcal{E}_S^{\text{indec}}$ *is empty if S contains the following subgraph:*



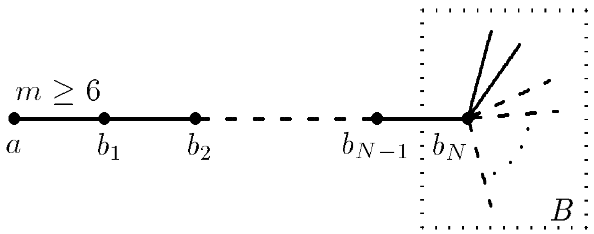
This leaves us to determine $\mathcal{E}_S^{\text{indec}}$ for S of the following two shapes:

(1)



$(n \geq 2)$

(2)



where B is a tree which contains only simple bonds such that b_N is adjoined to two or more elements of B (that is, $\operatorname{val}_B(b_N) \geq 2$).

Note that if S is of shape (1) or (2), Lemma (5.1) yields that each root in $\mathcal{E}_S^{\text{indec}}$ is preceded by $a + c_m b$. If S is of shape (1), it follows readily that each root in $\mathcal{E}_S^{\text{indec}}$ is preceded by $a + c_m(b_1 + \cdots + b_n)$, and the next result follows:

(5.3) Proposition *Suppose that S is of shape (1) with $m \geq 7$. Then*
$$\mathcal{E}_S^{\text{indec}} = \{a + c_m(b_1 + \cdots + b_n),\, (c_m^2 - 1)a + c_m(b_1 + \cdots + b_n)\}.$$

(5.4) Proposition *Suppose that S is of shape (1) and $m = 6$. Then the elements of $\mathcal{E}_S^{\text{indec}}$ are as follows:*

(i) $a + c_m(b_1 + \cdots + b_n)$, *and*

(ii) $2a + \sqrt{3}\big(2(b_1 + \cdots + b_{j-1}) + (b_j + \cdots + b_n)\big)$, *where* $j \in \{1, \ldots, n\}$.

It remains to determine $\mathcal{E}_S^{\text{indec}}$ for S of shape (2). In order to do so, we prove the following two assertions, which are valid for all $m \geq 4$, and since these statements will be used for the cases $m = 4$, 5, we prove the results in general.

(5.5) Proposition *Let $x \in S$, and let $S_0, \ldots, S_k$ be the $\{x\}$-components of S such that S_0 contains the bond of weight m; that is, $\{a, b\} \subseteq S_0$. Then*

$$\phi\colon (\beta_0, \ldots, \beta_k) \mapsto \beta_0 + c_m(\beta_1 + \cdots + \beta_k - kx)$$

defines a one-one correspondence between the set of $(k+1)$-tuples in $\Phi_{S_0}^+ \times \cdots \times \Phi_{S_k}^+$ such that the coefficient of x in the first component equals c_m, while for all other components the coefficient of x equals 1, and the set of roots in Φ_S^+ with coefficient c_m for x. Moreover, ϕ restricts to a one-one correspondence between the set of $(k+1)$-tuples in $\mathcal{E}_{S_0} \times \cdots \times \mathcal{E}_{S_k}$ such that x has coefficient c_m in the first component, and 1 in the others, and the set of roots in $\mathcal{E}_S$ with coefficient c_m for x.

Proof. We show first that ϕ is well defined. So let $(\beta_0, \ldots, \beta_k) \in \Phi_{S_0}^+ \times \cdots \times \Phi_{S_k}^+$ such that the coefficient of x in β_0 equals c_m, while the coefficient of x in β_j equals 1 for $j \in \{1, \ldots, k\}$. Lemma (3.5) implies that $x \preceq \beta_j$ for $j \in \{1, \ldots, k\}$, and hence there exist $w_j \in W_{S_j \setminus \{x\}}$ of length $\text{dp}(\beta_j) - 1$ such that $\beta_j = w_j \cdot x$. Define $\alpha = (w_1 \cdots w_k) \cdot \beta_0$; then the coefficient of x in α equals c_m, and it can be easily seen that α equals $\phi(\beta_0, \ldots, \beta_k)$. Therefore ϕ is well defined; it is clear that ϕ is one-one, and we show now that ϕ is onto.

Suppose that $\alpha \in \Phi_S^+$ has coefficient c_m for x, and let $\gamma \preceq \alpha$ be minimal such that the coefficient of x in α equals c_m. We deduce from (2.1)(ii) that $\{a, b\} \subseteq \text{supp}(\gamma)$. Since $\{a, b\} \subseteq S_0$, in particular $|\,\text{supp}(\gamma) \cap S_0| \geq 2$, and thus

$$(\text{supp}(\gamma) \cap S_0) \setminus \{x\} \neq \emptyset. \tag{$*$}$$

Now $r_x \cdot \gamma \prec \gamma$ by minimality of γ, and it follows by (2.1)(i) that the coefficient of x in $r_x \cdot \gamma$ equals 0 or 1. If $x \notin \text{supp}(r_x \cdot \gamma)$, then $\text{supp}(r_x \cdot \gamma) \subseteq S_i \setminus \{x\}$ for some i by connectedness of the support of $r_x \cdot \gamma$, and thus $\text{supp}(r_x \cdot \gamma) \subseteq S_0 \setminus \{x\}$ by $(*)$.

Assume now that the coefficient of x in $r_x \cdot \gamma$ equals 1. Let $z \in \text{supp}(\gamma)$ be adjoined to x, and denote the coefficient of z in γ by ν. If x and z are adjoined by a simple bond, it follows by (3.6)(i) applied to γ that $\nu \neq 1$; therefore (3.6)(i) applied to $r_x \cdot \gamma$ yields that $\nu \geq 2$. If x and z are adjoined by the bond of weight m, then $\nu \geq c_m$ by (3.6)(ii). Hence in any case, $\nu\langle z, x\rangle \leq -1$; since $\langle \gamma, x\rangle > 0$ as $r_x \cdot \gamma \prec \gamma$, we deduce that x is only adjoined to one $z \in \operatorname{supp}(\gamma)$. By $(*)$ we conclude that $\operatorname{supp}(\gamma) \subseteq S_0$.

Since γ has coefficient c_m for x, there exists a $w \in W_{S\setminus\{x\}}$ such that $\alpha = w \cdot \gamma$ and $\operatorname{dp}(\alpha) - \operatorname{dp}(\gamma) = l(w)$. As $W_{S\setminus\{x\}}$ is the direct product of $W_{S_0\setminus\{x\}}, \ldots, W_{S_k\setminus\{x\}}$, there exist $w_j \in W_{S_j\setminus\{x\}}$ for all $j \in \{0, \ldots, k\}$ such that w equals $w_0 \cdots w_k$ with length adding. Define $\beta = w_0 \cdot \gamma$; this is an element of $\Phi^+_{S_0}$ with coefficient c_m for x, and clearly

$$\alpha = (w_1 \cdots w_k) \cdot \beta = \phi(\beta, w_1 \cdot x, \ldots, w_k \cdot x).$$

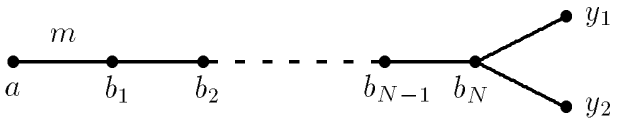
Hence ϕ is also onto. We show now for $w_i \in W_{S_i\setminus\{x\}}$ and $\beta \in \mathcal{E}_{S_0}$ with coefficient c_m for x that $\alpha = (w_1 \cdots w_k) \cdot \beta \notin \mathcal{E}$ if and only if $\beta \notin \mathcal{E}$ or $w_i \cdot x \notin \mathcal{E}$ for some $i \geq 1$. Note that a straightforward calculation yields for $i \geq 1$ and $\gamma \in \Phi_{S_i\setminus\{x\}}$, that

$$\langle \gamma, \alpha\rangle = c_m \langle \gamma, w_i \cdot x\rangle. \tag{$**$}$$

It can be easily seen that α is preceded by β; thus $\alpha \notin \mathcal{E}$ by (2.2)(iii) if $\beta \notin \mathcal{E}$. Next, suppose that $w_i \cdot x$ dominates $\gamma \in \Phi^+ \setminus \{w_i \cdot x\}$. Then $\gamma \in \Phi_{S_i\setminus\{x\}}$ as $x \notin \mathcal{E}$. Since $\langle \gamma, w_i \cdot x\rangle \geq 1$ by (2.2)(i), we deduce from $(**)$ that $\langle \alpha, \gamma\rangle \geq 1$, and it follows readily that $\alpha \notin \mathcal{E}$, as required.

Conversely, if α dominates some $\gamma \in \Phi^+ \setminus \{\alpha\}$, (2.2)(ii) implies that either β dominates $((w_1 \cdots w_k)^{-1}) \cdot \gamma$, or $((w_1 \cdots w_k)^{-1}) \cdot \gamma$ is negative. So $\beta \notin \mathcal{E}$ (as required), or $\gamma \in \Phi^+_{S_i\setminus\{x\}}$ for some $i \geq 1$. If $\gamma \in \Phi^+_{S_i\setminus\{x\}}$, then $(**)$ implies that $\langle \gamma, w_i \cdot x\rangle \geq \frac{1}{c_m} > \frac{1}{2}$. As S_i contains only simple bonds, this forces $\langle \gamma, w_i \cdot x\rangle \geq 1$; hence $w_i \cdot x$ dominates γ, and thus cannot be an element of $\mathcal{E}$, as required. □

(5.6) LEMMA *Suppose that S contains the following subgraph*



and let $\alpha_0 \in \Phi_S$ with coefficient for a greater than 1, coefficients for $b_1, \ldots, b_N$ greater than or equal to 2, and y_1, $y_2 \in \operatorname{supp}(\alpha_0)$. Then $\alpha_0 \notin \mathcal{E}$.

Proof. As in the proof of (3.1), we let $\alpha \preceq \alpha_0$ be minimal such that y_1, $y_2 \in \operatorname{supp}(\alpha)$, the coefficient of a in α is greater than 1, and the coefficients of $b_1, \ldots, b_N$ in α greater than or equal to 2. Further, let $x \in S$ with $r_x \cdot \alpha \prec \alpha$; again, it suffices to show that $\langle r_x \cdot \alpha, x\rangle \leq -1$.

Minimality of α forces $x = a$, or $x = y_1$, y_1, or $x = b_j$ for some $j \in \{1, \ldots, N\}$. If $x = a$, minimality of α together with (2.1)(ii) yield that the coefficient of a in $r_a \cdot \alpha$ equals

0 or 1. In the former case, it follows easily that $\langle r_a \cdot \alpha, a\rangle \leq -1$. If the coefficient of a in $r_a \cdot \alpha$ equals 1, an easy induction on depth yields that the coefficient of b_1 in α is an integer multiple of c_m, and thus greater than or equal to $2c_m$; whence again, $\langle r_a \cdot \alpha, a\rangle \leq -1$.

If $x = y_1$, minimality of α yields that $y_1 \notin \operatorname{supp}(r_{y_1} \cdot \alpha)$. Since the coefficient of b_N in $r_{y_1} \cdot \alpha$ is greater than or equal to 2, we deduce that $\langle r_{y_1} \cdot \alpha, y_1\rangle \leq -1$. Symmetrical arguments apply if $x = y_2$.

Finally, if $x = b_j$ for some $j \in \{1, \dots, N\}$, minimality of α together with (2.1)(ii) yield that the coefficient μ_j of b_j in $r_{b_j} \cdot \alpha$ equals 0, 1 or c_m. If $\mu_j \leq 1$, it can be easily seen that $\langle r_{b_j} \cdot \alpha, b_j\rangle \leq -1$, so suppose that $\mu_j = c_m$. If $j < N$, the coefficient of b_{j+1} in $r_{b_j} \cdot \alpha$ is an integer multiple of c_m by (5.5), and thus greater than or equal to $2c_m$; if $j = N$, the coefficients of y_1 and y_2 are integer multiples of c_m by (5.5), and we can deduce in both cases that $\langle r_{b_j} \cdot \alpha, b_j\rangle \leq -1$. □

The above shows that for S of shape (2) each element of $\mathcal{E}_S$ has coefficient 1 for a, or coefficient c_m for some b_j. Propositions (5.3), (5.4) and (5.5) enable us to describe the roots with coefficient c_m for some b_j, and we will now describe the set of roots in Φ_S with coefficient 1 for a. Note that $\{z\} \cup \big(S \setminus \{x\}\big)$ defined below contains only simple bonds, therefore $\mathcal{E}^{\text{indec}}_{\{z\}\cup(S\setminus\{x\})}$ has already been determined.

(5.7) Proposition *Since $\mathcal{E}_S$ does not depend on Π as long as $S \subseteq \Pi$, we may assume without loss of generality that there exists a $z \in \Pi \setminus S$ such that for $a_r \in S$,*

$$m_{r_z r} = m_{r r_z} = \begin{cases} 3 & \text{if } r = r_{b_1}, \\ 2 & \text{if } r \neq r_{b_1}. \end{cases}$$

Then $\phi\colon \beta \mapsto a + c_m(\beta - z)$ defines a bijection from the set of roots in $\Phi^+_{\{z\}\cup(S\setminus\{a\})}$ with coefficient 1 for z, to the set of roots in Φ^+_S with coefficient 1 for a. Moreover, ϕ restricts to a bijection from the set of roots in $\mathcal{E}_{\{z\}\cup(S\setminus\{a\})}$ with coefficient 1 for z, to the set of roots in $\mathcal{E}_S$ with coefficient 1 for a.

Proof. An easy calculation yields for $\beta \in \Phi$ and $y \in S \setminus \{a\}$ that

$$r_y \cdot \big(a + c_m(\beta - z)\big) = a + c_m(r_y \cdot \beta - z). \qquad (*)$$

Now let $\beta \in \Phi^+_{\{z\}\cup(S\setminus\{a\}}$ with coefficient 1 for z, and in accordance with (3.5) let $w \in W_{S\setminus\{a\}}$ be of length $\operatorname{dp}(\beta) - 1$ such that $\beta = w \cdot z$. The coefficient of a in $w \cdot a$ equals 1, and since $a = a + c_m(z - z)$, a straightforward induction on $l(w)$ using $(*)$ yields that $w \cdot a = a + c_m(w \cdot z - z)$; hence ϕ is well defined. By (3.5), every element of Φ^+_S with coefficient 1 for a can be written as $w \cdot a$ for some $w \in W_{S\setminus\{a\}}$, so the above also shows that ϕ is surjective. Since ϕ is certainly injective, ϕ is a bijection. As in the proof of (5.5), it follows readily that ϕ restricts to a bijection from the set of roots in $\mathcal{E}_{\{z\}\cup(S\setminus\{a\})}$ with coefficient 1 for z to the set of roots in $\mathcal{E}_S$ with coefficient 1 for a. □

(5.8) Proposition *Suppose that S is of shape (2) with $m \geq 7$. Then the elements of $\mathcal{E}_S^{\text{indec}}$ are as follows:*

(i) $a + c_m(\beta - z)$, *where* $\beta \in \mathcal{E}_{\{z\}\cup(S\setminus\{a\})}$ *with coefficient 1 for* z *(and* z *as in (5.7)), and*

(ii) $(c_m^2 - 1)a + c_m\gamma$, *where* $\gamma \in \mathcal{E}_{S\setminus\{a\}}$ *with coefficient 1 for* b_1.

(5.9) Proposition *Suppose that S is of shape (2) with $m = 6$. Then the elements of $\mathcal{E}_S^{\text{indec}}$ are as follows:*

(i) $a + c_m(\beta - z)$, *where* $\beta \in \mathcal{E}_{\{z\}\cup(S\setminus\{a\})}$ *with coefficient 1 for* z *(and* z *as in (5.7)), and*

(ii) $2a + \sqrt{3}(2(b_1 + \cdots + b_{n-1}) + \beta_n)$,
where $n \in \{1, \ldots, N\}$ *and* $\beta_n \in \mathcal{E}_{\{b_n,\ldots,b_N\}\cup B}$ *with coefficient 1 for* b_n.

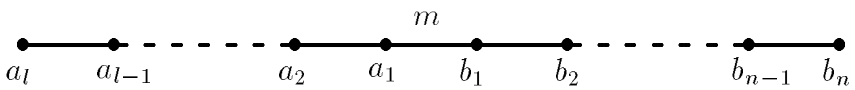
§6 One non-simple bond of weight $m \in \{4, 5\}$

Addressing now the final case, we assume for the duration of this section that $S \subseteq \Pi$ is a tree, and contains exactly one non-simple bond which has weight $m \in \{4, 5\}$. Let a and b be the vertices of the non-simple bond, and denote $2\cos(\frac{\pi}{m})$ by c_m.

(6.1) Definition For $\alpha \in \Phi_S$, define $\mathbf{C}_c(\alpha)$ to be the set of $x \in \Pi$ with coefficient c_m in α. If $\mathbf{C}_c(\alpha) \cap \mathcal{D}(\operatorname{supp}(\alpha))$ is empty, α is said to be *skew-indecomposable*, otherwise α is said to be *skew-decomposable*. The set of skew-indecomposable roots in $\mathcal{E}_S^{\text{indec}}$ is denoted by $\mathcal{E}_S^{\text{skew-ind}}$.

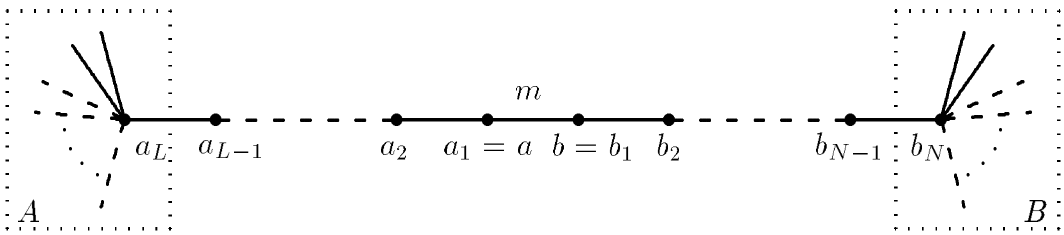
Proposition (5.5) together with Lemma (5.6) yield the next three propositions, which reduce the problem to the determination of $\mathcal{E}_S^{\text{skew-ind}}$ for S of the following shape:

(3)



with $m \in \{4, 5\}$. In fact, (6.2), (6.3) and (6.4) are valid for all $m \geq 4$; we chose to treat the case $m \geq 6$ separately because the statements simplify significantly in that case.

(6.2) Proposition *Suppose that S is of the following shape*

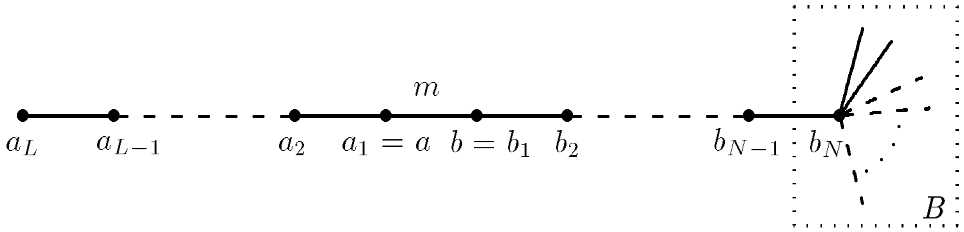


where A and B are trees which contain only simple bonds, such that $\mathrm{val}_A(a_L)$, $\mathrm{val}_B(b_N) \geq 2$. Then the elements of $\mathcal{E}_S^{\text{indec}}$ are

$$c_m\alpha_l - c_m a_l + \gamma_{l,n} + c_m\beta_n - c_m b_n,$$

where $l \in \{1, \ldots, L\}$ and $\alpha_l \in \mathcal{E}_{A\cup\{a_L,\ldots,a_l\}}$ with coefficient 1 for a_l, while $n \in \{1, \ldots, N\}$ and $\beta_n \in \mathcal{E}_{\{b_n,\ldots,b_N\}\cup B}$ with coefficient 1 for b_n, and further, $\gamma_{l,n} \in \mathcal{E}^{\text{skew-ind}}_{\{a_l,\ldots,a_1,b_1,\ldots,b_n\}}$ with coefficient c_m for a_l and b_n.

(6.3) PROPOSITION *Suppose that S is of the following shape:*



where B is a tree which only contains simple bonds, such that $\mathrm{val}_B(b_N) \geq 2$.

(1) If $L = 1$, the elements of $\mathcal{E}_S^{\text{indec}}$ are

(i) $\gamma_{1,n} + c_m\beta_n - c_m b_n$,
where $n \in \{1, \ldots, N\}$ and $\beta_n \in \mathcal{E}_{\{b_n,\ldots,b_N\}\cup B}$ with coefficient 1 for b_n, while $\gamma_{1,n} \in \mathcal{E}^{\text{skew-indec}}_{\{a,b_1,\ldots,b_n\}}$ with coefficient c_m for b_n, and

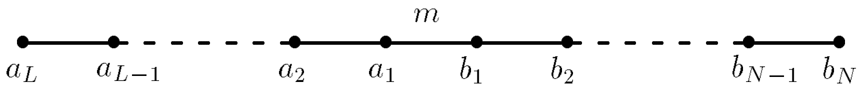
(ii) $a + c_m(\delta - z)$, where $\delta \in \mathcal{E}_{\{z\}\cup(S\setminus\{a\})}$ with coefficient 1 for z (where z as in (5.7)).

(2) If $L \geq 2$, the elements of $\mathcal{E}_S^{\text{indec}}$ are

$$c_m(a_L + \cdots + a_{l+1}) + \gamma_{l,n} + c_m\beta_n - c_m b_n,$$

where $l \in \{1, \ldots, L\}$, $n \in \{1, \ldots, N\}$ and $\beta_n \in \mathcal{E}_{\{b_n,\ldots,b_N\}\cup B}$ with coefficient 1 for b_n, while $\gamma_{l,n} \in \mathcal{E}^{\text{skew-ind}}_{\{a_l,\ldots,a_1,b_1,\ldots,b_n\}}$ with coefficient c_m for b_n and if $l < L$, coefficient c_m for a_l.

(6.4) PROPOSITION *Suppose that S is of the following shape:*



Then the elements of $\mathcal{E}_S^{\text{indec}}$ are

$$c_m(a_L + \cdots + a_{l+1}) + \gamma_{l,n} + c_m(b_{n+1} + \cdots + b_N),$$

where $l \in \{1, \ldots, L\}$ and $n \in \{1, \ldots, N\}$, while $\gamma_{l,n} \in \mathcal{E}^{\text{skew-ind}}_{\{a_l,\ldots,a_1,b_1,\ldots,b_n\}}$ with coefficient c_m for a_l if $l < L$, and coefficient c_m for b_n if $n < N$.

Suppose first that S is of shape (3) with $m = 4$. We proved in §5 that each element of $\mathcal{E}_S$ is preceded by $a_1 + \sqrt{2}b_1$ or $\sqrt{2}a_1 + b_1$. If $\alpha \in \Phi_S$ is preceded by $a_1 + \sqrt{2}b_1$, an easy induction on depth yields that the coefficients of the a_i in α are integers, and the coefficients of the b_j are integer multiples of $\sqrt{2}$. As in §3, it follows readily from (2.2)(iii),(iv) that if $\alpha = \sum_{x \in S} \lambda_x x$ and $\beta = \sum_{x \in S} \mu_x x$ are elementary roots preceded by $a_1 + \sqrt{2} b_1$ with $\mu_x \le \lambda_x$ for all $x \in S$, then $\beta \preceq \alpha$. In particular, each element of $\mathcal{E}_S^{\text{skew-ind}}$ which is preceded by $a_1 + \sqrt{2} b_1$ is preceded by $a_l + 2(a_{l-1} + \cdots + a_1) + \sqrt{2}\big(2(b_1 + \cdots + b_{n-1}) + b_n\big)$. A straightforward induction on depth now yields the following result:

(6.5) PROPOSITION *Suppose that S is of shape (3) with $m = 4$.*

(1) If $l, n \ge 3$, the elements of $\mathcal{E}_S^{\text{skew-ind}}$ are

$$a_l + 2(a_{l-1} + \cdots + a_1) \ + \ \sqrt{2}\big(2(b_1 + \cdots + b_{n-1}) + b_n\big) \text{ and}$$
$$\sqrt{2}\big(a_l + 2(a_{l-1} + \cdots + a_1)\big) + \qquad 2(b_1 + \cdots + b_{n-1}) + b_n.$$

(2) If $l = 2$ and $n \ge 3$, the elements of $\mathcal{E}_S^{\text{skew-ind}}$ are as follows:

(i) $a_2 + Ma + \sqrt{2}\big(M(b_1 + \cdots + b_{j(M-1)-1}) + (M-1)(b_{j(M-1)} + \cdots + b_{j(M-2)-1}) + \cdots$
$\cdots + 3(b_{j(3)} + \cdots + b_{j(2)-1}) + 2(b_{j(2)} + \cdots + b_{n-1}) + b_n\big)$
where $M \in \{2, \ldots, n+1\}$ and $0 < j(M-1) < \cdots < j(2) < n$,

(ii) $2a_2 + 3a + \sqrt{2}\big(3(b_1 + \cdots + b_{j-1}] + 2(b_j + \cdots + b_{n-1}) + b_n\big)$, *where $0 < j < n$, and*

(iii) $\sqrt{2}(a_2 + 2a_1) + 2(b_1 + \cdots + b_{n-1}) + b_n$.

(6.6) COROLLARY

(1) Suppose that S is of the shape described in (6.2) with $m = 4$. Then $\mathcal{E}_S^{\text{indec}}$ is empty.

(2) Suppose that S is of the shape described in (6.3) with $L \ge 3$ and $m = 4$. Then the elements of $\mathcal{E}_S^{\text{indec}}$ are

$$a_L + 2(a_{L-1} + \cdots + a_1) + \sqrt{2}\big(2(b_1 + \cdots + b_{n-1}) + \beta_n\big),$$

where $n \in \{1, \ldots, N\}$ and $\beta_n \in \mathcal{E}_{\{b_n, \ldots, b_N\} \cup B}$ with coefficient 1 for b_n.

(3) Suppose that S is of the shape described in (6.4) with $L, N \ge 3$ and $m = 4$. Then the elements of $\mathcal{E}_S^{\text{indec}}$ are as follows:

(i) $a_L + 2(a_{L-1} + \cdots + a_1) + \sqrt{2}\big(2(b_1 + \cdots + b_{n-1}) + (b_n + \cdots + b_N)\big)$,
where $n \in \{1, \ldots, N\}$ and

(ii) $\sqrt{2}\big((a_L + \cdots + a_l) + 2(a_{l-1} + \cdots + a_1)\big) + 2(b_1 + \cdots + b_{N-1}) + b_N$, *where $l \in \{1, \ldots, L\}$.*

Finally suppose that S is of shape (3) with $m = 5$. Proposition (2.1)(ii) yields that for $\alpha \in \Phi_S^+$, each coefficient of α is in $\mathbb{N}_0 + c_5 \mathbb{N}_0$; that is, equals $\lambda c_5 + \mu$ with $\lambda, \mu \in \mathbb{N}_0$. Note that the smallest possible nonnegative coefficients are: $0, 1, c_5, 2, c_5 + 1$.

(6.7) PROPOSITION *Suppose that S is of shape (3) with $m = 5$ and $l, n \ge 2$. Then $\mathcal{E}_S^{\text{skew-ind}} = \emptyset$.*

Proof. As in the proof of (3.1), we assume that α is a minimal root with coefficients for a_1, b_1 greater than or equal to 2, and a_2, $b_2 \in \operatorname{supp}(\alpha)$. Let $x \in S$ such that $r_x \cdot \alpha \prec \alpha$; we will show that $\langle r_x \cdot \alpha, x\rangle \leq -1$.

Denote the coefficient of a_i in α by λ_i, and the coefficient of b_j in α by μ_j. By minimality of α, we deduce that $x \in \{a_2, a_1, b_1, b_2\}$. If $x = a_2$, minimality of α yields that the coefficient of a_2 in $r_{a_2} \cdot \alpha$ equals 0, and since $\lambda_1 \geq 2$, it follows that $\langle r_{a_2} \cdot \alpha, a_2\rangle \leq -1$. Similarly, $\langle r_{b_2} \cdot \alpha, b_2\rangle \leq -1$ if $x = b_2$.

So we are left with $x \in \{a_1, b_1\}$, and we may assume without loss of generality that $x = a_1$. By minimality of α together with (2.1)(ii), the coefficient of a_1 in $r_{a_1} \cdot \alpha$ equals 0, 1 or c_5. If the coefficient of a_1 in $r_{a_1} \cdot \alpha$ is less than or equal to 1, then

$$\langle r_{a_1} \cdot \alpha, a_1\rangle \leq 1 + \lambda_2\Big(-\frac{1}{2}\Big) + \mu_1\Big(-\frac{c_5}{2}\Big) \leq 1 + \Big(-\frac{1}{2}\Big) + 2\Big(-\frac{c_5}{2}\Big) \leq -1,$$

as required.

Suppose now that the coefficient of a_1 in $r_{a_1} \cdot \alpha$ equals c_5, and assume for a contradiction that the coefficient of b in α equals 2. Let $\beta \preceq r_a \cdot \alpha$ be minimal such that the coefficient of a in β equals c_5, and the coefficient of b in β equals 2. Then $r_a \cdot \beta \prec \beta$ or $r_b \cdot \beta \prec \beta$ by minimality of β; but $\langle \beta, a\rangle \leq c_5 + \big(-\frac{c_5}{2}\big)2 = 0$, and thus $r_b \cdot \beta \prec \beta$. Denote the coefficient of b_2 in β by μ_2', and the coefficient of b_1 in $r_{b_1} \cdot \beta$ by μ_1'; then

$$\mu_1' = -2 - 2\Big(-\frac{c_5}{2}\Big)c_5 - 2\Big(-\frac{1}{2}\Big)\mu_2' = c_5 - 1 + \mu_2'.$$

This has to be less than 2 since $r_{b_1} \cdot \beta \prec \beta$, and so by (2.1)(ii) it must be equal to 0, 1 or c_5. Since $\mu_2' \in \mathbb{N}_0 + c_5\mathbb{N}_0$, we deduce that $\mu_2' = 1$ and $\mu_1' = c_5$, contradicting Corollary (3.6)(i). Therefore the coefficient of b in α is greater than 2, and thus greater than or equal to $c_5 + 1$. Hence

$$\langle r_a \cdot \alpha, a\rangle \leq c_5 + \Big(-\frac{1}{2}\Big)1 + \Big(-\frac{c_5}{2}\Big)(c_5 + 1) \leq -1,$$

as required. □

This leaves us to determine $\mathcal{E}_S^{\text{skew-ind}}$ for S of shape (3) with $l = 1$, $m = 5$ and $n \geq 4$.

(6.8) LEMMA *Suppose that S is of shape (3) with $m = 5$, $l = 1$ and $n \geq 3$. Then for each $\alpha \in \mathcal{E}_S^{\text{skew-ind}}$, there exists a $\gamma \in \mathcal{E}_{\{a_1,b_1,b_2,b_3\}}^{\text{skew-ind}}$ with $\gamma \preceq \alpha$. (Note that if $n \geq 4$, the coefficient of b_3 in γ has to be 1 or c_5.)*

Proof. If $n = 3$, the assertion is trivially true. Suppose now that $n \geq 4$, and let $\beta \preceq \alpha$ be minimal such that $\beta \in \mathcal{E}_S^{\text{skew-ind}}$. Then $r_x \cdot \beta \prec \beta$ for some $x \in S$. If $x = a$, minimality of β yields that $a_1 \notin \operatorname{supp}(r_{a_1} \cdot \beta)$; since the coefficient of b_1 in β is greater than or equal to 2, we find that $\langle r_a \cdot \beta, a_1\rangle \leq -1$, contradicting (2.2)(iii), (iv). Similarly $x \neq b_n$.

So $x = b_j$ for some $j \in \{1, \ldots, n-1\}$. Denote the coefficient of b_j in $r_{b_j} \cdot \beta$ by μ_j; then $\mu_j \in \{1, c_5\}$ by minimality of β and connectedness of the support of $r_{b_j} \cdot \beta$. If $\mu_j = 1$, Proposition (3.4) yields that the coefficient of b_{j+1} in β equals 1, while if $\mu_j = c_5$, Proposition (5.5) yields that the coefficient of b_{j+1} in β equals c_5. Since $\beta \in \mathcal{E}_S^{\text{skew-ind}}$, the coefficient of b_{j+1} in β has to be greater than or equal to 2 if $j+1 < n$; whence $j+1 = n$. Now the coefficients of b_{n-1} and b_n in $r_{b_{n-1}} \cdot \beta$ coincide, and it follows that $r_{b_n} r_{b_{n-1}} \cdot \beta$ is an element of $\mathcal{E}_{S \setminus \{b_n\}}^{\text{skew-ind}}$. By the inductive hypothesis, $r_{b_n} r_{b_{n-1}} \cdot \beta$ is preceded by a root $\gamma \in \mathcal{E}_{\{a_1, b_1, b_2, b_3\}}^{\text{skew-ind}}$, and therefore $\gamma \preceq \alpha$, as required. □

An induction on n similar to the one just demonstrated, together with an induction on depth yield the next assertion.

(6.9) Proposition *Suppose that S is of shape (3) with $m = 5$, $l = 1$ and $n \geq 4$. Then the elements of $\mathcal{E}_S^{\text{skew-ind}}$ are as follows:*

(i) $c_5 a_1 + (c_5+1)(b_1 + \cdots + b_{n-1}) + b_n$ *and* $c_5 a_1 + (c_5+1)(b_1 + \cdots + b_{n-1}) + c_5 b_n$,

(ii) $(c_5+1)a_1 + (2c_5+1)(b_1 + \cdots + b_{j-1}) + (c_5+1)(b_j + \cdots + b_{n-1}) + b_n$,
$(c_5+1)a_1 + (2c_5+1)(b_1 + \cdots + b_{j-1}) + (c_5+1)(b_j + \cdots + b_{n-1}) + c_5 b_n$,
$(c_5+1)a_1 + (2c_5+1)(b_1 + \cdots + b_{j-1}) + 2c_5(b_j + \cdots + b_{n-1}) + c_5 b_n$,
where $1 \leq j \leq n$,

(iii) $(2c_5+1)a_1 + (2c_5+1)(b_1 + \cdots + b_{j-1}) + (c_5+1)(b_j + \cdots + b_{n-1}) + b_n$,
$(2c_5+1)a_1 + (2c_5+1)(b_1 + \cdots + b_{j-1}) + (c_5+1)(b_j + \cdots + b_{n-1}) + c_5 b_n$,
$(2c_5+1)a_1 + (2c_5+1)(b_1 + \cdots + b_{j-1}) + 2c_5(b_j + \cdots + b_{n-1}) + c_5 b_n$,
where $2 \leq j \leq n$, *and*

(iv) $(2c_5+1)a_1 + (2c_5+2)b_1 + (c_5+1)(b_2 + \cdots + b_{n-1}) + b_n$,
$(2c_5+1)a_1 + (2c_5+2)b_1 + (c_5+1)(b_2 + \cdots + b_{n-1}) + c_5 b_n$,
$(2c_5+1)a_1 + (3c_5+1)b_1 + 2c_5(b_2 + \cdots + b_{n-1}) + c_5 b_n$,
$(2c_5+2)a_1 + (3c_5+1)b_1 + 2c_5(b_2 + \cdots + b_{n-1}) + c_5 b_n$.

(6.10) Corollary

(1) Suppose that S is of the shape described in (6.2) with $m = 5$. Then the elements of $\mathcal{E}_S^{\text{indec}}$ are as follows:

(i) $c_5 \alpha + (c_5+1)(b_1 + \cdots + b_{n-1}) + c_5 \beta_n$,
where $\alpha \in \mathcal{E}_{A \cup \{a_L, \ldots, a_1\}}$ with coefficient 1 for a_1, while $n \in \{1, \ldots, N\}$ and β_n is in $\mathcal{E}_{\{b_n, \ldots, b_N\} \cup B}$ with coefficient 1 for b_n, and

(ii) $c_5 \alpha_l + (c_5+1)(a_{l-1} + \cdots + a_1) + c_5 \beta$,
where $l \in \{1, \ldots, L\}$ and $\alpha_l \in \mathcal{E}_{A \cup \{a_L, \ldots, a_l\}}$ with coefficient 1 for a_l, while β is in $\mathcal{E}_{\{b_1, \ldots, b_N\} \cup B}$ with coefficient 1 for b_1.

(2) Suppose that S is of the shape described in (6.3) with $m = 5$ and $L \geq 2$. Then the elements of $\mathcal{E}_S^{\text{indec}}$ are as follows:

(i) $c_m(a_L + \cdots + a_1) + (c_5 + 1)(b_1 + \cdots + b_{n-1}) + c_m\beta_n$,
where $n \in \{1, \ldots, N\}$ *and* $\beta_n \in \mathcal{E}_{\{b_n,\ldots,b_N\}\cup B}$ *with coefficient 1 for* b_n,

(ii) $a_L + (c_5 + 1)(a_{L-1} + \cdots + a_1) + c_m\beta$,
where $\beta \in \mathcal{E}_{\{b_1,\ldots,b_N\}\cup B}$ *with coefficient 1 for* b_1, *and*

(iii) $c_m(a_L + \cdots + a_{l+1}) + (c_5 + 1)(a_l + \cdots + a_1) + c_m\beta$,
where $l \in \{1, \ldots, L\}$ *and* $\beta \in \mathcal{E}_{\{b_1,\ldots,b_N\}\cup B}$ *with coefficient 1 for* b_1.

(3) *Suppose that* S *is of the shape described in (6.4) with* $m = 5$ *and* $L, N \geq 2$. *Then the elements of* $\mathcal{E}_S^{\mathrm{indec}}$ *are as follows:*

(i) $c_m(a_L + \cdots + a_1) + (c_5 + 1)(b_1 + \cdots + b_{N-1}) + b_N$,

(ii) $c_m(a_L+\cdots+a_1)+(c_5+1)(b_1+\cdots+b_{n-1})+c_5(b_n+\cdots+b_N)$, *where* $n \in \{1, \ldots, N\}$,

(iii) $a_L + (c_5 + 1)(a_{L-1} + \cdots + a_1) + c_5(b_1 + \cdots + b_N)$,

(iv) $c_m(a_1+\cdots+a_l)+(c_5+1)(a_{l-1}+\cdots+a_1)+c_5(b_1+\cdots+b_N)$, *where* $l \in \{1, \ldots, L\}$.

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