ON CENTRALIZERS OF REFLECTIONS IN COXETER GROUPS

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Abstract

We show that the centralizer of a reflection in a Coxeter group is the semidirect product of a Coxeter group by a free group.

1. *Summary of background material*

Let (W, R) be a Coxeter system, as defined in [1] or [5], and for $r, s \in R$ let m_{rs} be the order of rs. Further, let V be an $\mathbb{R}$-vector space with basis $\Pi = \{a_r \mid r \in R\}$, and let $\langle\ ,\ \rangle: V \times V \to \mathbb{R}$ be a symmetrical bilinear form which satisfies $\langle a_r, a_s\rangle = -\cos(\pi/m_{rs})$ for all $r, s \in R$ such that m_{rs} is finite, and $\langle a_r, a_s\rangle \leq -1$ if m_{rs} is infinite. Then $r \cdot v = v - 2\langle v, a_r\rangle a_r$ (for all $r \in R$ and $v \in V$) defines a faithful action of W on V. The set $\Phi = \{w \cdot a_r \mid r \in R,\ w \in W\}$ is the *root system* of W in V, and the elements of Π are called *simple roots*. A root α is said to be *positive* (*negative*), if it can be written as $\sum_{a\in\Pi} \lambda_a a$ with all $\lambda_a \geq 0$ (all $\lambda_a \leq 0$). The sets of positive and negative roots are denoted by Φ^+ and Φ^- respectively, and it is well known that $\Phi = \Phi^+ \cup \Phi^-$.

Elements of W which are conjugate to elements of R are called *reflections*. For each $\alpha \in \Phi$, there is a reflection $r_\alpha \in W$ which satisfies $r_\alpha \cdot v = v - 2\langle \alpha, v\rangle v$ for all $v \in V$. Furthermore, $r_\alpha = wrw^{-1}$ whenever $r \in R$ and $w \in W$ satisfy $\alpha = w \cdot \alpha_r$.

For $w \in W$, the *length* $l(w)$ is defined to be the minimal integer n such that w equals $r_1 \cdots r_n$ for some $r_i \in R$. If $w \neq 1$, it is clear that there exists an $r \in R$ with $l(wr) = l(w) - 1$. We also define $N(w)$ to be the set of all $\alpha \in \Phi^+$ such that $w \cdot \alpha \in \Phi^-$.

The following proposition summarizes some well known properties.

Proposition 1. (i) *Let α and β be roots, and for each $n \in \mathbb{Z}$, let $\lambda_n, \mu_n \in \mathbb{R}$ be such that $(r_\alpha r_\beta)^n \cdot \alpha = \lambda_n \alpha + \mu_n \beta$. Then $(r_\alpha r_\beta)^n \cdot \beta = -\mu_n \alpha - \lambda_{n-1}\beta$ for all $n \in \mathbb{Z}$, and*

(a) $\lambda_n = \sin(\frac{(2n+1)\pi}{m}) / \sin(\frac{\pi}{m})$ *and* $\mu_n = \sin(\frac{2n\pi}{m}) / \sin(\frac{\pi}{m})$ *if* $\langle \alpha, \beta\rangle = -\cos(\frac{\pi}{m})$;

(b) $\lambda_n - 1 \geq \mu_n \geq \lambda_{n-1} + 1$ *if* $\langle \alpha, \beta\rangle \leq -1$.

(ii) *Let $\alpha \in \Phi^+$ and $w \in W$. Then $l(wr_\alpha) < l(w)$ if and only if $w \cdot \alpha \in \Phi^-$.*

(iii) *If $w \in W$, then $|N(w)| = l(w)$ and $N(w^{-1}) = -w \cdot N(w)$.*

(iv) *If* $u, w \in W$, *then* $N(uw) \subseteq N(w) \cup w^{-1} \cdot N(u)$, *and the following are equivalent:*

(a) $l(uw) = l(u) + l(w)$,

(b) $N(uw) = N(w) \cup w^{-1} \cdot N(u)$,

(c) $N(u) \cap N(w^{-1}) = \emptyset$.

For $\Gamma \subseteq \Phi$, define W_Γ to be the group generated by r_γ with $\gamma \in \Gamma$, and Φ_Γ to be the set of $w \cdot \gamma$ with $\gamma \in \Gamma$ and $w \in W_\Gamma$. The next theorem was proved independently by Deodhar in [3] and Dyer in [4].

THEOREM 2. *Let* $\Gamma \subseteq \Phi$. *Then* W_Γ *is a Coxeter group, and* Φ_Γ *is a root system for* W_Γ; *in particular, if* $w \in W_\Gamma$ *with* $\Phi_\Gamma \cap N(w) = \emptyset$, *then* $w = 1$.

2. *The centralizer of a reflection*

In this section we will prove a structure theorem for the centralizer of a reflection. It is clear that the centralizer of the reflection wrw^{-1} equals $wC(r)w^{-1}$, where $C(r)$ is the centralizer of r (and $w \in W$, $r \in R$). Hence it suffices to describe the structure of $C(r)$ for $r \in R$.

For $r \in R$, define $\Gamma(r)$ to be the set of roots γ with $\langle \gamma, a_r \rangle = 0$, and for $r, s \in R$, let $Y_{s,r}$ be the set of $u \in W$ with $a_s = u \cdot a_r$ and $N(u) \cap \Gamma(r) = \emptyset$. It follows from Proposition 1(iii) that $w^{-1} \in Y_{r,s}$ for $w \in Y_{s,r}$, and from Proposition 1(iv) that $uw \in Y_{t,r}$ for $u \in Y_{t,s}$ and $w \in Y_{s,r}$. Since clearly $1 \in Y_{r,r}$, we deduce that $Y_{r,r}$ is a group.

We also define the *odd Coxeter graph* of W to be the graph with vertex set R and edge set $\big\{\{r, s\} \mid r, s \in R \text{ such that } r \neq s \text{ and } m_{rs} \text{ is odd}\big\}$.

THEOREM. *Let* $r \in R$. *Then the centralizer of* r *in* W *is the semidirect product of* $W_{\Gamma(r)\cup\{\pm a_r\}}$ *by* $Y_{r,r}$; *that is,*

$$C(r) = W_{\Gamma(r)\cup\{\pm a_r\}} \rtimes Y_{r,r}.$$

Moreover, $W_{\Gamma(r)\cup\{\pm a_r\}}$ *is a Coxeter group, and* $Y_{r,r}$ *is free of rank* $e(r) - n(r) + 1$, *where* $e(r)$ *denotes the number of edges and* $n(r)$ *the number of vertices of the connected component of the odd Coxeter graph of* W *containing* r.

Proof. Since $uru^{-1} = r$ for $u \in C(r)$, we find that

$$-a_r = r \cdot a_r = uru^{-1} \cdot a_r = r_{u \cdot a_r} \cdot a_r = a_r - 2\langle u \cdot a_r, a_r \rangle u \cdot a_r.$$

Thus $a_r = \langle u \cdot a_r\,,\, a_r\rangle u \cdot a_r$, and as $\langle \alpha, \alpha\rangle = 1$ for $\alpha \in \Phi$, we deduce that $u \cdot a_r = \pm a_r$. So if $\gamma \in \Gamma(r)$, then $\langle u \cdot \gamma, a_r\rangle = \langle \gamma, u^{-1} \cdot a_r\rangle = \langle \gamma, \pm a_r\rangle = 0$. Therefore

$$u \cdot \Gamma(r) = \Gamma(r) \text{ and } u \cdot \{\pm a_r\} = \{\pm a_r\},$$

and it follows readily that $W_{\Gamma(r)\cup\{\pm a_r\}}$ is normal in $C(r)$. The above also yields that

$$\Phi_{\Gamma(r)\cup\{\pm a_r\}} = \Gamma(r) \cup \{\pm a_r\}; \tag{$*$}$$

hence $W_{\Gamma(r)\cup\{\pm a_r\}}$ is a Coxeter group with root system $\Gamma(r) \cup \{\pm a_r\}$ by Theorem 2. It remains to show that $Y_{r,r}$ is a complement of $W_{\Gamma(r)\cup\{\pm a_r\}}$ in $C(r)$, and that $Y_{r,r}$ is free of rank $e(r) - n(r) + 1$.

Let X_r be a set of coset representatives of $W_{\Gamma(r)\cup\{\pm a_r\}}$ in $C(r)$ of minimal length. For $u \in X_r$, the minimality of $l(u)$ yields that $l(ur) \geq l(u)$; so $u \cdot a_r \in \Phi^+$ by Proposition 1(ii), and hence $u \cdot a_r = a_r$ (as $u \cdot a_r = \pm a_r$ by the discussion above). Minimality of $l(u)$ yields further that $l(ur_\gamma) \geq l(u)$ for $\gamma \in \Gamma(r)$; so if γ is positive, then $u \cdot \gamma \in \Phi^+$ by Proposition 1(ii). Hence $N(u) \cap \Gamma(r) = \emptyset$, and thus $X_r \subseteq Y_{r,r}$.

To show that $X_r = Y_{r,r}$, let $w \in Y_{r,r}$, and let $u \in X_r$ be the representative of the coset $wW_{\Gamma(r)\cup\{\pm a_r\}}$. Then $u \in Y_{r,r}$, and thus $w^{-1}u \in Y_{r,r}$; that is, $(w^{-1}u) \cdot a_r = a_r$ and $\Gamma(r) \cap N(w^{-1}u) = \emptyset$. Thus by $(*)$,

$$\Phi_{\Gamma(r)\cup\{\pm a_r\}} \cap N(u^{-1}w) = \big(\Gamma(r) \cap N(u^{-1}w)\big) \cup \big(\{\pm a_r\} \cap N(u^{-1}w)\big) = \emptyset.$$

As $w^{-1}u \in W_{\Gamma(r)\cup\{\pm a_r\}}$ by construction, Theorem 2 now yields that $w^{-1}u = 1$. Hence $X_r = Y_{r,r}$, and it remains to show that $Y_{r,r}$ is free of rank $e(r) - n(r) + 1$.

It is clear that the elements of the fundamental group of a connected graph Ξ can be identified with paths in Ξ which start and end at a fixed vertex, and which never back-track upon themselves; that is, at no stage does the path traverse an edge and then immediately traverse it again in the opposite direction. It is well known (see, for example, [6], Chapter 6, Theorem (5.2), p.198) that this fundamental group is free of rank $\epsilon - \nu + 1$, where ϵ is the number of edges, and ν is the number of vertices of Ξ. (This follows from the fact that Ξ is homotopy equivalent to a graph on one vertex with $\epsilon - \nu + 1$ edges - topologically a bouquet of circles - as can be seen by shrinking a spanning tree of Ξ to a single vertex.)

Suppose now that the graph Ξ is the connected component of the odd Coxeter graph containing the vertex r. Define $\mathcal{F}$ to be the set of all $(l+1)$-tuples $(r_0, \ldots, r_l)$ with $r_0, \ldots, r_l \in R$ such that r_{i-1} and r_i are adjoined in the odd Coxeter graph for all $i \in \{1, \ldots, l\}$, and $r_{i-1} \neq r_{i+1}$ for all $i \in \{1, \ldots, l-1\}$. Further, let $\mathcal{F}_r$ be the set of

$(r_0, \ldots, r_l) \in \mathcal{F}$ with $r_0 = r_l = r$. The fundamental group of Ξ can be identified with $\mathcal{F}_r$, multiplication on $\mathcal{F}_r$ being defined by the rule

$$(r_0, \ldots, r_l) * (s_0, \ldots, s_m) = (r_0, \ldots, r_{l-i-1}, s_i, \ldots, s_m),$$

where i is the maximal integer such that $s_j = r_{l-j}$ for all $j \in \{0, \ldots, i\}$. Note that the identity of this group is (r). We will now construct an isomorphism $\pi_r \colon \mathcal{F}_r \to Y_{r,r}$.

If s, $t \in R$ are adjoined by a bond of weight $2n+1$, define $\pi(s,t) = (ts)^n$. Then $s\pi(s,t)$ is the uniquely determined word in $W_{\{a_s,a_t\}}$ of maximal length; it follows from Proposition 1(i)(a) that $N\big(\pi(s,t)\big) = \Phi^+_{\{a_s,a_t\}} \setminus \{a_t\}$ and that $\pi(s,t) \in Y_{s,t}$. Define $\pi \colon \mathcal{F} \to W$ by $\pi(r_0) = 1$, and for $l \geq 1$,

$$\pi(r_0, \ldots, r_l) = \pi(r_0, r_1)\pi(r_1, r_2)\pi(r_2, r_3) \cdots \pi(r_{l-2}, r_{l-1})\pi(r_{l-1},\, r_l).$$

Then $\pi(r_0, \ldots, r_l) \in Y_{r_0, r_l}$, since $\pi(r_{i-1}, r_i) \in Y_{r_{i-1}, r_i}$ for all i, and we conclude that π induces a homomorphism $\pi_r \colon \mathcal{F}_r \to Y_{r,r}$. The next two propositions yield that π_r is surjective and injective respectively, and therefore a group isomorphism, and this finishes the proof. $\square$

PROPOSITION. *Let* r, $s \in R$ *and* $w \in Y_{s,r}$. *Then there exists an* $(r_0, \ldots, r_l) \in \mathcal{F}$ *with* $r_0 = s$ *and* $r_l = r$ *such that* $w = \pi(r_0, \ldots, r_l)$.

Proof. If $w = 1$, the assertion is trivially true. So assume that $w \neq 1$, and proceed by induction on $l(w)$. Let $t \in R$ with $l(wt) < l(w)$; then $t \neq r$ by Proposition 1(ii) as $w \cdot a_r = a_s$ is positive. Next, let $d \in W$ be of minimal length such that $w = du$ for some $u \in W_{\{a_r,a_t\}}$. Since $w = (wt)t$, we know that $l(d) \leq l(wt) < l(w)$, and thus $u \neq 1$. Minimality of d forces $l(dr)$, $l(dt) \geq l(d)$, and thus $d \cdot a_r$, $d \cdot a_t \in \Phi^+$ by Proposition 1(ii). Further, let $\lambda, \mu \in \mathbb{R}$ such that $u \cdot a_r = \lambda a_r + \mu a_t$; then λ, $\mu \geq 0$ or λ, $\mu \leq 0$ by Proposition 1(i). Now

$$a_s = w \cdot a_r = d \cdot (u \cdot a_r) = \lambda(d \cdot a_r) + \mu(d \cdot a_t),$$

and since a_s is simple, we deduce that $\lambda = 0$ and $a_s = d \cdot a_t$, or $\mu = 0$ and $a_s = d \cdot a_r$. So $u \cdot a_r$ equals a_t or a_r, and we denote $u \cdot a_r$ by a_z.

Next, let $x \in R$ with $l(ux) < l(u)$; that is, $u \cdot a_x \in \Phi^-$. Since $u \in W_{\{a_r,a_t\}}$ and $u \cdot a_r = a_z$ is positive, we find that $x = t$; that is, $u \cdot a_t \in \Phi^-$. Linear independence of a_r and a_t yields that $u \cdot a_r$ and $u \cdot a_t$ are linearly independent; in particular, $u \cdot a_t$ does not equal $-u \cdot a_r = -a_z$, and thus $(zu) \cdot a_t \in \Phi^-$. Since $(zu) \cdot a_r = -a_z \in \Phi^-$, it follows that $N(zu)$ includes all roots of the form $\nu_r a_r + \nu_t a_t$ with $\nu_r,\ \nu_t \geq 0$. But $N(zu)$ is finite by Proposition 1(iii), and so Proposition 1(i)(b) implies that rt must have finite order. Furthermore, zu is the uniquely determined word in $W_{\{a_r,a_t\}}$ of maximal length.

Assume for a contradiction that the order of rt is even and equals $2m$. Then u equals $t(rt)^{m-1}$, and $(tr)^{\frac{m-1}{2}} \cdot a_t \in N(u)$ if m is odd, while $t(rt)^{\frac{m}{2}-1} \cdot a_r \in N(u)$ if m is even. It can be easily verified using Proposition 1(i) that these roots are perpendicular to a_r, and since $N(u) \subseteq N(w)$ by Proposition 1(iv), this contradicts $w \in Y_{s,r}$. Therefore rt must have odd order; moreover, $u = \pi(t,r)$ and $d \cdot a_t = d \cdot (u \cdot a_r) = w \cdot a_r = a_s$.

If $\beta \in N(d)$, then $\langle \beta, a_t \rangle = \langle u^{-1} \cdot \beta, u^{-1} \cdot a_t \rangle = \langle u^{-1} \cdot \beta, a_r \rangle$. Since $u^{-1} \cdot \beta \in N(w)$ by Proposition 1(iv), while $w \in Y_{s,r}$, we find that $\langle \beta, a_t \rangle \neq 0$, and thus $d \in Y_{s,t}$. By induction, there exists an $(r_0, \ldots, r_l) \in \mathcal{F}$ with $r_0 = s$ and $r_l = t$ such that d equals $\pi(r_0, \ldots, r_l)$. Since $l(dr) \geq l(d)$ by minimality of d, it follows that $r_{l-1} \neq r$ (if $l \geq 1$); hence $(r_0, \ldots, r_l, r) \in \mathcal{F}$ and $w = \pi(r_0, \ldots, r_l, r)$, as required. □

PROPOSITION. *Let* $(r_0, \ldots, r_l) \in \mathcal{F}$. *Then* $l\big(\pi(r_0, \ldots, r_l)\big) = \sum_{i=1}^{l} l\big(\pi(r_{i-1}, r_i)\big)$.

Proof. Assume for a contradiction that the assertion is not true, and let l be minimal such that there exists an $(r_0, \ldots, r_l) \in \mathcal{F}$ with $l\big(\pi(r_0, \ldots, r_l)\big) \neq \sum_{i=1}^{l} l\big(\pi(r_{i-1}, r_i)\big)$. Clearly $l \geq 2$, and we define $u_1 = \pi(r_0, r_1)$, $w = \pi(r_1, \ldots, r_{l-1})$ and $u_2 = \pi(r_{l-1}, r_l)$. Then $l(u_1 w) = \sum_{i=1}^{l-1} l\big(\pi(r_{i-1}, r_i)\big)$ by minimality of l, and our assumption becomes

$$l(u_1 w u_2) = l\big(\pi(r_0, \ldots, r_l)\big) \neq \sum_{i=1}^{l} l\big(\pi(r_{i-1}, r_i)\big) = l(u_1 w) + l(u_2).$$

Therefore $N(u_1 w) \cap N(u_2^{-1}) \neq \emptyset$ by Proposition 1(iv), and - again by Proposition 1(iv) - we deduce that

$$w^{-1} \cdot N(u_1) \cap N(u_2^{-1}) \neq \emptyset \text{ or } N(w) \cap N(u_2^{-1}) \neq \emptyset.$$

But minimality of l also yields that $l(w) = \sum_{i=2}^{l-1} l\big(\pi(r_{i-1}, r_i)\big)$ and $l(wu_2) = l(w) + l(u_2)$, and so $N(w) \cap N(u_2^{-1}) = \emptyset$ by Proposition 1(iv). In order to avoid double indices, denote the simple root corresponding to r_i by a_i for all i. Then $N(u_1) = \Phi^+_{\{a_0,a_1\}} \setminus \{a_1\}$ and $N(u_2^{-1}) = \Phi^+_{\{a_{l-1},a_l\}} \setminus \{a_{l-1}\}$, and thus there exist $\lambda > 0$, $\mu \geq 0$ and $x \geq 0, y > 0$ with

$$w^{-1} \cdot (\lambda a_0 + \mu a_1) = x a_{l-1} + y a_l.$$

Now $w^{-1} \cdot a_1 = a_{l-1}$ implies that $w^{-1} \cdot a_0 = \big(\frac{x-\mu}{\lambda}\big) a_{l-1} + \big(\frac{y}{\lambda}\big) a_l$. Since $\lambda,\ y > 0$, we find that $w^{-1} \cdot a_1$ is positive, and thus $x \geq \mu$. Symmetrically, $w \cdot a_{l-1} = a_1$ yields that $w \cdot a_l = \big(\frac{\lambda}{y}\big) a_0 + \big(\frac{\mu - x}{\lambda}\big) a_1$, and thus $\mu \geq x$. Therefore $\mu = x$ and $w \cdot a_l = a_0$.

It follows that $l > 2$, since otherwise $w = 1$ and $a_0 = w \cdot a_l = a_2$, contradicting $(r_0, \ldots, r_l) \in \mathcal{F}$. Set $u = \pi(r_2, \ldots, r_{l-1})$ and $n = \frac{m_{r_1 r_2} - 1}{2}$. Then $\pi(r_1, r_2) = (r_2 r_1)^n$, and thus $w = (r_2 r_1)^n u$ with $l(w) = l\big((r_2 r_1)^n\big) + l(u)$; hence by Proposition 1(iv),

$$N\big((r_2 r_1)^n\big) \cap N(u^{-1}) = \emptyset. \tag{1}$$

Next, define $\alpha = (r_1 r_2)^n \cdot a_0$ and $\beta = (r_1 r_2)^{n-2} r_1 \cdot a_0$. Since $r_0 \neq r_2$ (as $(r_0, \ldots, r_l) \in \mathcal{F}$), the coefficients of a_0 in α and β equal 1, and so α and β are positive. Furthermore,

$$(r_l u^{-1}) \cdot \alpha = r_l \cdot \big(u^{-1} \cdot \big((r_1 r_2)^n \cdot a_0\big)\big) = r_l \cdot \big(w^{-1} \cdot a_0\big) = r_l \cdot a_l = -a_l.$$

We will show that $(r_l u^{-1}) \cdot \beta$ is also negative, and that $\langle \alpha, \beta \rangle \leq -1$. This completes the proof, for then each root of the form $\nu_\alpha \alpha + \nu_\beta \beta$ with $\nu_\alpha,\, \nu_\beta \geq 0$ is an element of $N(r_l u^{-1})$; but Proposition 1(i)(b) yields that there are infinitely many roots of this form, and thus $N(r_l u^{-1})$ must have infinitely many elements, contradicting Proposition 1(iii).

Let $\gamma = (r_1 r_2)^{n-1} \cdot a_1$. A straightforward calculation yields that $\beta = r_\gamma \cdot \alpha$, and thus

$$(r_l u^{-1}) \cdot \beta = (r_l u^{-1}) \cdot \big(\alpha - 2\langle \alpha, \gamma \rangle \gamma\big) = -a_l - 2\langle \alpha, \gamma \rangle (r_l u^{-1}) \cdot \gamma.$$

So in order to show that $(r_l u^{-1}) \cdot \beta$ is negative, it suffices to show that $(r_l u^{-1}) \cdot \gamma$ is positive, and $\langle \alpha, \gamma \rangle \geq 0$. Proposition 1(i)(a) implies that $\gamma \in N\big((r_2 r_1)^n\big)$, and so $u^{-1} \cdot \gamma$ is positive by (1); since clearly $\gamma \neq \alpha = u \cdot a_l$, and thus $u^{-1} \cdot \gamma \neq a_l$, it follows that $(r_l u^{-1}) \cdot \gamma$ is positive. Next,

$$\langle \alpha, \gamma \rangle = \langle (r_1 r_2)^n \cdot a_0, (r_1 r_2)^{n-1} \cdot a_1 \rangle = \langle r_1 r_2 \cdot a_0, a_1 \rangle = -\langle a_0, a_1 \rangle + 2\langle a_0, a_2 \rangle \langle a_1, a_2 \rangle,$$

and since $(r_0, \ldots, r_l) \in \mathcal{F}$, we know that $\langle a_0, a_1 \rangle, \langle a_1, a_2 \rangle \leq -cos\big(\frac{\pi}{3}\big) = -\frac{1}{2}$, and $\langle a_0, a_2 \rangle \leq 0$. Therefore

$$\langle \alpha, \gamma \rangle \geq \frac{1}{2} - \langle a_0, a_2 \rangle, \tag{2}$$

and $\langle \alpha, \gamma \rangle > 0$, as required; it remains to show that $\langle \alpha, \beta \rangle \leq -1$.

A straightforward induction yields that $\alpha = a_0 + A a_1 + B a_2$, with

$$A = -\langle a_0, a_2 \rangle \left(\frac{2}{sin\big(\frac{\pi}{m}\big)} \sum_{i=1}^{n} sin\Big(\frac{2i\pi}{m}\Big) \right) - \langle a_0, a_1 \rangle \left(\frac{2}{sin\big(\frac{\pi}{m}\big)} \sum_{i=0}^{n-1} sin\Big(\frac{(2i+1)\pi}{m}\Big) \right),$$

where $m = m_{r_1 r_2}$; therefore $A = \frac{-\langle a_0, a_1 \rangle - \langle a_0, a_2 \rangle}{1 + \langle a_1, a_2 \rangle}$, and thus $A > 0$. Further,

$$\begin{aligned} a_l = u^{-1} \cdot \alpha &= u^{-1} \cdot a_0 + A\big(u^{-1} \cdot a_1\big) + B\big(u^{-1} \cdot a_2\big) \\ &= u^{-1} \cdot a_0 + A(u^{-1} \cdot a_1) + B a_{l-1}, \end{aligned}$$

and since a_l is simple, we deduce that $u^{-1} \cdot a_0$ and $u^{-1} \cdot a_1$ cannot both be positive. But $a_1 \in N\big((r_2 r_1)^n\big)$ by Proposition 1(i)(a), and thus $u^{-1} \cdot a_1$ is positive by (1); hence $a_0 \in N(u^{-1})$. Since $u^{-1} \in Y_{r_{l-2}, r_2}$, while $r_0 \neq r_2$ (as $(r_0, \ldots, r_l) \in \mathcal{F}$), this yields that $\langle a_0, a_2 \rangle \leq -cos\big(\frac{\pi}{3}\big) = -\frac{1}{2}$. Therefore $\langle \alpha, \gamma \rangle \geq 1$ by (2), and thus

$$\langle \alpha, \beta \rangle = \langle \alpha, r_\gamma \cdot \alpha \rangle = \langle \alpha, \alpha \rangle - 2\langle \alpha, \gamma \rangle^2 = 1 - 2\langle \alpha, \gamma \rangle^2 \leq -1,$$

as required.

□

ACKNOWLEDGEMENTS. This paper is taken from my dissertation. I would like to thank my supervisor Dr. R. B. Howlett for his support throughout my candidature and the writing of this paper. The work was completed while I was supported by an Overseas Postgraduate Research Award and a University Postgraduate Research Award.

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