The number of simple modules of the Hecke algebras of type $G(r, 1, n)$

Susumu Ariki and Andrew Mathas

Introduction

Let n and r be integers with $n \geq 0$ and $r \geq 1$. Let R be a commutative ring with 1 and let $q, Q_1, \ldots, Q_r$ be elements of R with q invertible. The cyclotomic Hecke algebra $\mathscr{H}_{R,n} = \mathscr{H}_{R,n}(q; \{Q_1, \ldots, Q_r\})$ of type $G(r, 1, n)$ is the unital associative R–algebra with generators $T_0, T_1, \ldots, T_{n-1}$ and relations

$$\begin{array}{rcll} (T_0 - Q_1) \cdots (T_0 - Q_r) & = & 0, & \\ T_0T_1T_0T_1 & = & T_1T_0T_1T_0, & \\ (T_i + 1)(T_i - q) & = & 0 & \text{for } 1 \leq i \leq n-1, \\ T_{i+1}T_iT_{i+1} & = & T_iT_{i+1}T_i & \text{for } 1 \leq i \leq n-2, \\ T_iT_j & = & T_jT_i & \text{for } 0 \leq i < j-1 \leq n-2. \end{array}$$

The first result of this paper says that, in a certain sense, the number of simple $\mathscr{H}_{R,n}$–modules is independent of the base field and independent of q. To state this precisely, let ℓ be the smallest positive integer such that $1 + q + \cdots + q^{\ell-1} = 0$; set $\ell = \infty$ if no such integer exits.

Theorem A *Suppose that R is a field and let $\{Q_1, \ldots, Q_r\} = S_0 \sqcup S_1 \sqcup \ldots \sqcup S_a$ be a partition of the parameter set such that $S_0 = \{\, Q_i \mid Q_i = 0\,\}$ and, for the remaining non–zero parameters, Q_i/Q_j is a power of q if and only if there exists an integer b such that Q_i and Q_j both belong to S_b. For $i = 1, 2, \ldots, a$ choose $s_i \in S_i$ and define n_{ij} to be the multiplicity of $s_i q^j$ in S_i. Then the number of simple $\mathscr{H}_{R,n}$–modules depends only upon ℓ and the integers n_{ij}.*

When $q = 1$ the simple $\mathscr{H}_{R,n}$–modules have been classified in [**19**] and it is easy to see that Theorem A is true in this case. When $q \neq 1$ the main result of [**7**] – see (1.2) – allows us to reduce to the case where all of Q_i are non–zero and Q_i/Q_j is always a power of q. The key idea now is to vary n and consider the (graded dual of the) direct sum of the Grothendieck groups of $\mathscr{H}_{R,n}$; by [**1**], this is an irreducible module of a Kac–Moody algebra when R is a field of

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characteristic zero. In fact, this is true independently of the characteristic of R, from which the result follows.

To date, few studies have been made of the modular representations of affine Hecke algebras in fields of positive characteristic. Theorem A, combined with results of Chriss and Ginzburg [**3**], yields a classification of the irreducible representations of the affine Hecke algebras of type A over arbitrary fields. Let $\hat{H}_{R,n}$ be the affine Hecke algebra associated to $GL(n,F)$ where F is a local field; the algebra $\hat{H}_{\mathbb{Z}[q,q^{-1}],n}$ is again a $\mathbb{Z}[q,q^{-1}]$-algebra.

A segment of length $d\geq 1$ is a sequence of d consecutive residues $[i,i+1,\dots,i+d-1]$ where $i,i+1,\dots,i+d-1\in\mathbb{Z}/\ell\mathbb{Z}$. A multisegment $\mathbf{m}$ is an unordered collection of segments; we denote by $|\mathbf{m}|$ the sum of the lengths of the segments in $\mathbf{m}$. A multisegment is aperiodic if for every d there exists an $i\in\mathbb{Z}/\ell\mathbb{Z}$ such that $[i,i+1,\dots,i+d-1]$ does not appear in $\mathbf{m}$. Denote by $\mathcal{M}_\ell$ the set of aperiodic multisegments and let $k_q^\times=k^\times/{<}q{>}$. For any field k, let

$$\mathcal{M}_\ell^n(k)=\{\,\underline{\lambda}\colon k_q^\times\longrightarrow\mathcal{M}_\ell\mid\sum_{x\in k_q^\times}|\underline{\lambda}(x)|=n\,\}.$$

Then, using Theorem A, we can deduce the following result.

Theorem B *Suppose that k is a field and that $q\neq 1$ has order ℓ in k. Let $(K,\mathcal{O},k)$ be a modular system such that q lifts to an element of order ℓ in $\mathcal{O}$.*

(i) *The simple $\hat{H}_{k,n}$-modules are indexed by $\mathcal{M}_\ell^n(k)$.*

(ii) *There is a set of $\hat{H}_{\mathcal{O},n}$-modules $\{\,D^{\underline{\lambda}}\mid\underline{\lambda}\in\mathcal{M}_\ell^n(k)\,\}$ such that*
  - (a) *$\{D^{\underline{\lambda}}\otimes K\}$ is a set of simple $\hat{H}_{K,n}$-modules; and,*
  - (b) *$D^{\underline{\lambda}}\otimes k$ has a simple head and the irreducible quotients form a complete set of simple $\hat{H}_{k,n}$-modules.*

Returning now to the simple $\mathscr{H}_{R,n}$–modules, integrable representations of Kac–Moody algebras have a basis indexed by the vertices of Kashiwara's crystal graph; by determining this graph we obtain our next result.

Theorem C *Suppose that R is a field, $q\neq 1$ and that $Q_1,\dots,Q_r$ are all non–zero elements of R. Then the irreducible $\mathscr{H}_{R,n}$–modules are indexed by the set of Kleshchev multipartitions.*

We define Kleshchev multipartitions below. We remark that we could include the case where some of the Q_i are 0 in Theorem C; however, this would overly complicate the definition of Kleshchev multipartitions.

Before giving the definition of Kleshchev multipartitions, we note the following consequence of Theorem C and the Weyl–Kac q-dimension formula. To do this we introduce formal power series $F_\ell(\{n_i\}_{i\in\mathbb{Z}/\ell\mathbb{Z}};x)$, when ℓ is finite, and $F_\infty(\{n_i\}_{i\in\mathbb{Z}};x)$, when ℓ is infinite.

We treat the case $\ell<\infty$ first. Given integers $n_0,\dots,n_{\ell-1}$ in $\mathbb{Z}/\ell\mathbb{Z}$ let $F_\ell(\{n_i\}_{i\in\mathbb{Z}/\ell\mathbb{Z}};x)$ be the rational function

$$\prod_{1\leq i<j\leq\ell}\frac{1-x^{N_{ji}}}{1-x^{j-i}}\prod_{k>0}\Big(\frac{1-x^{kN_\ell}}{1-x^{k\ell}}\Big)^{\ell-1}\prod_{1\leq i<j\leq\ell}\frac{1-(x^{N_{ij}}+x^{N_{ji}})x^{kN_\ell}+x^{2kN_\ell}}{1-(x^{i-j}+x^{j-i})x^{k\ell}+x^{2k\ell}}$$

where $N_i = \sum_{j=0}^{i-1}(n_j+1)$ and $N_{ij} = N_i - N_j$. This is invariant a under cyclic permutation of the indices.

Suppose next that $\ell = \infty$. Then, given integers $\dots, n_{-1}, n_0, n_1, \dots$ such that only finitely many of them are non-zero, let $F_\infty(\{n_i\}_{i\in\mathbb{Z}}; x)$ be the rational function

$$\prod_{i<j} \frac{1-x^{N_{ji}}}{1-x^{j-i}}$$

where $N_{ji} = \sum_{i\le k<j}(n_k+1)$. This is invariant under the shift operation on the indices.

Finally, we define $F_\ell(x) = F_\ell(\{n_i\}; x)$ where $n_i = \delta_{i0}$ for all i.

The following theorem gives the precise number of Kleshchev diagrams associated with a parameter set $\{q, Q_1, \dots, Q_r\}$; as we explain in the final section, this has important consequences for the classification of the irreducible modules of the finite groups of Lie type in non–defining characteristic.

Theorem D *Suppose that R is a field, $q \neq 1$ and define integers n_{ij} relative to a partitioning of the parameter set $\{Q_1, \dots, Q_r\} = S_0 \sqcup S_1 \sqcup \dots \sqcup S_a$ as in Theorem A. Then the generating function for the number of simple $\mathscr{H}_{R,n}$–modules is*

$$F_\ell(x) \prod_{i=1}^{a} F_\ell(\{n_{ij}\}; x).$$

By [**7**], the simple modules of $\mathscr{H}_{R,n}(q; S_0)$ are indexed by multipartitions of the form $((0), \dots, (0), \mu)$ where μ is an ℓ–restricted partition; so, in fact, $F_\ell(x)$ is the generating function for the set of ℓ–restricted partitions. For example,

$$\begin{aligned} F_3(x) &= \prod_{k>0} \frac{(1-x^{4k-2})(1-x^{4k-1})^2(1-x^{4k})^2(1-x^{4k+1})^2(1-x^{4k+2})}{(1-x^{3k-2})(1-x^{3k-1})^2(1-x^{3k})^2(1-x^{3k+1})^2(1-x^{3k+2})} \\ &= \prod_{k>0} \frac{(1-x^{4k-1})(1-x^{4k})(1-x^{4k+1})(1-x^{4k+2})}{(1-x^{3k-1})(1-x^{3k})(1-x^{3k+1})^2(1-x^{3k+2})} \\ &= \prod_{k>0} \frac{1}{(1-x^{3k+1})(1-x^{3k+2})} \\ &= \prod_{k>0} \frac{1-x^{3k}}{1-x^k} \end{aligned}$$

We now define the set of Kleshchev multipartitions; in order to do this we need some notation. A partition of an integer m is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non–negative integers such that $\lambda_1 \ge \lambda_2 \ge \cdots$ and $\sum_i \lambda_i = m$; we write $|\lambda| = m$. A multipartition of n (with r–components), is an ordered r–tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions such that $|\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. Let Π_n be the set of all multipartitions of n.

The diagram of $\lambda \in \Pi_n$ is the set

$$[\lambda] = \{\, (a, b, c) \mid 1 \le c \le r \text{ and } 1 \le b \le \lambda_a^{(c)} \,\}.$$

The elements of $[\lambda]$ are the nodes of λ; more generally, a node is any element of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. If $\gamma=(a,b,c)$ and $\gamma'=(a',b',c')$ are two nodes then γ is below γ', or γ' is above γ, if either $c>c'$, or $c=c'$ and $a>a'$. If $\gamma=(a,b,c)\in[\lambda]$ is a node then its residue, relative to $\{q,Q_1,\dots,Q_r\}$, is $\operatorname{res}(\gamma)=q^{b-a}Q_c$, an element of R. If $\operatorname{res}(\gamma)=x$ then we say that γ is an x–node.

Write $\lambda\to\mu$ if μ is a multipartition of $n+1$ such that $[\lambda]\subset[\mu]$. In this case $[\mu]=[\lambda]\cup\{\gamma\}$ for some node γ. We say that γ is an addable node of λ and a removable node of μ. If $\operatorname{res}(\gamma)=x$ then we shall also write $\lambda\overset{x}{\to}\mu$.

Let μ be a multipartition and suppose that η is a removable node of μ with $\operatorname{res}(\eta)=x$. Then η is a normal x–node if whenever γ is an addable x–node in μ which is below η then there are more removable x–nodes between γ and η than there are addable x–nodes. In addition, η is good if it is the highest normal x–node in μ.

If η is a good x–node of μ and λ is the multipartition such that $[\mu]=[\lambda]\cup\{\eta\}$ we write $\lambda\overset{x}{\twoheadrightarrow}\mu$. Notice that if $\lambda\overset{x}{\twoheadrightarrow}\mu$ then $\lambda\overset{x}{\to}\mu$.

Definition *Suppose that $n\geq 0$ and that $Q_1,\dots,Q_r$ are all non–zero. The set $\mathscr{K}_n=\mathscr{K}_n(q,Q_1\dots,Q_r)$ of* Kleshchev multipartitions *is defined inductively as follows.*

(i) $\mathscr{K}_0=\{((0),\dots,(0))\}$*; and*

(ii) $\mathscr{K}_{n+1}=\{\,\mu\in\Pi_{n+1}\mid\lambda\overset{x}{\twoheadrightarrow}\mu \text{ for some } \lambda\in\mathscr{K}_n \text{ and some } x\in R\,\}$.

There exist multipartitions which have no good or normal nodes so it is not completely obvious that $\mathscr{K}_n\neq\varnothing$. However, $((0),\dots,(0),(1^n))\in\mathscr{K}_n$ for all n, so $\mathscr{K}_n$ is always non–empty.

Example Suppose that $q=\sqrt[3]{1}$, $Q_1=Q_2=q$ and $Q_3=1$. Then $\ell=3$ and $\operatorname{res}(\gamma)\in\{\,q^i\mid 0\leq i<\ell\,\}$ for all nodes γ. Let $\lambda=\big((5,4,2),(6,3,2,1),(6,5,3)\big)$; although one can use the definition directly to find the good nodes in λ the following procedure is easier. In the diagram below we record the residues in λ by writing i if the residue is q^i.

$$\left(\ \begin{array}{|c|c|c|c|c|}\hline 1&2&0&1&2\\\hline 0&1&2&0\\\cline{1-4} 2&0\\\cline{1-2}\end{array}\ ,\quad \begin{array}{|c|c|c|c|c|c|}\hline 1&2&0&1&2&0\\\hline 0&1&2\\\cline{1-3} 2&0\\\cline{1-2} 1\\\cline{1-1}\end{array}\ ,\quad \begin{array}{|c|c|c|c|c|c|}\hline 0&1&2&0&1&2\\\hline 2&0&1&2&0\\\cline{1-5} 1&2&0\\\cline{1-3}\end{array}\ \right)$$

We find the good node in λ of residue $q^0=1$. Reading nodes from the bottom up, we obtain the sequence

$$AR\quad R\quad AARARR\quad R\quad A$$

where each "A" corresponds to an addable q^0–node and each "R" to a removable q^0–node. From this sequence, remove all occurrences of the string "AR", and keep on doing this until all such strings have been deleted. The "R"s that remain are the normal q^0–nodes of λ and the highest of these is the good q^0–node. In this example, all of the grouped letters disappear; so the normal nodes are the circled nodes in the diagram and the good q^0–node is $(2,4,1)$. Notice that λ is not a Kleshchev multipartition because the node $(3,3,3)$ can never be good.

When $r=1$, Π_n can be identified with the set of partitions of n and it is not difficult to see that $\mathscr{K}_n$ is the set of ℓ–restricted partitions. The definition of good and normal nodes in this case is due to Kleshchev [**16**] who used them to describe the socle of the restriction of the irreducible modules for the symmetric group in characteristic ℓ. When $r>1$ we do not know of a simple description of the set $\mathscr{K}_n$ (unless $q=-1$; see [**19**]).

1 Specht modules and Grothendieck groups

Let $\mathscr{H}_{R,n}$ be a cyclotomic Hecke algebra, as defined in the introduction. When $R=\mathbb{Q}(q,Q_1,\dots,Q_r)$ the simple $\mathscr{H}_{R,n}$–modules were constructed by the first author and Koike in [**2**]; for arbitrary fields the set of simple modules was first constructed by Graham and Lehrer [**11**], as we now describe (see also [**6**]).

Given two multipartitions $\lambda=(\lambda^{(1)},\dots,\lambda^{(r)})$ and $\mu=(\mu^{(1)},\dots,\mu^{(r)})$ say that λ **dominates** μ, and write $\lambda\trianglerighteq\mu$, if for $a=1,2\dots,r$ and for all $i\ge 1$

$$\sum_{b=1}^{a-1}|\lambda^{(b)}|+\sum_{j=1}^{i}\lambda_j^{(a)}\ge\sum_{b=1}^{a-1}|\mu^{(b)}|+\sum_{j=1}^{i}\mu_j^{(a)}.$$

If $\lambda\trianglerighteq\mu$ and $\lambda\ne\mu$ we write $\lambda\triangleright\mu$. This defines a partial order on the set of multipartitions.

Graham and Lehrer showed that for any ring R and any multipartition λ of n there exists a right $\mathscr{H}_{R,n}$–module S_R^λ, called a **Specht module** (or **cell module**). When $R=\mathbb{Q}(q,Q_1,\dots,Q_r)$, the Specht modules are simple $\mathscr{H}_{R,n}$–modules and coincide with the simple modules defined in [**2**]. In general, S_R^λ possesses a natural bilinear form and the radical, $\operatorname{rad}S_R^\lambda$, of this form is an $\mathscr{H}_{R,n}$–submodule of S_R^λ. Let R be a field and $D_R^\lambda=S_R^\lambda/\operatorname{rad}S_R^\lambda$; then D_R^λ is either absolutely irreducible or 0.

Thus, given partitions λ and μ of n with $D_R^\mu\ne 0$ we can talk of the composition multiplicity $d_{\lambda\mu}$ of D_R^μ in S_R^λ. Graham and Lehrer proved the following fundamental result.

1.1 (Graham–Lehrer [11]) *Suppose that R is a field.*

(i) *Then $\{\,D_R^\lambda \mid \lambda\in\Pi_n$ and $D_R^\lambda\ne 0\,\}$ is a complete set of non–isomorphic irreducible $\mathscr{H}_{R,n}$–modules. Moreover, whenever D_R^λ is non–zero it is absolutely irreducible.*

(ii) *Let λ and μ be multipartitions of n and suppose that $D_R^\mu\ne 0$. Then $d_{\mu\mu}=1$ and $d_{\lambda\mu}\ne 0$ only if $\lambda\trianglerighteq\mu$.*

(iii) *The Hecke algebra $\mathscr{H}_{R,n}$ is semisimple if and only if $S_R^\lambda=D_R^\lambda$ for all multipartitions $\lambda\in\Pi_n$.*

Key to the proof of our main results is the following reduction theorem.

1.2 (Dipper–Mathas [7]) *Suppose that R is a field and let $\{Q_1,\dots,Q_r\} = S_0 \sqcup S_1 \sqcup \dots \sqcup S_a$ be a partition of the parameter set as in Theorem A; write $S_k = \{Q_{k_1},\dots,Q_{k_{r_k}}\}$ for $k=0,1,\dots,a$. Then the module categories of the algebras $\mathscr{H}_{R,n}$ and*

$$\bigoplus_{n_0+\dots+n_a=n} \mathscr{H}_{R,n_0}(q;S_0)\otimes_R\dots\otimes_R\mathscr{H}_{R,n_a}(q;S_a)$$

are Morita equivalent. Moreover, if $\lambda=(\lambda^{(1)},\dots,\lambda^{(r)})$ is a multipartition of n and $\lambda_k=(\lambda^{(k_1)},\dots,\lambda^{(k_{r_k})})$ then $D_R^\lambda\neq 0$ if and only if the $\mathscr{H}_{R,n_0}(q;S_0)\otimes_R \dots\otimes_R\mathscr{H}_{R,n_a}(q;S_a)$–module $D_R^{\lambda_0}\otimes\dots\otimes D_R^{\lambda_a}$ is non–zero, where $n_k=|\lambda_k|$.

As remarked in the introduction, it is also shown in [**7**] that D_R^λ is a simple $\mathscr{H}_{R,n}(q;S_0)$–module if and only if $\lambda=((0),\dots,(0),\lambda^{(r)})$ where $\lambda^{(r)}$ is an ℓ-restricted partition. Thus, in order to prove Theorems A, C and D we are reduced to the following situation.

1.3 *Henceforth, we assume that R is a field, $q\neq 1$ and that there exist integers n_i such that $Q_i=q^{n_i}$ for $i=1,2,\dots,r$. Let ℓ be the smallest positive integer such that $q^\ell=1$; set $\ell=\infty$ if no such integer exists. If ℓ is finite let $I_\ell=\{0,1,\dots,\ell-1\}$; otherwise, let $I_\ell=\mathbb{Z}$.*

We also write $\lambda\xrightarrow{i}\mu$ and $\lambda\overset{i}{\twoheadrightarrow}\mu$ rather than $\lambda\xrightarrow{q^i}\mu$ and $\lambda\overset{q^i}{\twoheadrightarrow}\mu$ respectively.

For $i=1,2,\dots,n$ let $L_i=q^{1-i}T_{i-1}\dots T_1T_0T_1,\dots T_{i-1}$ and set $c_n=L_1+L_2+\dots+L_n$. Because c_n is a symmetric polynomial in the elements $L_1,\dots,L_n$ it follows from [**2**, Lemma 3.3] that c_n belongs to the centre of $\mathscr{H}_{R,n}$. The next result describes how c_n acts upon the Specht modules.

1.4 Lemma *Suppose that R is a field and let λ be a multipartition of n. Then c_n is central in $\mathscr{H}_{R,n}$ and acts on the Specht module S_R^λ as multiplication by*

$$c_n(\lambda) := \sum_{\gamma\in[\lambda]} \operatorname{res}(\gamma).$$

Proof. When $R=\mathbb{C}(q,Q_1,\dots,Q_r)$ the Specht module S_R^λ is irreducible and c_n must act upon S_R^λ as multiplication by a scalar because it is central; that this scalar is $c_n(\lambda)$ follows from [**2**, Prop. 3.16]. By restriction, the Lemma also holds when $R=\mathbb{Z}[q,Q_1,\dots,Q_r]$; hence, the general case follows by specialization. ◻

Let $\mathfrak{u}_{R,n}=\mathscr{G}_0(\mathscr{H}_{R,n})\otimes_\mathbb{Z}\mathbb{C}$ be the Grothendieck group of finitely generated $\mathscr{H}_{R,n}$–modules modulo short exact sequences, with coefficients extended to $\mathbb{C}$; see [**4**, §16]. Given a finitely generated $\mathscr{H}_{R,n}$–module let $[M]$ denote its equivalence class in $\mathfrak{u}_{R,n}$. By (1.1), $\mathfrak{u}_{R,n}$ is an abelian group which is free as a $\mathbb{C}$–module with basis $\{\, [D^\mu_R] \mid \mu \in \Pi_n \text{ and } D^\mu_R \neq 0 \,\}$. Furthermore, $[S^\lambda_R] = \sum_\lambda d_{\lambda\mu}[D^\mu_R]$ if $\mu \in \Pi_n$ and $D^\mu_R \neq 0$.

Given any $\mathscr{H}_{R,n}$–module M let $\operatorname{Res} M$ be the restriction of M to $\mathscr{H}_{R,n-1}$. Then Res is an exact functor from the category of right $\mathscr{H}_{R,n}$–modules to the category of right $\mathscr{H}_{R,n-1}$–modules. Similarly, we have an adjoint induction functor Ind given by $\operatorname{Ind} M = M \otimes_{\mathscr{H}_{R,n}} \mathscr{H}_{R,n+1}$. These functors induce maps between the corresponding Grothendieck groups via $\operatorname{Res}[M] = [\operatorname{Res} M]$ and $\operatorname{Ind}[M] = [\operatorname{Ind} M]$.

1.5 Proposition *Suppose that λ is a multipartition of n. Then $\operatorname{Res} S^\lambda_R$ has a filtration with composition factors precisely the Specht modules S^ν_R where ν is a multipartition of $n-1$ and $\nu \to \lambda$.*

Proof. By [**6**, 3.28] the Specht module S^λ_R has a basis $\{m_\mathfrak{t}\}$ indexed by the standard λ–tableaux; see [**6**] for the definition of standard tableaux. If $\mathfrak{t}$ is a standard λ–tableau let $\mathfrak{t}\downarrow n-1$ be the tableau obtained by deleting the node labelled n and let $|\mathfrak{t}\downarrow n-1|$ be the corresponding multipartition. By repeating the proof of [**6**, 3.15,3.18], if $h \in \mathscr{H}_{R,n-1}$ then $m_\mathfrak{t} h$ is a linear combination of terms $m_\mathfrak{v}$ such that $|\mathfrak{v}\downarrow n-1| \trianglerighteq |\mathfrak{t}\downarrow n-1|$; that is, n occupies either the same node or a lower node in $\mathfrak{v}$ than it does in $\mathfrak{t}$. Therefore, by extending the dominance order on the set $\{\, \nu \mid \nu \to \lambda \,\}$ to a total order, we can define an $\mathscr{H}_{R,n-1}$–stable filtration of S^λ_R where for each quotient there exists a multipartition ν such that $\nu \to \lambda$ and the quotient has a basis indexed by the λ–tableau $\mathfrak{t}$ such that $|\mathfrak{t}\downarrow n-1| = \nu$. As in [**6**, 3.20], for each such ν, we can define a map from S^ν_R into the corresponding quotient of the filtration which maps $m_\mathfrak{s}$ to the image of $m_\mathfrak{t}$ in the quotient where $\mathfrak{s} = \mathfrak{t}\downarrow n-1$. From what we have said, this is an $\mathscr{H}_{R,n-1}$–module homomorphism; by a counting argument, it is actually an isomorphism. □

1.6 Corollary *Let λ be a multipartition of n. Then*

$$\operatorname{Res}[S^\lambda_R] = \sum_{\nu\to\lambda} [S^\nu_R] \quad \text{and} \quad \operatorname{Ind}[S^\lambda_R] = \sum_{\lambda\to\mu} [S^\mu_R].$$

Proof. The restriction formula is a direct consequence of Proposition 1.5. The induction formula follows by adjointness. □

By (1.1), $[D^\mu_R]$ is a linear combination of Specht modules $[S^\lambda_R]$ so this completely determines the maps Res and Ind at the level of Grothendieck groups.

Now consider the element c_n and suppose that $D^\mu_R \neq 0$. By Lemma 1.4, c_n is central and acts on S^μ_R, and hence also on D^μ_R, as multiplication by $c_n(\mu)$. Let M be a $\mathscr{H}_{R,n}$-module and let $M = \oplus_x M_x$ be the decomposition of M into a direct sum of generalized eigenspaces M_x for the action of c_n. Then, given $i \in I_\ell$ we have functors i–Res and i–Ind given by

$$\text{(1.7)} \qquad i\text{–Res} M = \bigoplus_{x\in R} (\operatorname{Res} M_x)_{x-q^i} \quad \text{and} \quad i\text{–Ind} M = \bigoplus_{x\in R} (\operatorname{Ind} M_x)_{x+q^i}.$$

Then Res $= \sum_i i$–Res and Ind $= \sum_i i$–Ind. In turn, these functors induce homomorphisms i–Res : $\mathfrak{u}_{R,n} \longrightarrow \mathfrak{u}_{R,n-1}$ and i–Ind : $\mathfrak{u}_{R,n} \longrightarrow \mathfrak{u}_{R,n+1}$ given by i–Res$[M] = [i$–Res$M]$ and i–Ind$[M] = [i$–Ind$M]$. Comparing the definitions and Corollary 1.6 we can rephrase this as follows.

1.8 Corollary *Suppose that λ is a multipartition of n and let $i \in I_\ell$. Then the homomorphisms i–Res : $\mathfrak{u}_{R,n} \longrightarrow \mathfrak{u}_{R,n-1}$ and i–Ind : $\mathfrak{u}_{R,n} \longrightarrow \mathfrak{u}_{R,n+1}$ are completely determined by*

$$i\text{–Res}[S_R^\lambda] = \sum_{\nu \xrightarrow{i} \lambda} [S_R^\nu] \quad \text{and} \quad i\text{–Ind}[S_R^\lambda] = \sum_{\lambda \xrightarrow{i} \mu} [S_R^\mu].$$

Although we are most interested $\mathfrak{u}_{R,n}$ it is more useful to consider $\mathfrak{u}_{R,n}^0$, the Grothendieck group of finitely generated *projective* modules, again with coefficients extended to $\mathbb{C}$. If P is a projective $\mathscr{H}_{R,n}$–module write $[\![P]\!]$ for its image in $\mathfrak{u}_{R,n}^0$. For each multipartition μ with $D_R^\mu \neq 0$, up to isomorphism, there is a uniquely determined projective indecomposable module P_R^μ and $\mathfrak{u}_{R,n}^0$ is the free abelian group which is free as a $\mathbb{C}$–module with basis $\{\, [\![P_R^\mu]\!] \mid \mu \in \Pi_n \text{ and } D_R^\mu \neq 0 \,\}$.

It is easy to see that the functors Res and Ind take projectives to projectives and so again induce maps between $\mathfrak{u}_{R,n}^0$ and $\mathfrak{u}_{R,n\pm 1}^0$. Moreover, by Lemma 1.4, each projective decomposes into a sum of generalized eigenspaces of c_n, which are again projective modules, so we also have homomorphisms i–Res and i–Ind which are defined using (1.7) exactly as before.

The next Lemma is standard.

1.9 Lemma *Let* $\mathfrak{u}_{R,n}^* = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{u}_{R,n}, \mathbb{C})$.

(i) *The abelian groups $\mathfrak{u}_{R,n}^0$ and $\mathfrak{u}_{R,n}^*$ are canonically isomorphic.*

(ii) *The natural pairing $\mathfrak{u}_{R,n}^0 \times \mathfrak{u}_{R,n} \to \mathbb{C}$ is determined by $([\![P_R^\lambda]\!], [D_R^\mu]) = \delta_{\lambda\mu}$.*

(iii) *If P is a projective $\mathscr{H}_{R,n}$–module and M is any $\mathscr{H}_{R,n+1}$–module then*

$$(i\text{–Ind}[\![P]\!], [M]) = ([\![P]\!], i\text{–Res}[M]).$$

(iv) *If P is a projective $\mathscr{H}_{R,n}$–module and M is any $\mathscr{H}_{R,n-1}$–module then*

$$(i\text{–Res}[\![P]\!], [M]) = ([\![P]\!], i\text{–Ind}[M]).$$

Proof. (i) Given any projective $\mathscr{H}_{R,n}$–module P the functor $\operatorname{Hom}_{\mathscr{H}_{R,n}}(P, \_)$ is exact; hence, we can define a map $d_P \in \mathfrak{u}_{R,n}^*$ by $d_P[M] = \dim \operatorname{Hom}_{\mathscr{H}_{R,n}}(P, M)$. It follows that $P \mapsto d_P$ is an isomorphism of abelian groups.

(ii) The natural pairing between $\mathfrak{u}_{R,n}^*$ and $\mathfrak{u}_{R,n}$ is given by evaluation; therefore, using the notation of (i),

$$([\![P_R^\lambda]\!], [D_R^\mu]) = d_{P_R^\lambda}[D_R^\mu] = \dim \operatorname{Hom}_{\mathscr{H}_{R,n}}(P_R^\lambda, D_R^\mu) = \delta_{\lambda\mu}.$$

(iii) Using the adjointness of i–Res and i–Ind we find that

$$\begin{aligned}(i\text{–Ind}[\![P]\!],[M]) &= \dim \mathrm{Hom}_{\mathscr{H}_{R,n}}(i\text{–Ind}P, M) = \dim \mathrm{Hom}_{\mathscr{H}_{R,n}}(P, i\text{–Res}M)\\ &= ([\![P]\!], i\text{–Res}[M]).\end{aligned}$$

The proof of (iv) is similar. □

Henceforth, we identify $\mathrm{u}^0_{R,n}$ and $\mathrm{u}^*_{R,n}$.

There is a natural homomorphism $\mathbf{c}:\mathrm{u}^0_{R,n}\longrightarrow \mathrm{u}_{R,n}$ given by $[\![P]\!]\mapsto [P]$. The map $\mathbf{c}$ is the so–called "Cartan map" and it is best understood as follows. Let $\mathscr{F}_n$ be the free abelian group with $\mathbb{C}$–basis the set of symbols $\{\,[\![S^\lambda]\!] \mid \lambda\in\Pi_n\,\}$. We define group homomorphisms Res, i–Res, Ind and i–Ind from $\mathscr{F}_n$ to $\mathscr{F}_{n\pm1}$ by formulas in Corollary 1.8; importantly, these maps are independent of the field R.

By [**11**, Theorem 3.7], the projective indecomposable P^μ_R has a filtration with each composition factor isomorphic to some Specht module such that S^λ_R occurs with multiplicity $d_{\lambda\mu}$. Since the $[\![P^\mu_R]\!]$ are linearly independent, we have a well–defined *injective* homomorphism of abelian groups $\mathbf{e}:\mathrm{u}^0_{R,n}\longrightarrow\mathscr{F}_n$ given by $\mathbf{e}[\![P^\mu_R]\!]=\sum_\lambda d_{\lambda\mu}[\![S^\lambda]\!]$. Similarly, there is a well–defined *surjective* homomorphism $\mathbf{d}:\mathscr{F}_n\longrightarrow \mathrm{u}_{R,n}$ given by $\mathbf{d}[\![S^\lambda]\!]=[S^\lambda_R]=\sum_\mu d_{\lambda\mu}[D^\mu_R]$. Moreover, by [**11**, Theorem 3.7] the diagram

$$\begin{tikzcd}
\mathrm{u}^0_{R,n} \arrow[dr, "\mathbf{e}"] \arrow[dd, "\mathbf{c}"'] & \\
& \mathscr{F}_n \arrow[dl, "\mathbf{d}"] \\
\mathrm{u}_{R,n} &
\end{tikzcd}$$

commutes. By (1.1)(iii), $\mathscr{F}_n$ is isomorphic to the Grothendieck group of a semisimple Hecke algebra, so this is really just the Cartan–Brauer *cde*–triangle for $\mathscr{H}_{R,n}$ (see [**4**, §18]).

Finally, it follows directly from the definitions that all of the maps in the Cartan triangle commute with the homomorphisms i–Res and i–Ind. We summarize these results in the following Lemma.

1.10 Lemma *Suppose that R is a field and that $i\in I_\ell$. Then all of the squares and triangles in the diagram*

$$\begin{tikzcd}
\mathrm{u}^0_{R,n-1} \arrow[dr, "\mathbf{e}"] \arrow[dd, "\mathbf{c}"'] & & \mathrm{u}^0_{R,n} \arrow[ll, "i\text{–Res}"'] \arrow[rr, "i\text{–Ind}"] \arrow[dr, "\mathbf{e}"] \arrow[dd] & & \mathrm{u}^0_{R,n+1} \arrow[dr, "\mathbf{e}"] \arrow[dd] & \\
& \mathscr{F}_{n-1} \arrow[dl, "\mathbf{d}"] & & \mathscr{F}_n \arrow[ll, "i\text{–Res}"'] \arrow[rr, "i\text{–Ind}"] \arrow[dl, "\mathbf{d}"] & & \mathscr{F}_{n+1} \arrow[dl, "\mathbf{d}"] \\
\mathrm{u}_{R,n-1} & & \mathrm{u}_{R,n} \arrow[ll, "i\text{–Res}"'] \arrow[rr, "i\text{–Ind}"] & & \mathrm{u}_{R,n+1} &
\end{tikzcd}$$

commute (the vertical arrows are the Cartan maps $\mathbf{c}$*).*

2 The Kac–Moody algebra

Throughout this section we maintain the assumptions of (1.3). In particular, ℓ is the smallest positive integer such that $q^\ell = 1$; $\ell = \infty$ if no such integer exists.

If ℓ is finite let $U(\mathfrak{g})$ be the Kac–Moody algebra of type $A^{(1)}_{\ell-1}$; if $\ell = \infty$ let $U(\mathfrak{g})$ be the Kac–Moody algebra of type A_∞. We refer the reader to Kac's book [**14**] for the standard properties of Kac–Moody algebras.

The Kac–Moody algebra $U(\mathfrak{g})$ is generated by elements e_i, f_i, h_i and d where $i \in I_\ell$. Let $U(\mathfrak{h})$ be the Cartan subalgebra of $U(\mathfrak{g})$, the subalgebra of $U(\mathfrak{g})$ generated by h_i and d, and for $i \in I_\ell$ let $\Lambda_i, \alpha_i \in U(\mathfrak{h})^*$ be the fundamental and simple weights of $U(\mathfrak{g})$. We may choose d so that $\Lambda_i(d) = 0$ and $\alpha_i(d) = \delta_{i0}$ for all $i \in I_\ell$.

The following result is a theorem of Hayashi [**12**] when $r = 1$ (see also [**20**]); the general case follows easily; see [**1**, 4.5]. We give the proof of a slightly stronger result in Proposition 2.5 below.

2.1 Definition *Given $i \in I_\ell$ let $N_i(\lambda) = \#\{\,\mu \mid \lambda \xrightarrow{i} \mu\,\} - \#\{\,\nu \mid \nu \xrightarrow{i} \lambda\,\}$ and let $N_d(\lambda) = \#\{\,(a,b,c) \in [\lambda] \mid \operatorname{res}(a,b,c) = 0\,\}$.*

2.2 Lemma *Let $\mathscr{F} = \bigoplus_{n\ge 0} \mathscr{F}_n$. Then $\mathscr{F}$ becomes a $U(\mathfrak{g})$–module with action*

$$e_i[\![S^\lambda]\!] = \sum_{\nu \xrightarrow{i} \lambda} [\![S^\nu]\!], \quad f_i[\![S^\lambda]\!] = \sum_{\lambda \xrightarrow{i} \mu} [\![S^\mu]\!], \quad h_i[\![S^\lambda]\!] = N_i(\lambda)[\![S^\lambda]\!],$$

for each $i \in I_\ell$, and $d[\![S^\lambda]\!] = -N_d(\lambda)[\![S^\lambda]\!]$.

Comparing Lemma 2.2 with Corollary 1.8, we can identify i–Ind and i–Res with f_i and e_i on $\bigoplus \mathfrak{u}^0_{R,n}$.

For $j = 1, 2, \dots, r$ recall that $Q_j = q^{n_j}$ for some integer n_j. When ℓ is finite $q^\ell = 1$ we may assume that $0 \le n_j < \ell$ for all j. In this way we associate the dominant weight $\Lambda = \Lambda_{n_1} + \cdots + \Lambda_{n_r}$ with $\mathscr{H}_{R,n}$. Notice that the multiplicity of Λ_i in Λ is equal to the number of Q_j such that $Q_j = q^i$.

2.3 Theorem *Let R be a field and set $\mathfrak{u}^0_R = \bigoplus_{n\ge 0} \mathfrak{u}^0_{R,n}$. Then $\mathfrak{u}^0_R$ is isomorphic to the integrable highest weight $U(\mathfrak{g})$–module of highest weight Λ.*

Proof. When $R = \mathbb{C}$ this is just [**1**, Theorem 4.4(i)]: the crucial point in the proof is to show that $\mathfrak{u}^0_R$ is cyclic, which was proved by counting orbits in quiver spaces which are known to parametrize both the irreducible representations of the affine Hecke algebras of type A and also a basis of the quantum algebras of affine type A. Let R be a field of characteristic 0 and let R' be an extension of R. By (1.1)(i), D^μ_R is absolutely irreducible so $D^\mu_{R'} \cong D^\mu_R \otimes_R R'$ and i–Ind and i–Res act in the same way upon $[D^\mu_R]$ and $[D^\mu_{R'}]$ for all μ. Thus, we may identify $\mathfrak{u}^0_R$ and $\mathfrak{u}^0_{R'}$ as $U(\mathfrak{g})$–modules. In conclusion, the Theorem holds for fields of characteristic 0 by embedding them into a suitable extension of $\mathbb{C}$.

Now suppose that R is a field of positive characteristic. By construction, $\mathfrak{u}_R^0$ is a $\mathbb{C}$–submodule of $\mathscr{F}$. However, by Lemma 1.10 and Lemma 2.2, if D_R^μ is non–zero then both $f_i[\![P_R^\mu]\!]$ and $e_i[\![P_R^\mu]\!]$ are elements of $\mathfrak{u}_R^0$; so, $\mathfrak{u}_R^0$ is actually a $U(\mathfrak{g})$–submodule of $\mathscr{F}$. Notice also that $\mathscr{F}$ is integrable since each multipartition has a finite number of addable and removable nodes; hence, $\mathfrak{u}_R^0$ is also integrable.

Let $u_\Lambda = [\![S^{((0),\dots,(0))}]\!]$ be the vector in $\mathscr{F}$ corresponding to the empty multipartition; then $U(\mathfrak{g})\mathfrak{u}_\Lambda \subseteq \mathfrak{u}_R^0$. Now u_Λ is annihilated by e_i and by d and $h_i u_\Lambda = \Lambda(h_i)u_\Lambda$ for all $i \in I_\ell$; thus, $U(\mathfrak{g})u_\Lambda$ is an integrable highest weight module of highest weight Λ and to complete the proof we must show that $\mathfrak{u}_R^0 = U(\mathfrak{g})u_\Lambda$. To do this, take a modular system $(K, \mathcal{O}, R)$ where K is a field of characteristic 0 and q' is an element of order ℓ in $\mathcal{O} \subset K$ which is in the preimage of q. As in section 2, we have operators i–Res and i–Ind for $\mathfrak{u}_K^0$ and $\mathfrak{u}_R^0$ respectively; by Corollary 1.8 these are defined by their action on Specht modules and by Corollary 1.6 they are both restrictions of the same operators acting on $\mathscr{F}$. By lifting idempotents, $\mathfrak{u}_R^0$ is a submodule of $\mathfrak{u}_K^0$. However, we have already shown that $\mathfrak{u}_K^0$ is the integrable highest weight module of weight Λ; so any submodule of $\mathfrak{u}_K^0$ which contains the highest weight vector coincides with the module itself. Thus, $\mathfrak{u}_R^0 = U(\mathfrak{g})\mathfrak{u}_\Lambda$ as required. □

Note that Theorem 2.3 is completely independent of the field R; hence, this completes the proof of Theorem A.

Proof of Theorem B. Let k be a field. We first notice that any simple $\hat{H}_{k,n}$-module is a simple module for some $\mathscr{H}_{k,n}$. Hence we can use (1.2) to reduce to the case where the support of $\underline{\lambda}$ is a single $<q>$-orbit in k. In this case, the simple $\hat{H}_{\mathbb{C},n}$–modules are indexed by aperiodic multisegments; see [**3**, Theorem 8.6.12] and [**1**, Proposition 4.3]. Therefore, by an similar argument to that of the first paragraph of the proof of Theorem 2.3, we deduce that the simple $\hat{H}_{k,n}$–modules are indexed by aperiodic multisegments whenever k is a field of characteristic 0.

Now suppose that k is a field of positive characteristic and, as in [**1**], let $\mathfrak{U}_k^*$ be the graded dual of the direct sum of Grothendieck groups of $\hat{H}_{k,n}$–modules; cf. the definition of $\mathfrak{u}_k^0$. Then $\mathfrak{U}_k^*$ is a $U^-(\mathfrak{g})$–module and $\mathfrak{u}_k^*$ is a quotient of $\mathfrak{U}_k^*$ by [**1**, Lemma 4.1]. By fixing n and taking the highest weight Λ sufficiently large, the degree n part of $\mathfrak{U}_k^*$ is isomorphic to the degree n part of $\mathfrak{u}_k^*$. Since we already know that the dimension of the degree n part of $\mathfrak{U}_k^*$ is equal to the number of aperiodic multisegments of size n, Theorem A tells us that the simple $\hat{H}_{k,n}$–modules are also indexed by aperiodic multisegments of size n. We have proved (i).

To prove (ii), take a modular system $(K, \mathcal{O}, k)$ where q lifts to an element of the same order in $\mathcal{O}$. Recall that the isomorphism $\mathfrak{u}_K^0 \simeq \mathfrak{u}_k^0$, which induces an isomorphism from $\mathfrak{u}_k$ to $\mathfrak{u}_K$ by Lemma 1.9(i), is given by the modular reduction procedure. Hence, [**10**] gives (ii). □

Now we turn to the proof of Theorem C. By Theorem A, we can restrict ourselves to the case $R = \mathbb{C}$. The main result of [**1**] states that if $R = \mathbb{C}$ then the canonical basis of $\mathfrak{u}_{\mathbb{C}}^0$ coincides with the basis of $\mathfrak{u}_{\mathbb{C}}^0$ given by the principal indecomposable $\mathscr{H}_{\mathbb{C},n}$–modules. The canonical basis of $\mathfrak{u}_{\mathbb{C}}^0$ is indexed by the vertices of the crystal graph of $\mathfrak{u}_{\mathbb{C}}^0$; to describe this we next introduce the quantized enveloping algebra of $U(\mathfrak{g})$.

Let v be an indeterminate over $\mathbb{C}$ and let $\mathbf{U}_v(\mathfrak{g})$ be the quantized enveloping algebra of $U(\mathfrak{g})$. This is a $\mathbb{C}(v)$–algebra generated by elements E_i, F_i, K_{h_i} and K_d (for $i \in I_\ell$), which are subject to the quantized Serre relations [**18**, 1.4.3].

We first show how $\mathscr{F}_v = \mathscr{F} \otimes_{\mathbb{C}} \mathbb{C}(v)$ may be endowed with the structure of a $\mathbf{U}_v(\mathfrak{g})$–module. For convenience, consider $\mathscr{F}_v$ to be the free $\mathbb{C}(v)$–module with basis $\{\,\lambda \mid \lambda \in \Pi_n \text{ for some } n \geq 0\,\}$.

2.4 Definition (cf. [17]) *Let λ, ν and μ be multipartitions such that $\nu \xrightarrow{i} \lambda$ and $\lambda \xrightarrow{i} \mu$ for some $i \in I_\ell$.*
(i) $N_i^r(\nu,\lambda) = \#\{\,\nu \xrightarrow{i} \alpha \mid \alpha \rhd \lambda\,\} - \#\{\,\beta \xrightarrow{i} \lambda \mid \nu \rhd \beta\,\}$.
(ii) $N_i^l(\lambda,\mu) = \#\{\,\lambda \xrightarrow{i} \alpha \mid \mu \rhd \alpha\,\} - \#\{\,\beta \xrightarrow{i} \mu \mid \beta \rhd \lambda\,\}$.

These definitions can be rephrased in terms of the addable and removable nodes of the introduction (cf. Lemma 2.6 below).

2.5 Proposition *The $\mathbb{C}(v)$–module $\mathscr{F}_v$ is an integrable $\mathbf{U}_v(\mathfrak{g})$–module with action determined by*

$$K_{h_i}\lambda = v^{N_i(\lambda)}\lambda, \qquad K_d\lambda = v^{-N_d(\lambda)}\lambda$$
$$E_i\lambda = \sum_{\nu \xrightarrow{i} \lambda} v^{-N_i^r(\nu,\lambda)}\nu, \qquad F_i\lambda = \sum_{\lambda \xrightarrow{i} \mu} v^{N_i^l(\lambda,\mu)}\mu$$

where $i \in I_\ell$ and λ is multipartition of k.

Proof. First consider the case where $r = 1$. In this case $\mathscr{F}_v$ is the Fock space $\mathscr{F}_{Q_1}^{(1)}$; that is, the free $\mathbb{C}(v)$–module with basis the set of all partitions of all integers), and this action was discovered by Hayashi [**12**] (cf. [**17**]). In fact, he considered only the case where $Q_1 = 1$; however, the general case is easily derived from this.

Suppose now that $r > 1$ and identify $\mathscr{F}_v$ with $\mathscr{F}_{Q_1}^{(1)} \otimes \cdots \otimes \mathscr{F}_{Q_r}^{(1)}$ via the $\mathbb{C}(v)$–linear map which sends $(\lambda^{(1)}, \dots, \lambda^{(r)})$ to $\lambda^{(1)} \otimes \cdots \otimes \lambda^{(r)}$. Now, $\mathbf{U}_v(\mathfrak{g})$ is a Hopf algebra having a standard comultiplication map Δ; because of its compatibility with the representation theory of $\mathscr{H}_{R,n}$ we use the twisted coproduct map $\Delta' = w_0 \Delta w_0$, where w_0 is the longest element of $\mathfrak{S}_\ell$. Then Δ' is the $\mathbb{C}(v)$–linear map determined by

$$\Delta'(K_h) = K_h \otimes K_h, \quad \Delta'(E_i) = E_i \otimes 1 + K_{-h_i} \otimes E_i, \text{ and } \Delta'(F_i) = F_i \otimes K_{h_i} + 1 \otimes F_i,$$

for all $i \in I_\ell$ and all $h \in U(\mathfrak{h})$. The coproduct map Δ' induces an action of

$\mathbf{U}_v(\mathfrak{g})$ upon $\mathscr{F}_v$. For example, using the obvious notation,

$$\begin{aligned} E_i\lambda &= E_i\lambda^{(1)}\otimes\lambda^{(2)}\otimes\cdots\otimes\lambda^{(r)} + K_{-h_i}\lambda^{(1)}\otimes E_i\lambda^{(2)}\otimes\cdots\otimes\lambda^{(r)} \\ &\quad + \cdots + K_{-h_i}\lambda^{(1)}\otimes\cdots\otimes K_{-h_i}\lambda^{(r-1)}\otimes E_i\lambda^{(r)} \\ &= \Big(\sum_{\mu^{(1)}} v^{-N_i^r(\lambda^{(1)},\mu^{(1)})}\mu^{(1)}\Big)\otimes\lambda^{(2)}\otimes\cdots\otimes\lambda^{(r)} \\ &\quad + \cdots + v^{\sum_{j=1}^{r-1}-N_i(\lambda^{(j)})}\lambda^{(1)}\otimes\cdots\otimes\lambda^{(r-1)}\otimes\Big(\sum_{\mu^{(r)}} v^{-N_i^r(\lambda^{(r)},\mu^{(r)})}\mu\Big) \\ &= \sum_{\lambda\xrightarrow{i}\mu} v^{-N_i^r(\lambda,\mu)}\mu, \end{aligned}$$

as required. The other calculations are similar.

Finally, $\mathscr{F}_v$ is integrable because each basis element λ is a weight vector and $E_i^{(k)}\lambda = F_i^{(k)}\lambda = 0$ for all sufficiently large k since λ has only a finite number of addable and removable nodes. □

By the Proposition, the empty multipartition $u_\Lambda = \big((0),\dots,(0)\big)$ in $\mathscr{F}_v$ is a highest weight vector of weight Λ. Now integrable $\mathbf{U}_v(\mathfrak{g})$–modules are completely reducible [**18**, 6.2.2], so the highest weight module $L(\Lambda) := \mathbf{U}_v(\mathfrak{g})\cdot u_\Lambda$ is a direct summand of $\mathscr{F}_v$. In order to describe the crystal graph of $L(\lambda)$ we need two lemmas.

2.6 Lemma *Suppose that λ and μ are multipartitions with $\lambda \xrightarrow{i} \mu$ for some $i \in I_\ell$. Let η be the addable node of λ such that $[\mu] = [\lambda]\cup\{\eta\}$. Then η is normal if and only if $N_i^l(\lambda,\mu) \le 0$ and $N_i^l(\lambda,\mu) < N_i^l(\lambda,\nu)$ whenever $\lambda \xrightarrow{i} \nu$ and $\mu \rhd \nu$.*

Proof. The addable i–nodes below η are in bijection with those multipartitions α such that $\lambda \xrightarrow{i} \alpha$ and $\mu \rhd \alpha$; similarly, the removable i–nodes below η correspond to the multipartitions β with $\beta \xrightarrow{i} \mu$ and $\beta \rhd \lambda$. Hence, the Lemma is just a translation of the original definition given in the introduction into the notation of Definition 2.4. □

If $i \in I_\ell$ and $k \ge 0$ let $E_i^{(k)} = E_i^k/[k]_v^!$ and $F_i^{(k)} = F_i^k/[k]_v^!$ where $[k]_v^! = [1]_1[2]_v\dots[k]_v$ where $[k]_v = (v^k - v^{-k})/(v - v^{-1})$.

Suppose that $i \in I_\ell$ and that there exist multipartitions $\lambda, \lambda_1, \lambda_2, \dots, \lambda_k = \mu$ with $\lambda \xrightarrow{i} \lambda_1 \xrightarrow{i} \cdots \xrightarrow{i} \lambda_k = \mu$; then we write $\lambda \xrightarrow{i:k} \mu$.

2.7 Lemma *Let λ be a multipartition and suppose $i \in I_\ell$ and $k \ge 0$. Then*

$$E_i^{(k)}\lambda = \sum_{\nu\xrightarrow{i:k}\lambda} v^{-N_i^r(\nu,\lambda)}\nu \qquad \textit{and} \qquad F_i^{(k)}\lambda = \sum_{\lambda\xrightarrow{i:k}\mu} v^{N_i^l(\lambda,\mu)}\mu,$$

where

$$N_i^r(\nu,\lambda)=\sum_{\gamma\in[\lambda]/[\nu]}\#\Big\{\gamma'\;\Big|\;\begin{matrix}\gamma' \text{ an addable}\\ i\text{–node of } \lambda \textit{ above } \gamma\end{matrix}\Big\}-\#\Big\{\gamma'\;\Big|\;\begin{matrix}\gamma' \text{ a removable}\\ i\text{–node of } \nu \textit{ above } \gamma\end{matrix}\Big\},$$

and

$$N_i^l(\lambda,\mu)=\sum_{\gamma\in[\mu]/[\lambda]}\#\Big\{\gamma'\;\Big|\;\begin{matrix}\gamma' \text{ an addable}\\ i\text{–node of } \mu \textit{ below } \gamma\end{matrix}\Big\}-\#\Big\{\gamma'\;\Big|\;\begin{matrix}\gamma' \text{ a removable}\\ i\text{–node of } \lambda \textit{ below } \gamma\end{matrix}\Big\}.$$

Proof. We only prove the formula for $F_i^{(k)}\lambda$, the proof for $E_i^{(k)}\lambda$ being almost identical. When $k=1$ the lemma is a rephrasing of Proposition 2.5; cf. Lemma 2.6. We proceed now by induction on $k>1$.

Suppose that $F_i^{(k+1)}\lambda\neq 0$ and let μ be a multipartition such that $\lambda\xrightarrow{i:k+1}\mu$. Let $[\mu]\setminus[\lambda]=\{\gamma_0,\gamma_1,\dots,\gamma_k\}$ where these nodes are ordered so that γ_l is below γ_m whenever $l<m$. For $s=0,1,\dots,k$ let μ_s be the multipartition such that $[\mu]=[\mu_s]\cup\{\gamma_s\}$ and set

$$N_i^l(\mu,\lambda;\gamma_s)=\#\Big\{\gamma'\;\Big|\;\begin{matrix}\gamma' \text{ an addable}\\ i\text{–node of } \mu \textit{ below } \gamma_s\end{matrix}\Big\}-\#\Big\{\gamma'\;\Big|\;\begin{matrix}\gamma' \text{ a removable}\\ i\text{–node of } \lambda \textit{ below } \gamma_s\end{matrix}\Big\};$$

then $N_i^l(\lambda,\mu)=\sum_{s=0}^k N_i^l(\mu,\lambda;\gamma_s)$. By induction on k, the coefficient of μ_s in $F_i^{(k)}\lambda$ is $v^{N_i^l(\lambda,\mu_s)}$. Therefore, the multipartition μ appears in $F_iF_i^{(k)}\lambda$ with coefficient $\sum_{s=0}^k v^{N_i^l(\lambda,\mu_s)+N_i^l(\mu_s,\mu)}$. Now $N_i^l(\mu_s,\mu)=N_i^l(\lambda,\mu;\gamma_s)-s$ and

$$\begin{aligned}N_i^l(\lambda,\mu_s)&=\sum_{t=0}^{s-1}N_i^l(\lambda,\mu_s;\gamma_t)+\sum_{t=s+1}^{k}N_i^l(\lambda,\mu_s;\gamma_t)\\&=\sum_{t=0}^{s-1}N_i^l(\lambda,\mu;\gamma_t)+\sum_{t=s+1}^{k}\big(N_i^l(\lambda,\mu;\gamma_t)+1\big)\\&=N_i^l(\lambda,\mu)-N_i^l(\lambda,\mu;\gamma_s)+k-s.\end{aligned}$$

Therefore, the coefficient of μ in $F_iF_i^{(k)}\lambda$ is $\sum_{s=0}^k v^{N_i^l(\lambda,\mu)+k-2s}$. This proves the Lemma because $F_iF_i^{(k)}\lambda=[k+1]_vF_i^{(k+1)}\lambda$ and $[k+1]_v=v^k+v^{k-2}+\dots+v^{-k}$. ◻

Given $i\in I_\ell$ let $\mathbf{U}_i$ be the subalgebra of $\mathbf{U}_v(\mathfrak{g})$ generated by E_i, F_i, and $K_{\pm h_i}$. Every element x of $\mathscr{F}_v$ can be written in a unique way as a linear combination $x=\sum_\Gamma x_\Gamma$ where $\Gamma\in\mathfrak{h}^*$ and $K_hx_\Gamma=v^{\Gamma(h)}x$ for all $h\in\mathfrak{h}$. If $x=x_\Gamma$ for some $\Gamma\in\mathfrak{h}^*$, because E_i and F_i both act as locally nilpotent operators on $\mathscr{F}$, it follows from [**18**, 16.1.4] that

$$x\;=\;\sum_{s\geq 0}F_i^{(s)}x_s,$$

where each x_s is uniquely determined by the conditions that $E_i x_s = 0$ and $K_i x_s = v^{\Gamma(h_i)+2s} x_s$. The Kashiwara operators [**15**] $\tilde{E}_i$ and $\tilde{F}_i$ are defined by

$$\tilde{E}_i x = \sum_{s\geq 1} F_i^{(s-1)} x_s \text{ and } \tilde{F}_i x = \sum_{s\geq 0} F_i^{(s+1)} x_s. \tag{2.8}$$

Let A be the ring of rational functions in $\mathbb{C}(v)$ which do not have a pole at 0. Let $\mathscr{F}_A = \oplus_\lambda A\lambda$ where λ runs over all multipartitions of all integers. In the proof below we implicitly assume Kashiwara's result that the lattice $\mathscr{F}_A$ is stable under the action of $\tilde{E}_i$ and $\tilde{F}_i$; this can be proven by induction on λ using an extension of our argument.

2.9 Theorem (cf. [13, Theorem 3.6]) *Suppose that μ is a multipartition of n and let $i \in I_\ell$.*
(i) *If μ has no good node of residue i then $\tilde{E}_i\mu = 0 \pmod{v\mathscr{F}_A}$.*
(ii) *If η is the good node of residue i in μ and $\mu = \lambda \cup \{\eta\}$ then*

$$\tilde{F}_i\lambda = \mu \pmod{v\mathscr{F}_A} \text{ and } \tilde{E}_i\mu = \lambda \pmod{v\mathscr{F}_A}.$$

Proof. (i) Suppose that μ is a multipartition which does not have a good node of residue i. To show that $\tilde{E}_i\mu = 0 \pmod{v\mathscr{F}_A}$ we introduce an element $\Omega_i(\mu)$ of $\mathscr{F}_A$ below.

Let $\gamma_1, \gamma_2, \dots, \gamma_k$ be the removable i–nodes of μ and for $j = 1, 2, \dots, k$ write $N_i^l(\gamma_j) = N_i^l(\mu_j, \mu)$ where μ_j is the multipartition obtained from μ by deleting γ_j. Suppose that $N_i^l(\gamma_j) < 0$ for some j. Then there exists a node γ with $N_i^l(\gamma) = -1$ and, by Lemma 2.6, the lowest such node is normal. However, by assumption, μ has no normal nodes so this shows that $N_i^l(\gamma_j) > 0$ for all j.

Therefore, from the definition of $N_i^l(\mu_j, \mu)$, there are at least as many addable i–nodes below each γ_j as there are removable i–nodes. Let $\tilde{\gamma}_j$ be the highest addable i–node which is below γ_j such that $N_i^l(\gamma_j) = N_i^l(\tilde{\gamma}_j) + 1$. We claim that the nodes $\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_k$ are all distinct. To see this consider the sequence of A's and R's corresponding to the addable and removable i–nodes for μ, as in the example from the introduction. By assumption, once all of the subsequences of form AR have been removed, no R's remain in this sequence. It is easy to see that each pair $(\tilde{\gamma}_j, \gamma_j)$ corresponds to one of the deleted subsequences AR; this establishes our claim.

Now let

$$\Omega_i(\mu) = \sum_{J\subset\{1,2,\dots,k\}} (-v)^{|J|}\mu_J,$$

where μ_J is the multipartition obtained from μ by moving γ_j to $\tilde{\gamma}_j$ for each $j \in J$. The only multipartitions appearing in $E_i\Omega_i(\mu)$ are those of the form μ_{JK} where $[\mu_{JK}] = [\mu] \cup \{\, \tilde{\gamma}_j \mid j \in J \,\} \setminus \{\, \gamma_j \mid j \notin K \,\}$ for disjoint subsets J, K of $\{1, 2, \dots, k\}$ such that $J \cup K \cup \{l\} = \{1, 2, \dots, k\}$ for some $l \notin J \cup K$. The coefficient of μ_{JK} in $E_i\Omega_i(\mu)$ is

$$(-v)^{|J|+1} v^{-N_i^r(\mu_{J\cup\{l\}K}, \mu_{JK})} + (-v)^{|J|} v^{-N_i^r(\mu_{JK\cup\{l\}}, \mu_{JK})} = 0.$$

Therefore, $E_i\Omega_i(\mu)=0$ and hence $\tilde E_i\Omega_i(\mu)=0$ by (2.8). Consequently, $\tilde E_i\mu\equiv \tilde E_i\Omega_i(\mu)\equiv 0 \pmod{v\mathscr{F}_A}$.

(ii) Rather than computing $\tilde F_i\lambda$ directly, we again assume that λ has no normal node of residue i and calculate $\tilde F_i^t\lambda$ for $t\ge 0$. Maintain the notation from part (i) and let $\mathscr{A}_i(\lambda_J)$ be the set of addable i–nodes of λ_J, and write

$$\mathscr{A}_i=\{\alpha_1,\dots,\alpha_s\}=\bigcap_J \mathscr{A}_i(\lambda_J)=\{\,\alpha\in\mathscr{A}_i(\lambda)\mid \alpha\ne\tilde\gamma_j \text{ for any } j\,\},$$

where we order the elements of $\mathscr{A}_i$ so that α_l is below α_m whenever $l<m$. Note that $s=|\mathscr{A}_i|=N_i(\lambda)$ and that $N_i^l(\alpha_m)=m$ for all m. If $K\subseteq\mathscr{A}_i$ write $\lambda\cup K$ for the multipartition with diagram $[\lambda]\cup K$. Now $\lambda\equiv\Omega_i(\lambda)\pmod{v\mathscr{F}_A}$; so, if $t\ge 0$ then $\tilde F_i^t\lambda\equiv\tilde F_i^t\Omega_i(\lambda)=F_i^{(t)}\Omega_i(\lambda)$ by (2.8), since $E_i\Omega_i(\lambda)=0$. Therefore, using Lemma 2.7, we find that

$$\begin{aligned}
\tilde F_i^t\lambda &\equiv F_i^{(t)}\Omega_i(\lambda) \pmod{v\mathscr{F}_A}\\
&=\sum_{J\subset\{1,2,\dots,k\}}(-v)^{|J|}\sum_{\substack{K\subset\mathscr{A}_i(\lambda_J)\\|K|=t}} v^{N_i^l(\lambda_J,\lambda_J\cup K)}(\lambda_J\cup K)\\
&=\sum_{J\subset\{1,2,\dots,k\}}(-v)^{|J|}\sum_{\substack{K\subset\mathscr{A}_i\\|K|=t}} v^{N_i^l(\lambda_J,\lambda_J\cup K)}(\lambda_J\cup K)\\
&=\sum_{\substack{K\subset\mathscr{A}_i\\|K|=t}} v^{N_i^l(\lambda_J,\lambda_J\cup K)}\sum_{J\subset\{1,2,\dots,k\}}(-v)^{|J|}(\lambda_J\cup K)\\
&\equiv\begin{cases}\lambda\cup\{\alpha_1,\alpha_2,\dots,\alpha_t\} \pmod{v\mathscr{F}_A}, & \text{if } t\le s=N_i(\lambda),\\ 0 \pmod{v\mathscr{F}_A}, & \text{otherwise.}\end{cases}
\end{aligned}$$

To complete the proof it remains to observe that α_t is the good node of residue i in $\lambda\cup\{\alpha_1,\dots,\alpha_t\}$; thus, $\tilde F_i\cdot\lambda\cup\{\alpha_1,\dots,\alpha_{t-1}\}=\lambda\cup\{\alpha_1,\dots,\alpha_t\}$ for $t=1,2,\dots,s$ proving (ii). □

We can now describe the crystal graph of $L(\Lambda)$. Let $\mathbf{U}_A$ be the Lusztig–Kostant $\mathbb{Z}[v,v^{-1}]$–form of $\mathbf{U}_v(\mathfrak{g})$, with coefficients extended to A and let $L(\Lambda)_A=\mathbf{U}_Au_\Lambda$. Then Kashiwara [**15**] has shown that $L(\Lambda)_A$ is a free A–submodule of $L(\Lambda)$ which is stable under the action of $\tilde E_i$ and $\tilde F_i$, for all $i\in I_\ell$, and that

$$\mathbf{B}_0(\Lambda)=\{\,\tilde F_{i_1}\dots\tilde F_{i_k}u_\Lambda+v\mathrm{u}_A^0\mid i_1,\dots,i_k\in I_\ell\,\}\setminus\{0\}$$

is a basis of $L(\Lambda)_A/vL(\Lambda)_A$. Moreover, if $b,b'\in\mathbf{B}_0(\Lambda)$ then $\tilde F_ib=b'$ if and only if $\tilde E_ib'=b$.

2.10 Definition (Kashiwara [15]) *The* crystal graph *of $L(\Lambda)$ is the edge labelled directed graph with vertex set $\mathbf{B}_0(\Lambda)$ and edges $b\xrightarrow{i}b'$ whenever $\tilde F_ib=b'$ for $b,b'\in\mathbf{B}_0(\Lambda)$ and $i\in I_\ell$.*

Kashiwara proved that $\mathbf{B}_0(\Lambda)$ is a connected graph. Consequently, from Theorem 2.9 we obtain the following result.

2.11 Corollary *The crystal graph of $L(\Lambda)$ is the graph with vertices the set $\mathscr{K} = \bigcup_{n\geq 0}\mathscr{K}_n$ of Kleshchev multipartitions and edges $\lambda \overset{i}{\twoheadrightarrow} \mu$ given by adjoining good nodes.*

Kashiwara and Lusztig have shown that $\mathbf{B}_0(\Lambda)$ lifts to the canonical basis (or global basis) of $L(\Lambda) = L(\Lambda)_A \otimes_A \mathbb{C}(v)$, so we can specialize the canonical basis to give a basis of $\mathfrak{u}_R^0$ indexed by $\mathbf{B}_0(\Lambda)$. This shows that, under the assumptions of (1.3), the Kleshchev multipartitions $\mathscr{K}_n$ index the simple modules of $\mathscr{H}_{R,n}$. Thus, by (1.2), we have proved Theorem C.

Jimbo *et al* [**13**] (see also [**8**]) have given a different description of the crystal graph of $L(\Lambda)$. The point of this construction is the following conjecture.

2.12 Conjecture *Suppose that R is a field, $q \neq 1$ and that all of the parameters $Q_1, \dots, Q_r$ are non–zero. Then $D_R^\lambda \neq 0$ if and only if λ is a Kleshchev multipartition.*

This conjecture is known to be true when $r = 1$ [**5**] and when $q = -1$ [**19**]. Graham and Lehrer [**11**] have given sufficient conditions for D_R^λ to be non–zero and they have conjectured that these conditions are also necessary. It seems likely that the Kleshchev multipartitions are precisely the multipartitions which satisfy the Graham–Lehrer conjecture.

It remains to prove Theorem D. Recall the Weyl-Kac q-dimension formula [**14**, Proposition 10.10]. Since the positive root system for $A_{\ell-1}^{(1)}$ is explicitly known [**14**, Exercise 6.5], this implies Theorem D when $\ell < \infty$. The case A_∞ is also obvious.

3 Applications to the groups of Lie type

In this final section, we comment on the application of our results to the classification of the irreducible representations of finite classical groups in non–defining characteristic. In general, these modules are parametrized by triples (L, X, ϕ) where L is a Levi subgroup of G, X a cuspidal simple module of L and ϕ is a simple module of the endomorphism ring $\mathrm{End}_G(R_L^G(X))$ of the Harish-Chandra induced module $R_L^G(X)$; see [**10**, Theorem 2.4].

Assume that G is one of the following groups: $GU_n(q)$, $Sp_{2n}(q)$, $CSp_{2n}(q)$ or $SO_{2n+1}(q)$. Then Geck, Hiß and Malle [**10**, Proposition 4.4] have shown that $\mathrm{End}_G(R_L^G(X))$ is isomorphic to a product of Hecke algebras of type B. Thus, given a cuspidal simple module, by determining the appropriate parameters for the associated Hecke algebra, our Theorem D gives the exact number of simple modules in the Harish–Chandra series of (L, X). Since the total number of simple G–modules is known [**9**] it is therefore sufficient to find enough cuspidal simple modules in order to complete the classification of the simple G–modules. This is not possible using the classification of the simple $\mathscr{H}_{R,n}$–modules proposed by Graham and Lehrer [**11**], because their classification does not give a way of counting the number of simple modules.

Thus, Theorem D solves problem (b) from the introduction of [**10**], modulo determination of certain parameter values. We can therefore focus on problem (a) of [**10**]; that is, the problem of finding enough cuspidal simple modules and determining the parameters for them.

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S.A. TOKYO UNIVERSITY OF MERCANTILE MARINE, TOKYO 135-8533 JAPAN. `ariki@ipc.tosho-u.ac.jp`

A.M. UNIVERSITY OF SYDNEY, SYDNEY NSW 2006. AUSTRALIA. `a.mathas@maths.usyd.edu.au`